CHAPTER 3

Functions, Limits, and Continuity

The notions described in this chapter historically followed the introduction of differentiation and integration. These concepts were established, developed, and applied in the 1700s on a strong mechanical basis but a weak theoretical foundation. In the 1800s, the theoretical inadequacies were resolved with the mathematical invention of limits. Precise definitions of derivatives and integrals were formulated. Many mathematicians, including Bolzano, introduced rigorous proofs free of geometry. Elegant notation, such as the $\varepsilon - \delta$ form of Weierstrass, became available. As a bonus, clear definitions of irrational numbers were made. Also, unexpected properties of infinite sets of real numbers were found by Cantor and other mathematicians.

This chapter sets forth the notion of the limit of a function, concepts that followed, and how these ideas made possible the rigorization of analysis.

Functions

A function is composed of a domain set, a range set, and a rule of correspondence that assigns exactly one element of the range to each element of the domain.

This definition of a function places no restrictions on the nature of the elements of the two sets. However, in our early exploration of the calculus, these elements are real numbers. The rule of correspondence can take various forms, but in advanced calculus it most often is an equation or a set of equations.

If the elements of the domain and range are represented by x and y, respectively, and f symbolizes the function, then the rule of correspondence takes the form $y = f(x)$.

The distinction between f and $f(x)$ should be kept in mind. f denotes the function as defined in the first paragraph. y and $f(x)$ are different symbols for the range (or image) values corresponding to domain values x. However, a common practice that provides an expediency in presentation is to read $f(x)$ as "the image of x with respect to the function f" and then use it when referring to the function. (For example, it is simpler to write $\sin x$ than "the sine function, the image value of which is $\sin x$.") This deviation from precise notation appears in the text because of its value in exhibiting the ideas.

The domain variable x is called the *independent* variable. The variable y representing the corresponding set of values in the range, is the *dependent* variable.

Note: There is nothing exclusive about the use of x, y, and f to represent domain, range, and function. Many other letters are employed.

There are many ways to relate the elements of two sets. (Not all of them correspond a unique range value to a given domain value.) For example, given the equation $y^2 = x$, there are two choices of y for each positive value of x. As another example, the pairs (a, b), (a, c), (a, d), and (a, e) can be formed, and again the correspondence to a domain value is not unique. Because of such possibilities, some texts, especially older ones, distinguish between multiple-valued and single-valued functions. This viewpoint is not consistent with our definition or modern presentations. In order that there be no ambiquity, the calculus and its applications require a single image associated with each domain value. A multiple-valued rule of correspondence gives rise to a collection of functions (i.e., single-valued). Thus, the rule $y^2 = x$ is replaced by the pair of rules $y = -x^{1/2}$

and the functions they generate through the establishment of domains. (See the following section on graphs for pictorial illustrations.)

EXAMPLES. 1. If to each number in $-1 \leqq x \leqq 1$ we associate a number y given by x^2, then the interval $-1 \leqq x \leqq 1$ is the domain. The rule $y = x^2$ generates the range $-1 \leqq y \leqq 1$. The totality is a function f.

The functional image of x is given by $y = f(x) = x^2$. For example, $f\left(-\frac{1}{3}\right) = \left(-\frac{1}{3}\right)^2 = \frac{1}{9}$ is the image of $-\frac{1}{3}$ with respect to the function f.

2. The sequences of Chapter 2 may be interpreted as functions. For infinite sequences, consider the domain as the set of positive integers. The rule is the definition of u_n, and the range is generated by this rule. To illustrate, let $u_n = \frac{1}{n}$ with $n = 1, 2, \ldots$. Then the range contains the elements $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$. If the function is denoted by f, then we may write $f(n) = \frac{1}{n}$.

As you read this chapter, reviewing Chapter 2 will be very useful.

