From Equation (1), integrating with respect to x keeping y constant, we have

$$\phi = x^3y = 2xy^2 + f(y)$$

where $f(y)$ is the "constant" of integration. Substituting this into Equation (2) yields

$$x^3 - 4xy + F'(y) = x^3 - 4xy + 6y^2$$

from which $F'(y) = 6y^2$, i.e., $f(y) = 2y^3 + c$.

Hence, the required function is $\phi = x^3y - 2xy^2 + 2y^3 + c$, where c is an arbitrary constant.

Note that by Theorem 3, Page 130, the existence of such a function is guaranteed, since if $P = 3x^2y - 2y^2$ and $Q = x^3 - 4xy + 6y^2$, then $\partial P/\partial y = 3x^2 - 4y = \partial Q/\partial x$ identically. If $\partial P/\partial y \neq \partial Q/\partial x$, this function would not exist and the given expression would not be an exact differential.

Method 2:

$$\begin{aligned}(3x^2y - 2y^2)\,dx + (x^3 - 4xy + 6y^2)\,dy &= (3x^2y\,dx + x^3\,dy) - (2y^2\,dx + 4xy\,dy) + 6y^2\,dy \\ &= d(x^3y) - d(2xy^2) + d(2y^3) = d(x^3y - 2xy^2 + 2y^3) \\ &= d(x^3y - 2xy^2 + 2y^3 + c)\end{aligned}$$

Then the required function is $x^3y - 2xy^2 + 2y^3 + c$.

This method, called the *grouping method*, is based on our ability to recognize exact differential combinations and is less than Method 1. Naturally, before attempting to apply any method, we should determine whether the given expression is an exact differential by using Theorem 3, Page 130. See Theorem 4, Page 130.

Differentiation of composite functions

6.17. Let $z = f(x, y)$ and $x = \phi(t)$, $y = \psi(t)$ where f, ϕ, ψ are assumed differentiable. Prove

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Using the results of Problem 6.14, we have

$$\frac{dz}{dt} = \lim_{\Delta t \to 0}\frac{\Delta z}{\Delta t} = \lim_{\Delta t \to 0}\left\{\frac{\partial z}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta t} + \varepsilon_1\frac{\Delta x}{\Delta t} + \varepsilon_2\frac{\Delta y}{\Delta t}\right\} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

since, as $\Delta t \to 0$, we have $\Delta x \to 0, \Delta y \to 0, \varepsilon_1 \to 0, \dfrac{\Delta x}{\Delta t} \to \dfrac{dx}{dt}, \dfrac{\Delta y}{\Delta t} \to \dfrac{dy}{dt}$.

6.18. If $z = e^{xy^2}$, $x = t\cos t$, $y = t\sin t$, compute dz/dt at $t = \pi/2$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (y^2e^{xy^2})(-t\sin t + \cos t) + (2xye^{xy^2})(t\cos t + \sin t)$$

At $t = \pi/2$, $x = 0$, $y = \pi/2$. Then $\left.\dfrac{dz}{dt}\right|_{t=\pi/2} = (\pi^2/4)(-\pi/2) + (0)(1) = -\pi^3/8$.

Another method: Substitute x and y to obtain $z = e^{t^3\sin^2 t\cos t}$ and then differentiate.

6.19. If $z = f(x, y)$ where $x = \phi(u, \upsilon)$ and $y = \psi(u, \upsilon)$, prove the following:

$$(a)\ \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \qquad (b)\ \frac{\partial z}{\partial \upsilon} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \upsilon} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \upsilon}$$

(a) From Problem 6.14, assuming the differentiability of f, ϕ, ψ, we have

$$\frac{\partial z}{\partial u} = \lim_{\Delta u \to 0}\frac{\Delta z}{\Delta u} = \lim_{\Delta u \to 0}\left\{\frac{\partial z}{\partial x}\frac{\Delta x}{\Delta u} + \frac{\partial z}{\partial y}\frac{\Delta y}{\Delta u} + \varepsilon_1\frac{\Delta x}{\Delta u} + \varepsilon_2\frac{\Delta y}{\Delta u}\right\} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

(b) The result is proved as in (a) by replacing Δu by $\Delta \upsilon$ and letting $\Delta \upsilon \to 0$.

