Theorems on Power Series

Theorem 9 A power series converges uniformly and absolutely in any interval which lies *entirely within* its interval of convergence.

Theorem 10 A power series can be differentiated or integrated term by term over any interval lying entirely within the interval of convergence. Also, the sum of a convergent power series is continuous in any interval lying entirely within its interval of convergence.

This follows at once from Theorem 9 and the theorem on uniformly convergent series on Pages 284 and 285. The results can be extended to include endpoints of the interval of convergence by the following theorems.

Theorem 11 *Abel's theorem.* When a power series converges up to and including an endpoint of its interval of convergence, the interval of uniform convergence also extends so far as to include this endpoint. See Problem 11.42.

Theorem 12 *Abel's limit theorem.* If $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = x_0$, which may be an interior point or an endpoint of the interval of convergence, then

$$\lim_{x \to x_0} \left\{ \sum_{n=0}^{\infty} a_n x^n \right\} = \sum_{n=0}^{\infty} \left\{ \lim_{x \to x_0} a_n x^n \right\} = \sum_{n=0}^{\infty} a_n x_0^n \tag{10}$$

If x_0 is an endpoint, we must use $x \to x_0 +$ or $x \to x_0 -$ in Equation (10) according as x_0 is a left- or a right-hand endpoint.

This follows at once from Theorem 11 and Theorem 6 on the continuity of sums of uniformly convergent series.

Operations With Power Series

In the following theorems we assume that all power series are convergent in some interval.

Theorem 13 Two power series can be added or subtracted term by term for each value of x common to their intervals of convergence.

Theorem 14 Two power series, for example, $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$, can be multiplied to obtain $\sum_{n=0}^{\infty} c_n x^n$ where

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0 \tag{11}$$

the result being valid for each x within the common interval of convergence.

Theorem 15 If the power series $\sum_{n=0}^{\infty} a_n x^n$ is divided by the power series $\Sigma b_n x^n$ where $b_0 \neq 0$, the quotient can be written as a power series which converges for sufficiently small values of x.

Theorem 16 If $y = \sum_{n=0}^{\infty} a_n x^n$, then by substituting $x = \sum_{n=0}^{\infty} b_n y^n$, we can obtain the coefficients b_n in terms of a_n. This process is often called *reversion of series.*

Expansion of Functions in Power Series

This section gets at the heart of the use of infinite series in analysis. Functions are represented through them. Certain forms bear the names of mathematicians of the eighteenth and early nineteenth centuries who did so much to develop these ideas.

A simple way (and one often used to gain information in mathematics) to explore series representation of functions is to assume such a representation exists and then discover the details. Of course, whatever is found must be confirmed in a rigorous manner. Therefore, assume

$$f(x) = A_0 + A_1(x-c) + A_2(x-c)^2 + \cdots + A_n(x-c)^n + \cdots$$

Notice that the coefficients A_n can be identified with derivatives of f. In particular,

$$A_0 = f(c), A_1 = f'(c), A_2 = \frac{1}{2!}f''(c), \cdots, A_n = \frac{1}{n!}f^{(n)}(c), \cdots$$

This suggests that a series representation of f is

$$f(x) = f(c) + f'(c)(x-c) + \frac{1}{2!}f''(x-c)^2 + \cdots + \frac{1}{n!}f^{(n)}(c)(x-c)^n \cdots$$

A first step in formalizing series representation of a function f, for which the first n derivatives exist, is accomplished by introducing *Taylor polynomials* of the function.

$$P_0(x) = f(c) \qquad P_1(x) = f(c) + f'(c)(x-c),$$

$$P_2(x) = f(c) + f'(c)(x-c) + \frac{1}{2!}f''(c)(x-c)^2,$$

$$P_n(x) = f(c) + f'(c)(x-c) + \cdots \frac{1}{n!}f^{(n)}(c)(x-c)^n \tag{12}$$

Taylor's Theorem

Let f and its derivatives $f', f'', \ldots, f^{(n)}$ exist and be continuous in a closed interval $a \le x \le b$ and suppose that $f^{(n+1)}$ exists in the open interval $a < x < b$. Then for c in $[a, b]$,

$$f(x) = P_n(x) + R_n(x)$$

where the remainder $R_n(x)$ may be represented in any of the three following ways.