3. With each time t after the year 1800 we can associate a value P for the population of the United States. The correspondence between P and t defines a function—say, F—and we can write $P = F(t)$.
4. For the present, both the domain and the range of a function have been restricted to sets of real numbers. Eventually this limitation will be removed. To get the flavor for greater generality, think of a map of the world on a globe with circles of latitude and longitude as coordinate curves. Assume there is a rule that corresponds this domain to a range that is a region of a plane endowed with a rectangular Cartesian coordinate system. (Thus, a flat map usable for navigation and other purposes is created.) The points of the domain are expressed as pairs of numbers (θ, ϕ), and those of the range by pairs (x, y). These sets and a rule of correspondence constitute a function whose independent and dependent variables are not single real numbers; rather, they are pairs of real numbers.

Graph of a Function

A function f establishes a set of ordered pairs (x, y) of real numbers. The plot of these pairs $[x, f(x)]$ in a coordinate system is the graph of f. The result can be thought of as a pictorial representation of the function.

For example, the graphs of the functions described by $y = x^2$, $-1 \leqq x \leqq 1$, and $y^2 = x$, $0 \leqq x \leqq 1$, $y \geqq 0$ appear in Figure 3.1.

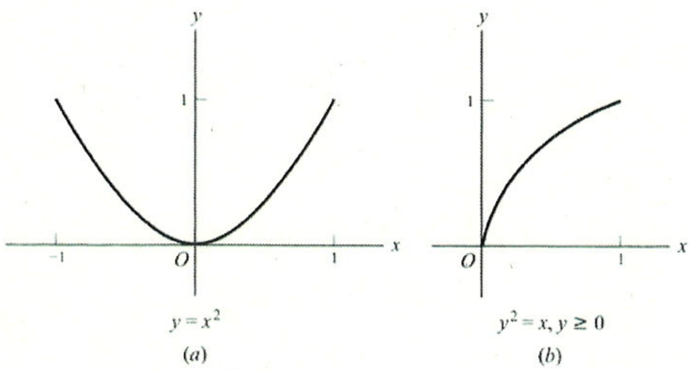

Figure 3.1

Bounded Functions

If there is a constant M such that $f(x) \leqq M$ for all x in an interval (or other set of numbers), we say that f is *bounded above* in the interval (or the set) and call M an *upper bound* of the function.

If a constant m exists such that $f(x) \geqq m$ for all x in an interval, we say that $f(x)$ is *bounded below* in the interval and call m a *lower bound.*

If $m \leqq f(x) \leqq M$ in an interval, we call $f(x)$ *bounded.* Frequently, when we wish to indicate that a function is bounded, we write $|f(x)| < P$.

EXAMPLES. 1. $f(x) = 3 + x$ is bounded in $-1 \leqq x \leqq 1$. An upper bound is 4 (or any number greater than 4). A lower bound is 2 (or any number less than 2).

2. $f(x) = 1/x$ is not bounded in $0 < x < 4$, since, by choosing x sufficiently close to zero, $f(x)$ can be made as large as we wish, so that there is no upper bound. However, a lower bound is given by $\frac{1}{4}$ (or any number less than $\frac{1}{4}$).

If $f(x)$ has an upper bound, it has a *least upper bound* (l.u.b.); if it has a lower bound, it has a *greatest lower bound* (g.l.b.). (See Chapter 1 for these definitions.)

Monotonic Functions

A function is called *monotonic increasing* in an interval if for any two points x_1 and x_2 in the interval $x_1 < x_2$, $f(x_1) \leqq f(x_2)$. If $(f(x_1) < f(x_2)$, the function is called *strictly increasing*.

Similarly, if $f(x_1) \geqq f(x_2)$ whenever $x_1 < x_2$, then $f(x)$ is *monotonic decreasing,* while if $f(x_1) > f(x_2)$, it is *strictly decreasing.*

Inverse Functions, Principal Values

Suppose y is the range variable of a function f with domain variable x. Furthermore, let the correspondence between the domain and range values be one-to-one. Then a new function f^{-1}, called the *inverse function* of f, can be created by interchanging the domain and range of f. This information is contained in the form $x = f^{-1}(y)$.

