Calculus 1: Final Exam

Solve, justifying your answers, the following exercises.

Exercise 1. Let $\mathcal{C}$ the curve in the space parametrized by

$$\begin{aligned} f :& \mathbb{R} \longrightarrow \mathbb{R}^3 \\ & t \longmapsto (t^2 \sin t, e^t, t^2 - \cos t + \sin t) \end{aligned}$$

1. compute velocity and acceleration of a particle moving along the curve $\mathcal{C}$ at time $t = 0$; (8)
2. is it moving or stationary along the x axis at the instant $t = 0$? (2)
3. compute the tangent line to $\mathbb{C}$ at $t = \frac{\pi}{2}$; (7.5)
4. compute the normal plane to $\mathbb{C}$ at $t = \frac{\pi}{2}$. (7.5)

Exercise 2. Let $z = e^{f(x,y)}$ be a two variable function.

1. Find maxima, minima and saddle point when $z = e^{-\frac{1}{3}x^3 + x - y^2}$ (20);
2. show that for all functions $z = f(x, y)$, the two functions $z = e^{f(x,y)}$ and $z = f(x, y)$ have the same critical points (5);
3. show that, if P is a critical point and $H(f)_{|P} = H(f(x_0, y_0))$ is the Hessian of $f(x, y)$ evaluated in P, then $H(e^{f(x_0,y_0)}) = e^{f(x_0,y_0)} H(f(x_0, y_0))$ (5).
4. What can we conclude about maxima, minima and saddle points of $z = e^{f(x,y)}$ and $z = f(x, y)$? (2)

Exercise 3. Consider the surface $\mathcal{S}$ defined by $x^4yz^2 - 3x^2 = 2y^2z - 5y - 4z$.

1. Find the tangent plane to $\mathcal{S}$ at the point (1,0,1); (7.5)
2. find the normal line to $\mathcal{S}$ at the point (1,0,1); (7.5)
3. find the tangent line at the point (1,0,1) to the level curve $\mathcal{C}$ obtained intersecting $\mathcal{S}$ with the plane π defined by the equation $z = 1$. (10)

Exercise 4. Consider the two variables function defined by $f(x, y) = \log(\sqrt{y - x^2})$.

1. Compute the domain of f (4) and draw it (that is show it graphically). (3) Is it open or closed? (3)
2. consider polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ $(r > 0)$. By the chain rule, compute the partial derivatives $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$ in terms of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$; (5)
3. what do level curves $c = \log(\sqrt{y - x^2})$ (c constant) look like? (that is are they circles, hyperbola or parabola etc..)(3)