Calculus 1: Middle test exam

Solve, **justifying your answers**, the following exercises.

Exercise 1. Solve the following exercises on sequences.

1. A sequence $\{a_n\}_{n\in\mathbb{N}}$ is increasing if $a_n \leq a_{n+1}$, decreasing if $a_n \geq a_{n+1}$. Is the sequence $\{a_n\}_{n\in\mathbb{N}}$ increasing, decreasing or neither when:
$$a_n = (1+\frac{1}{n})^n(8); \quad a_n = \frac{3n}{n+1}(4).$$
2. Compute the limits of the following sequences:
$$\lim_{n\to\infty}(\frac{n^3+16}{22n^2(n+2\sqrt{n})}+\frac{2^n}{3^n})(5); \quad \lim_{n\to\infty}(2\times 4^n+\frac{7^n}{2})^{\frac{1}{2n}}.(8)$$

Exercise 2. Compute the following limits:
$$\lim_{x\to 0} x^{\sin x}(7); \quad \lim_{x\to 1}\frac{(x-1)\ln x-\frac{(x-1)^2}{x}}{(e^x-e)^3}.(6)$$

Exercise 3. Given two real functions $f,g:\mathbb{R}\longrightarrow\mathbb{R}$ write the rule for the derivative $D\frac{f(g(x))}{h(x)}$ (6).

Compute the derivative of the function $f(x)=\frac{\sqrt{1+x^2}}{\tan^{-1}x}$. (10)

Exercise 4. Answer to the following exercises on continuity and differentiability:

1. Give an example of a function $f(x)$ continuous, but not differentiable in a point (motivate the answer!); (6)
2. Give an example of a function $f(x)$ continuous and differentiable in a point, but not differentiable two times in the point; (8)
3. for $n\in\mathbb{N}$ consider the function:
$$f(x)=\begin{cases} e^{\frac{1}{x}} & x<0 \\ 0 & x=0 \\ x^a\ln x & x>0 \end{cases}$$
   - study for which values of $a\in\mathbb{N}$ the function $f(x)$ is continuous in $x=0$; (6)
   - study for which values of $a\in\mathbb{N}$ the function $f(x)$ is differentiable in $x=0$. (8)

Exercise 5. Verify if the following inequalities hold using Taylor series:

1. $x+2\geq e^{x+1}$ for $x>0$; (10)
2. $e^{\frac{x}{2}}-1\leq\frac{x}{2}e^x$ for $x>0$. (10)