As you work with the inverse function, it often is convenient to rename the domain variable as x and use y to symbolize the images; then the notation is $y = f^{-1}(x)$. In particular, this allows graphical expression of the inverse function with its domain on the horizontal axis.

Note: f^{-1} does *not* mean f to the negative one power. When used with functions, the notation f^{-1} always designates the inverse function to f.

If the domain and range elements of f are not in one-to-one correspondence (this would mean that distinct domain elements have the same image), then a collection of one-to-one functions may be created. Each of them is called a *branch*. It is often convenient to choose one of these branches, called the *principal branch*, and denote it as the inverse function f^{-1}. The range values of f that compose the principal branch, and hence the domain of f^{-1}, are called the *principal values*. (As will be seen in the section on elementary functions, it is common practice to specify these principal values for that class of functions.)

EXAMPLE. Suppose f is generated by $y = \sin x$ and the domain is $-\infty \leqq x \leqq \infty$. Then there are an infinite number of domain values that have the same image. (A finite portion of the graph is illustrated in Figure 3.2(a). In Figure 3.2(b) the graph is rotated about a line at 45° so that the x axis rotates into the y axis. Then the variables are interchanged so that the x axis is once again the horizontal one. We see that the image of an x value is not unique. Therefore, a set of principal values must be chosen to establish an inverse function. A choice of a branch is accomplished by restricting the domain of the starting function, $\sin x$. For example, choose $-\frac{\pi}{2} \leqq x \leqq \frac{\pi}{2}$. Then there is a one-to-one correspondence between the elements of this domain and the images in

$-1 \leq x \leq 1$. Thus, f^{-1} may be defined with this interval as its domain. This idea is illustrated in Figure 3.2(*c*) and (*d*). With the domain of f^{-1} represented on the horizontal axis and by the variable x, we write $y = \sin^{-1} x$, $-1 \leq x \leq 1$.

If $x = -\dfrac{1}{2}$, then the corresponding range value is $y = -\dfrac{\pi}{6}$.

Note: In algebra, b^{-1} means $\dfrac{1}{b}$ and the fact that bb^{-1} produces the identity element 1 is simply a rule of algebra generalized from arithmetic. Use of a similar exponential notation for inverse functions is justified in that corresponding algebraic characteristics are displayed by $f^{-1}[f(x)] = x$ and $f[f^{-1}(x)] = x$.

