Calculus 1: Middle test exam

Solve, **justifying your answers**, the following exercises.

Exercise 1. A sequence $\{a_n\}_{n\in\mathbb{N}}$ is increasing if $a_n \leq a_{n+1}$, decreasing if $a_n \geq a_{n+1}$. Is the sequence $\{a_n\}_{n\in\mathbb{N}}$ increasing, decreasing or neither when:

$$a_n = \frac{n^2-3}{n^4-2}\cos n\pi (6); \quad a_n = \frac{n^2}{n^2+2}(6).$$

Solution Exercise 1. A sequence $\{a_n\}_{n\in\mathbb{N}}$ is increasing if $a_n \leq a_{n+1}$, decreasing if $a_n \geq a_{n+1}$.

1. $a_n = \frac{n^2-3}{n^4-2}\cos n\pi$ is clearly neither increasing nor decreasing as $\cos n\pi = -1$ if $n = 2k+1$ and $\cos n\pi = 1$ if $n = 2k$. That is this is an alternating sequence.
2. If $a_n = \frac{n^2}{n^2+2}$, the following equalities hold:

$$a_{n+1} - a_n = \frac{(n+1)^2}{(n+1)^2+2} - \frac{n^2}{n^2+2} =$$
$$\frac{n^4+2n^2+2n^3+4n+n^2+2-(n^4+2n^3+3n^2)}{(n^2+2n+3)(n^2+2)} = \frac{4n+2}{(n^2+2n+3)(n^2+2)} > 0$$

that is a_n is an increasing sequence.

Exercise 2. Compute the following limits:

$$\lim_{x\to 0}(\frac{1}{x} - \frac{1}{\tan x})(7); \quad \lim_{x\to 1}(\frac{1}{\ln x} - \frac{1}{x-1})(6)$$
$$\lim_{x\to\infty}(\sqrt{x+\sqrt{x}} - \sqrt{x-\sqrt{x}})(7); \quad \lim_{x\to 1}\frac{x^2 \mid x-1 \mid \ln(x+1)}{e^{\sqrt{x^2+2x}}(x-1)}.(6)$$

Solution Exercise 2.

$$\lim_{x\to 0}(\frac{1}{x} - \frac{1}{\tan x}) = \lim_{x\to 0}(\frac{\tan x - x}{x\tan x})$$

and applying De L'Hopital we get

$$\lim_{x\to 0}(\frac{\tan x - x}{x\tan x}) = \lim_{x\to 0}[(\frac{\cos^2 x}{\cos^2 x})(\frac{1-\cos^2 x}{\sin x\cos x + x})] = \lim_{x\to 0}(\frac{2\cos x\sin x}{\cos^2 x - \sin^2 x + 1}) = 0.$$

For the second limit we have

$$\lim_{x\to 1}(\frac{1}{\ln x} - \frac{1}{x-1}) = \lim_{x\to 1}(\frac{x-1-\ln x}{x\ln x})$$

and applying De L'Hopital we get

$$\lim_{x\to 1}(\frac{x-1-\ln x}{(x-1)\ln x}) = \lim_{x\to 1}(\frac{1-\frac{1}{x}}{\ln x + \frac{x-1}{x}}) = \lim_{x\to 1}[(\frac{x}{x})\frac{x-1}{x\ln x + x - 1}] = \lim_{x\to 1}\frac{1}{\ln x + 1 + 1} = \frac{1}{2}$$

For the third limit we have

$$\lim_{x\to\infty}(\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}) = \lim_{x\to\infty}(\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}})\frac{\sqrt{x+\sqrt{x}}+\sqrt{x-\sqrt{x}}}{\sqrt{x+\sqrt{x}}+\sqrt{x-\sqrt{x}}} =$$

$$= \lim_{x\to\infty}\frac{x+\sqrt{x}-x+\sqrt{x}}{\sqrt{x+\sqrt{x}}+\sqrt{x-\sqrt{x}}} = \lim_{x\to\infty}\frac{2\sqrt{x}}{\sqrt{x}(\sqrt{1+\frac{1}{\sqrt{x}}}+\sqrt{1-\frac{1}{\sqrt{x}}})} = 1$$

The forth limits is:

$$\lim_{x\to 1}\frac{x^2 \mid x-1 \mid \ln(x+1)}{e^{\sqrt{x^2+2x}}(x-1)} = \quad \lim_{x\to 1}[(\frac{x^2\ln(x+1)}{e^{\sqrt{x^2+2x}}})\frac{\mid x-1 \mid}{x-1}]$$

since

$$\lim_{x\to 1}(\frac{x^2\ln(x+1)}{e^{\sqrt{x^2+2x}}}) = (\frac{\ln(2)}{e^{\sqrt{3}}}$$

and

$$\lim_{x\to 1}\frac{\mid x-1 \mid}{x-1} = \lim_{x\to 0}\frac{\mid x \mid}{x}$$

that doesn't exist, we get that the forth limits doesn't exist.

