

Calculus 1: Final Exam (Sample) Solution

Solve, justifying your answers, the following exercises at home. We will correct in class next thursday.

Exercise 1. Let \mathcal{C} the curve in the space parametrized by

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R}^3 \\ t &\longmapsto (t^3 - \sin 2t, te^t, t^2 - \cos t) \end{aligned}$$

1. compute velocity and acceleration of a particle moving along the curve \mathcal{C} at time $t = 0$; (10)
2. compute the tangent line to \mathcal{C} at $t = \frac{\pi}{2}$; (7.5)
3. compute the normal plane to \mathcal{C} at $t = \frac{\pi}{2}$. (7.5)

Solution Exercise 1.

1. Velocity and acceleration of a particle moving along the curve \mathcal{C} at time $t = 0$ are, respectively, the first and second derivatives vectors of the curve \mathcal{C} evaluated at $t = 0$, that is:

$$\begin{aligned} \frac{d}{dt}(t^3 - \sin 2t, te^t, t^2 - \cos t)|_{t=0} &= (3t^2 - 2 \cos 2t, e^t + te^t, 2t + \sin t)|_{t=0} = (-2, 1, 0) \\ \frac{d}{dt}(3t^2 - 2 \cos 2t, e^t + te^t, 2t + \sin t)|_{t=0} &= (6t + 4 \sin 2t, 2e^t + te^t, 2 + \cos t)|_{t=0} = (0, 2, 3). \end{aligned}$$

2. The tangent line to \mathcal{C} at $t = \frac{\pi}{2}$ has equation:

$$\frac{x - x_0}{\frac{dx}{dt}|_{t=\frac{\pi}{2}}} = \frac{y - y_0}{\frac{dy}{dt}|_{t=\frac{\pi}{2}}} = \frac{z - z_0}{\frac{dz}{dt}|_{t=\frac{\pi}{2}}}$$

where $(x_0, y_0, z_0) = f(\frac{\pi}{2})$. That is:

$$\frac{x - (\frac{\pi}{2})^3}{\frac{3\pi^2}{4} + 2} = \frac{y - \frac{\pi}{2}e^{\frac{\pi}{2}}}{e^{\frac{\pi}{2}} + \frac{\pi}{2}e^{\frac{\pi}{2}}} = \frac{z - (\frac{\pi}{2})^2}{\pi + 1}.$$

3. The normal plane to \mathcal{C} at $t = \frac{\pi}{2}$ has equation

$$\frac{d}{dt}(t^3 - \sin 2t, te^t, t^2 - \cos t)|_{t=\frac{\pi}{2}} \cdot (x - x_0, y - y_0, z - z_0) = 0$$

where $(x_0, y_0, z_0) = f(\frac{\pi}{2})$ and \cdot is the scalar product. Hence we get:

$$\left(\frac{3\pi^2}{4} + 2\right)\left(x - \left(\frac{\pi}{2}\right)^3\right) + \left(e^{\frac{\pi}{2}} + \frac{\pi}{2}e^{\frac{\pi}{2}}\right)\left(y - \frac{\pi}{2}e^{\frac{\pi}{2}}\right) + (\pi + 1)\left(z - \left(\frac{\pi}{2}\right)^2\right) = 0.$$

Exercise 2. (25) Find maxima, minima and saddle point of the function $z = f(x, y)$ defined by:

$$f(x, y) = x^2 + y^4 + 2y^2 - 4xy + 1$$

Solution Exercise 2. Maxima, minima and saddle point of a function $z = f(x, y)$ have to zeros of the gradient of f :

$$\nabla f(x, y) = (2x - 4y, 4y^3 + 4y - 4x) = (0, 0)$$

that is $P = (0, 0)$, $N = (2, 1)$, $M = (-2, -1)$. In order to classify these points we need to study the determinant of the Hessian matrix:

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ -4 & 12y^2 + 4 \end{pmatrix}$$

evaluated in each point. We have that $\det H(0, 0) = -16 < 0$ hence $(0, 0)$ is a saddle point, while $\det H(2, 1) = \det H(-2, -1) = 16 > 0$ and $\frac{\partial^2 f}{\partial x^2} = 2 > 0$ hence $(2, 1)$ and $(-2, -1)$ are minimum points.

Exercise 3. Consider the surface \mathcal{S} defined by $x^3y^2z - 3z^2 = 4xy - 5z + 2x$.