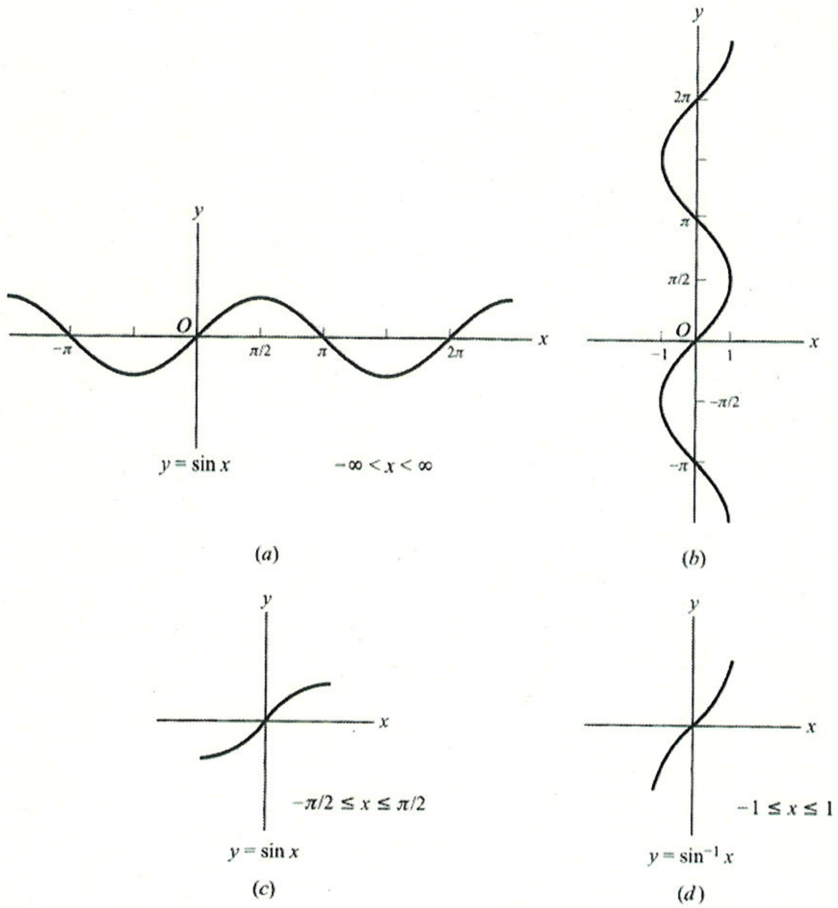

Figure 3.2

Maxima and Minima

The seventeenth-century development of the calculus was strongly motivated by questions concerning extreme values of functions. Of most importance to the calculus and its applications were the notions of *local extrema*, called the *relative maximum* and *relative minimum*.

If the graph of a function were compared to a path over hills and through valleys, the local extrema would be the high and low points along the way. This intuitive view is given mathematical precision by the following definition.

Definition If there exists an open interval (a, b) containing c such that $f(x) < f(c)$ for all x other than c in the interval, then $f(c)$ is a *relative maximum* of f. If $f(x) > f(c)$ for all x in (a, b) other than c, then $f(c)$ is a *relative minimum* of f. (See Figure 3.3.)

Functions may have any number of relative extrema. On the other hand, they may have none, as in the case of the strictly increasing and decreasing functions previously defined.

Definition If c is in the domain of f and for all x in the domain of the function $f(x) \leqq f(c)$; then $f(c)$ is an *absolute maximum* of the function f. If for all x in the domain $f(x) \geqq f(c)$, then $f(c)$ is an *absolute minimum* of f. (See Figure 3.3.)

Note: If defined on closed intervals, the strictly increasing and decreasing functions possess *absolute extrema*.

Absolute extrema are not necessarily unique. For example, if the graph of a function is a horizontal line, then every point is an *absolute maximum* and an *absolute minimum*.

Note: A *point of inflection* is also represented in Figure 3.3. There is an overlap with relative extrema in representation of such points through derivatives that will be addressed in the problem set of Chapter 4.

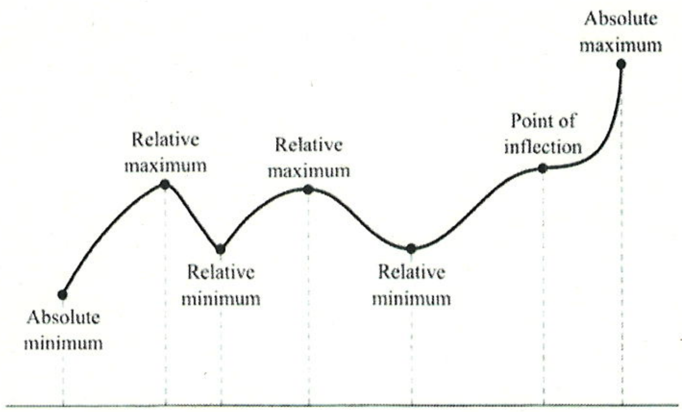

Figure 3.3

Types of Functions

It is worth realizing that there is a fundamental pool of functions at the foundation of calculus and advanced calculus. These are called *elementary functions*. Either they are generated from a real variable x by the fundamental operations of algebra, including powers and roots, or they have relatively simple geometric interpretations. As the title "elementary functions" suggests, there is a more general category of functions (which, in fact, are dependent on the elementary ones). Some of these will be explored later in this book. The *elementary functions* are described as follows.