6.20. Prove that $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$ even if x and y are dependent variables.

Suppose x and y depend on three variables u, υ, w, for example. Then

$$dx = x_u\,du + x_\upsilon\,d\upsilon + x_w dw \quad (1)$$

$$dy = y_u du + y_\upsilon\,d\upsilon + y_w\,dw \quad (2)$$

Thus,

$$z_x dx + z_y dy = (z_x x_u + z_y y_u)\,du + (z_x x_\upsilon + z_{y}y_{\upsilon})\,d\upsilon + (z_x x_w + z_y y_w)\,dw = z_u du + z_\upsilon d\upsilon + z_w = dz$$

using obvious generalizations from Problem 6.19.

6.21. If $T = x^3 - xy + y^3$, $x = \rho\cos\phi$, and $y = \rho\sin\phi$, find (a) $\partial T/\partial\rho$, $\partial T/\partial\rho$ and (b) $\partial T/\partial\phi$.

$$\frac{\partial T}{\partial\rho} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial\rho} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial\rho} = (3x^2 - y)(\cos\phi) + (3y^2 - x)(\sin\phi)$$

$$\frac{\partial T}{\partial\phi} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial\phi} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial\phi} = (3x^2 - y)(-\rho\sin\phi) + (3y^2 - x)(\rho\cos\phi)$$

This may also be worked by direct substitution of x and y in T.

6.22. If $U = z\sin y/x$ where $x = 3r^2 + 2s$, $y = 4r - 2s^3$, and $z = 2r^2 - 3s^2$, find (a) $\partial U/\partial r$ and (b) $\partial U/\partial s$.

(a) $$\frac{\partial U}{\partial r} = \frac{\partial U}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial U}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial U}{\partial z}\frac{\partial z}{\partial r}$$

$$= \left\{\left(z\cos\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)\right\}(6r) + \left\{\left(z\cos\frac{y}{x}\right)\left(\frac{1}{x}\right)\right\}(4) + \left(\sin\frac{y}{x}\right)(4r)$$

$$= -\frac{6ryz}{x^2}\cos\frac{y}{x} + \frac{4z}{x}\cos\frac{y}{x} + 4r\sin\frac{y}{x}$$

(b) $$\frac{\partial U}{\partial s} = \frac{\partial U}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial U}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial U}{\partial z}\frac{\partial z}{\partial s}$$

$$= \left\{\left(z\cos\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)\right\}(2) + \left\{\left(z\cos\frac{y}{x}\right)\left(\frac{1}{x}\right)\right\}(-6s^2) + \left(\sin\frac{y}{x}\right)(-6s)$$

$$= -\frac{2yz}{x^2}\cos\frac{y}{x} - \frac{6s^2z}{x}\cos\frac{y}{x} - 6s\sin\frac{y}{x}$$

6.23. If $x = \rho\cos\phi$, $y = \rho\sin\phi$, shown that $\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 = \left(\frac{\partial V}{\partial\rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial V}{\partial\phi}\right)^2$.

Using the subscript notation for partial derivatives, we have

$$V_\rho = V_x x_\rho + V_y y_\rho = V_x\cos\phi + V_y\sin\phi \quad (1)$$

$$V_\phi = V_x x_\phi + V_y y_\phi = V_x(-\rho\sin\phi) + V_y(\rho\cos\phi) \quad (2)$$

Dividing both sides of Equation (2) by ρ, we have

$$\frac{1}{\rho}V_\phi = -V_x\sin\phi + V_y\cos\phi \quad (3)$$

Then from Equations (1) and (3), we have

$$V_\rho^2 + \frac{1}{\rho^2}V_\phi^2 = (V_x\cos\phi + V_y\sin\phi)^2 + (-V_x\sin\phi + V_y\cos\phi)^2 = V_x^2 + V_y^2$$