1. Find the tangent plane to \mathcal{S} at the point $(1, 1, 0)$; (7.5)
2. find the normal line to \mathcal{S} at the point $(1, 1, 0)$; (7.5)
3. find the tangent line at the point $(1, 1, 0)$ to the level curve \mathcal{C} obtained intersecting \mathcal{S} with the plane π defined by the equation $z = 0$. (10)

Solution Exercise 3. The surface \mathcal{S} is defined by $F(x, y, z) =$ with

$$F(x, y, z) = x^3y^2z - 3z^2 - 4xy + 5z - 2x.$$

1. The tangent plane to \mathcal{S} at the point $(1, 1, 0)$ has equation

$$\nabla F|_{(1,1,0)} \cdot (x - 1, y - 1, z) = 0$$

where \cdot is the scalar product and ∇F is the gradient of F , that is $\nabla F = (3x^2y^2z - 4y - 2, 2x^3y^2 - 4x, x^3y^2 - 6z + 5)$, evaluated in the point $(1, 1, 0)$. Hence we get:

$$-6(x - 1) - 4(y - 1) + 6z = 0.$$

2. The normal line to \mathcal{S} at the point $(1,1,0)$ has equation:

$$\frac{x-1}{\frac{\partial F}{\partial x}|_{(1,1,0)}} = \frac{y-1}{\frac{\partial F}{\partial y}|_{(1,1,0)}} = \frac{z}{\frac{\partial F}{\partial z}|_{(1,1,0)}}$$

that is:

$$\frac{x-1}{-6} = \frac{y-1}{-4} = \frac{z}{6}.$$

3. The tangent line at the point $(1,1,0)$ to the level curve \mathcal{C} obtained intersecting \mathcal{S} with the plane π defined by the equation $z = 0$ has equation

$$\begin{cases} \frac{x-1}{\frac{\partial F}{\partial y}|_{(1,1,0)}} = \frac{y-1}{-\frac{\partial F}{\partial x}|_{(1,1,0)}} \\ z = 0 \end{cases}$$

from whom we get:

$$\begin{cases} \frac{x-1}{-4} = \frac{y-1}{6} \\ z = 0 \end{cases}.$$

Exercise 4. Consider the two variables function defined by $f(x, y) = \log(\sqrt{x^2 + y^2})$.

1. Compute the domain of f ; (5)
2. compute the Laplacian $\Delta f(x, y) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$; (5)
3. consider polar coordinates $x = r \cos \theta$, $y = r \sin \theta$ ($r > 0$). By the chain rule, compute the partial derivatives $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$ in terms of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$; (5)
4. show that for any function $z = f(x, y)$ the following equality holds: (10)

$$\left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

Solution Exercise 4.

1. The domain of the two variables function defined by $f(x, y) = \log(\sqrt{x^2 + y^2})$ is given by $\sqrt{x^2 + y^2} > 0$, that is $\mathbb{R}^2 \setminus (0, 0)$.
2. The gradient of the function f is

$$\nabla f(x, y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right),$$

hence the Laplacian

$$\Delta f(x, y) = \left(\frac{y^2 - x^2}{(x^2 + y^2)^2}, \frac{x^2 - y^2}{(x^2 + y^2)^2}\right).$$

3. consider polar coordinates $x = r \cos \theta$, $y = r \sin \theta$ ($r > 0$). By the chain rule

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r}$$

and

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta}.$$

Hence, by simple computations we get:

$$\frac{\partial f}{\partial r} = \frac{x}{x^2 + y^2} \cos \theta + \frac{y}{x^2 + y^2} \sin \theta = \frac{r \cos^2 \theta}{r^2} + \frac{r \sin^2 \theta}{r^2} = \frac{1}{r}$$

and

$$\frac{\partial f}{\partial \theta} = \frac{r \cos \theta}{r^2} r(-\sin \theta) + \frac{r \sin \theta}{r^2} r \cos \theta = 0.$$

4. If $z = f(x, y)$ is a function, then

$$\begin{aligned} \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial f}{\partial \theta}\right)^2 &= \left(\frac{\partial f}{\partial x} \cos \theta + \frac{\partial f}{\partial y} \sin \theta\right)^2 + \frac{1}{r^2} \left(\frac{\partial f}{\partial x} (-r \sin \theta) + \frac{\partial f}{\partial y} (r \cos \theta)\right)^2 = \\ &= \left(\frac{\partial f}{\partial x}\right)^2 \cos^2 \theta + \left(\frac{\partial f}{\partial y}\right)^2 \sin^2 \theta + 2 \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \cos \theta \sin \theta + \left(\frac{\partial f}{\partial x}\right)^2 \sin^2 \theta + \\ &+ \left(\frac{\partial f}{\partial y}\right)^2 \cos^2 \theta - 2 \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \cos \theta \sin \theta = \\ &= \left(\frac{\partial f}{\partial x}\right)^2 (\cos^2 \theta + \sin^2 \theta) + \left(\frac{\partial f}{\partial y}\right)^2 (\cos^2 \theta + \sin^2 \theta) = \\ &= \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2. \end{aligned}$$