1. **Polynomial functions** have the form

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \tag{1}$$

where $a_0, \ldots, a_n$ are constants and n is a positive integer called the *degree* of the polynomial if $a_0 \neq 0$.

The *fundamental theorem of algebra* states that in the field of complex numbers every polynomial equation has at least one root. As a consequence of this theorem, it can be proved that every nth-degree polynomial has n roots in the complex field. When complex numbers are admitted, the polynomial theoretically may be expressed as the product of n linear factors; with our restriction to real numbers, it is possible that $2k$ of the roots may be complex. In this case, the k factors generating them will be quadratic. (The corresponding roots are in complex conjugate pairs.) The polynomial $x^3 - 5x^2 + 11x - 15 = (x - 3)(x^2 - 2x + 5)$ illustrates this thought.

2. **Algebraic functions** are functions $y = f(x)$ satisfying an equation of the form

$$p_0(x)\,y^n + p_1(x)\,y^{n-1} + \cdots + p_{n-1}(x)\,y + p_n(x) = 0 \tag{2}$$

where $p_0(x), \ldots, p_n(x)$ are polynomials in x.

If the function can be expressed as the quotient of two polynomials, i.e., $P(x)/Q(x)$ where $P(x)$ and $Q(x)$ are polynomials, it is called a *rational algebraic function*; otherwise, it is an *irrational algebraic function.*

3. **Transcendental functions** are functions which are not algebraic; i.e., they do not satisfy equations of the form of Equation (2).

Note the analogy with real numbers, polynomials corresponding to integers, rational functions to rational numbers, and so on.

Transcendental Functions

The following are sometimes called *elementary transcendental functions.*

1. **Exponential function:** $f(x) = a^x$, $a \neq 0, 1$. For properties, see Page 4.
2. **Logarithmic function:** $f(x) = \log_a x$, $a \neq 0, 1$. This and the exponential function are inverse functions. If $a = e = 2.71828\ldots$, called the *natural base of logarithms*, we write $f(x) = \log_e x = \text{In } x$, called the *natural logarithm* of x. For properties, see Page 4.
3. **Trigonometric functions** (also called *circular functions* because of their geometric interpretation with respect to the unit circle):

$$\sin x,\ \cos x,\ \tan x = \frac{\sin x}{\cos x},\ \csc x = \frac{1}{\sin x}, \sec x = \frac{1}{\cos x}, \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

The variable x is generally expressed in radians (π radians = 180°). For real values of x, $\sin x$ and $\cos x$ lie between -1 and 1 inclusive.

The following are some properties of these functions:

$\sin^2 x + \cos^2 x = 1 \quad 1 + \tan^2 x = \text{sce}^2 x \qquad 1 + \cot^2 x = \csc^2 x$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \qquad \sin(-x) = -\sin x$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y \qquad \cos(-x) = \cos x$

$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y} \qquad \tan(-x) = -\tan x$

4. **Inverse trigonometric functions.** The following is a list of the inverse trigonometric functions and their principal values:

(*a*) $y = \sin^{-1} x$, $(-\pi/2 \leqq y \leqq \pi/2)$ (*d*) $y = \csc^{-1} x = \sin^{-1} 1/x$, $(-\pi/2 \leqq y \leqq \pi/2)$

(*b*) $y = \cos^{-1} x$, $(0 \leqq y \leqq \pi)$ (*e*) $y = \sec^{-1} x = \cos^{-1} 1/x$, $(0 \leqq y \leqq \pi)$

(*c*) $y = \tan^{-1} x$, $(-\pi/2 < y < \pi/2)$ (*f*) $y = \cot^{-1} x = \pi/2 - \tan^{-1} x$, $(0 < y < \pi)$

5. **Hyperbolic functions** are defined in terms of exponential functions as follows. These functions may be interpreted geometrically, much as the trigonometric functions but with respect to the unit hyperbola.