Exercise 3. Given three real functions $f, g, h : \mathbb{R} \longrightarrow \mathbb{R}$ write the rule for the derivative $D\frac{f(g(x))}{h(x)}$ (6).
Find the definition domain and compute the derivative of the function $f(x) = \frac{x^3\ln(1+x^2)}{e^{\sqrt{x^2-2}}}$ where it is defined. (10)

Solution Exercise 3. If $y = g(x)$ and $f'(y) = D(f(y))$ is the derivative of the function f with respect the variable y we get the formula

$$D\frac{f(g(x))}{h(x)} = \frac{f'(y)g'(x)h(x) - h'(x)f(g(x))}{h^2(x)} \quad .$$

The function $f(x) = \frac{x^3\ln(1+x^2)}{e^{\sqrt{x^2-2}}}$ is defined for $x^2 - 2 \geq 0$ that is $x \leq -\sqrt{2}$ $x \geq \sqrt{2}$. It is not difficult to see that the derivative is defined where the function is defined and we get

$$D(\frac{x^3\ln(1+x^2)}{e^{\sqrt{x^2-2}}}) = \frac{(3x^2\ln(1+x^2) + x^3\frac{2x}{1+x^2})e^{\sqrt{x^2-2}} - e^{\sqrt{x^2-2}}x^3\ln(1+x^2)\frac{x}{\sqrt{x^2-2}}}{e^{2\sqrt{x^2-2}}}.$$

Exercise 4. Give the definition of continuity and differentiability for a function $f : A \subset \mathbb{R} \longrightarrow \mathbb{R}$. (8) Use the definition of continuity and differentiability to justify your answer to the following questions.

1. Let
$$g(x) = \begin{cases} x^2 + 2x & x < -1 \\ 0 & -1 \leq x \leq 1 \\ \ln x & x > 1 \end{cases} .$$
Where is $g(x)$ not continuous? (4) Where is $g(x)$ not differentiable? (4) Use the definition of continuity and differentiability to justify your answer.

2. Consider the function:
$$f(x) = \begin{cases} 2cx & x < 4 \\ x + a & x = 4 \\ x^2 - 6 & x > 4 \end{cases}$$
   (a) study for which values of $a, c \in \mathbb{R}$ the function $f(x)$ is continuous in $x = 4$; (6)
   (b) study for which values of $a, c \in \mathbb{R}$ the function $f(x)$ is differentiable in $x = 4$. (6)

Solution Exercise 4. A function $f : A \subset \mathbb{R} \longrightarrow \mathbb{R}$ is continuous in A if for any $x_0 \in A$ $\lim_{x \to x_0} f(x) = f(x_0)$ while it is differentiable in A if for any $x_0 \in A$ it exists the $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$.

1. From definition of continuity it is an easy computation to check that $g(x)$ is continuous in $\mathbb{R} \setminus \{-1\}$. By computing the limit of the increment $\frac{f(x)-f(x_0)}{x-x_0}$ for $x_0 = 1$ we obtain that the limit from the right and from the left are different, that is $g(x)$ is differentiable in $\mathbb{R} \setminus \{-1, 1\}$.

2. Since $\lim x \to 4^+ = 10$ and $\lim x \to 4^- = 8c$ and, by continuity, we have to have $f(4) = 4 + a = 10 = 8c$ we get that the function is continuous in $x =$ if and only if $a = 6$ and $c = \frac{5}{4}$. As the function is continuous if and only if $a = 6$ and $c = \frac{5}{4}$, we have that for sure is not differentiable if those two conditions are not satisfied. If $a = 6$ and $c = \frac{5}{4}$, by an easy computation on the limit of the increment in $x = 4$ we get that the function is not differentiable in $x = 4$.

Exercise 5. Taylor series.

1. Compute the approximation of the value of $\cos 0.3$ and the error using Taylor polynomial up to the 5th order; (10)

2. Prove the inequality $1 + 2x \leq e^{2x}$ using Taylor series. (10)

Solution Exercise 5.

1. $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + R_5$ where R_5 is the Lagrange remainder $R_5 = \frac{1}{5!}\sin\xi x^5$ and we get $\cos 0.3 = 1 - \frac{(0.3)^2}{2} + \frac{(0.3)^4}{24} + R_5$ with $R_5 \leq \frac{1}{5!}(0.3)^5$.

2. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and hence $e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$. Then if $x \geq 0$ obviously $1 + 2x < 1 + 2x + \sum_{n=2}^{\infty} \frac{(2x)^n}{n!}$ since $\sum_{n=2}^{\infty} \frac{(2x)^n}{n!} > 0$. Moreover if $1 + 2x \leq 0$ the inequality is obvious since $e^{2x} > 0$ for any $x \in \mathbb{R}$. We need only to check the case $\frac{-1}{2} < x < 0$. Since the series is alternanting we need to prove that $\frac{x^{2n}}{2n!} + \frac{x^{2n+1}}{(2n+1)!} > 0$.

$$\frac{x^{2n}}{2n!} + \frac{x^{2n+1}}{(2n+1)!} = \frac{(2n+1)x^n + x^{2n+1}}{(2n+1)!}$$

and by $\frac{-1}{2} < x < 0$ obviously $\frac{(2n+1)x^n + x^{2n+1}}{(2n+1)!} > 0$.