(*a*) $\sinh x = \dfrac{e^x - e^{-x}}{2}$ (*d*) $\text{csch}\ x = \dfrac{1}{\sinh x} = \dfrac{2}{e^x - e^{-x}}$

(*b*) $\cosh x = \dfrac{e^x + e^{-x}}{2}$ (*e*) $\text{sech}\ x = \dfrac{1}{\cosh x} = \dfrac{2}{e^x + e^{-x}}$

(*c*) $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ (*f*) $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

The following are some properties of these functions:

$\cosh^2 x - \sinh^2 x = 1$ $\quad 1 - \tanh^2 x = \text{sech}^2 x$ $\quad \coth^2 x - 1 = \text{csch}^2 x$

$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$ $\quad \sinh(-x) = -\sinh x$

$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$ $\quad \cosh(-x) = \cosh x$

$\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$ $\quad \tanh(-x) = -\tanh x$

6. **Inverse hyperbolic functions.** If $x = \sinh y$, then $y = \sinh^{-1} x$ is the *inverse hyperbolic sine* of x. The following list gives the principal values of the inverse hyperbolic functions in terms of natural logarithms and the domains for which they are real.

(*a*) $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$, all x $\quad$ (*d*) $\text{csch}^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{x^2+1}}{|x|}\right), x \neq 0$

(*b*) $\cosh^{-1} x = \ln(x + \sqrt{x^2-1}), x \geqq 1$ $\quad$ (*e*) $\text{sech}^{-1} x = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right), 0 < x \leqq 1$

(*c*) $\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right), |x| < 1$ $\quad$ (*f*) $\coth^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{x+1}{x-1}\right), |x| > 1$

Limits of Functions

Let $f(x)$ be defined and single-valued for all values of x near $x = x_0$ with the possible exception of $x = x_0$ itself (i.e., in a deleted δ neighborhood of x_0). We say that the number l is the *limit of $f(x)$ as x approaches x_0* and write $\lim_{x \to x_0} f(x) = l$ if for any positive number ϵ (however small) we can find some positive number δ (usually depending on ϵ) such that $|f(x) - l| < \epsilon$ whenever $0 < |x - x_0| < \delta$. In such a case we also say that $f(x)$ approaches l as x approaches x_0 and write $f(x) \to l$ as $x \to x_0$.

In words, this means that we can make $f(x)$ arbitrarily close to l by choosing x sufficiently close to x_0.

EXAMPLE. Let $f(x) = \begin{cases} x^2 & \text{if } x \neq 2 \\ 0 & \text{if } x = 2 \end{cases}$. Then as x gets closer to 2 (i.e., x approaches 2), $f(x)$ gets closer to 4. We thus *suspect* that $\lim_{x \to 2} f(x) = 4$. To *prove* this we must see whether the preceding definition of limit (with $l = 4$) is satisfied. For this proof, see Problem 3.10.

Note that $\lim_{x \to 2} f(x) = f(2)$; i.e., the limit of $f(x)$ as $x \to 2$ is not the same as the value of $f(x)$ *at* $x = 2$, since $f(2) = 0$ by definition. The limit would, in fact, be 4 even if $f(x)$ were not defined at $x = 2$.

When the limit of a function exists, it is unique; i.e., it is the only one (see Problem 3.17).

Right- and Left-Hand Limits

In the definition of limit, no restriction was made as to how x should approach x_0. It is sometimes found convenient to restrict this approach. Considering x and x_0 as points on the real axis where x_0 is fixed and x is moving, then x can approach x_0 from the right or from the left. We indicate these respective approaches by writing $x \to x_0+$ and $x \to x_0-$.

If $\lim_{x \to x_0} +f(x) = l_1$ and $\lim_{x \to x_{0^-}} f(x) = l_2$, we call l_1 and l_{l2}, respectively, the *right- and left-hand limits* of f at x_0 and denote them by $f(x_0+)$ or $f(x_0 + 0)$ and $f(x_0-)$ or $f(x_0 - 0)$. The ϵ, δ definitions of limit of $f(x)$ as