For each n there exists ξ such that

$$R_n(x) = \frac{1}{(n+1)!}f^{(n+1)}(\xi)(x-c)^{n+1} \qquad \text{(Lagrange form)} \tag{13}$$

(ξ is between c and x.)

(The theorem with this remainder is a mean value theorem. Also, it is called Taylor's formula.)

For each n there exists ξ such that

$$R_n(x) = \frac{1}{n!}f^{(n+1)}(\xi)(x-\xi)^n(x-c) \qquad \text{(Cauchy form)} \tag{14}$$

$$R_n(x) = \frac{1}{n!}\int_c^x (x-t)^n f^{(n+1)}(t)\,dt \qquad \text{(Integral form)} \tag{15}$$

If all the derivatives of f exist, then the following form, without remainder, may be explored:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!}f^{(n)}(c)(x-c)^n \tag{16}$$

This infinite series is called a Taylor series, although when $c = 0$, it can also be referred to as a MacLaurin series or expansion.

We might be tempted to believe that if all derivatives of $f(x)$ exist at $x = c$, the expansion shown here would be valid. This, however, is not necessarily the case, for although one can then *formally* obtain the series on the right of the expansion, the resulting series may not converge to $f(x)$. For an example of this see Problem 11.108.

Precise conditions under which the series converges to $f(x)$ are best obtained by means of the theory of functions of a complex variable. (See Chapter 16.)

The determination of values of functions at desired arguments is conveniently approached through Taylor polynomials.

EXAMPLE. The value of $\sin x$ may be determined geometrically for $0, \frac{\pi}{6}$, and an infinite number of other arguments. To obtain values for other real number arguments, a Taylor series may be expanded about any of these points. For example, let $c = 0$ and evaluate several derivatives there; i.e., $f(0) = \sin 0 = 0, f'(0) = \cos 0 = 1$, $f''(0) = -\sin 0 = 0, f'''(0) = -\cos 0 = -1, f^{iv}(0) = \sin 0 = 0, f^{v}(0) = \cos 0 = 1$.

Thus, the MacLaurin expansion to five terms is

$$\sin x = 0 + x - 0 - \frac{1}{3!}x^3 + 0 - \frac{1}{51}x^5 + \cdots$$

Since the fourth term is 0, the Taylor polynomials P_3 and P_4 are equal, i.e.,

$$P_3(x) = p_4(x) = x - \frac{x^3}{3!}$$

and the Lagrange remainder is

$$R_4(x) = \frac{1}{5!}\cos\xi \; x^5$$

Suppose an approximation of the value of sin .3 is required. Then

$$P_4(.3) = .3 - \frac{1}{6}(.3)^3 \approx .2945.$$

The accuracy of this approximation can be determined from examination of the remainder. In particular (remember $|\cos\xi| \le 1$),

$$|R_4| = \left|\frac{1}{5!}\cos\xi \; (.3)^5\right| \le \frac{1}{120}\frac{243}{10^5} < .000021$$

Thus, the approximation $P_4(.3)$ for sin .3 is correct to four decimal places.

Additional insight into the process of approximation of functional values results by constructing a graph of $P_4(x)$ and comparing it to $y = \sin x$. (See Figure 11.2.)

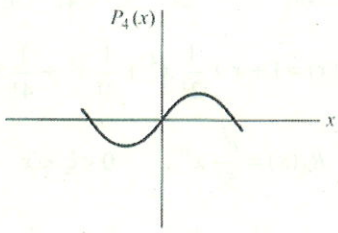

Figure 11.2

The roots of the equation are $0, \pm\sqrt{6}$. Examination of the first and second derivatives reveals a relative maximum at $x = \sqrt{2}$ and a relative minimum at $x = -\sqrt{2}$. The graph is a local approximation of the sin curve. The reader can show that $P_6(x)$ produces an even better approximation.

(For an example of series approximation of an integral, see the example that follows.)

Some Important Power Series

The following series, convergent to the given function in the indicated intervals, are frequently employed in practice:

1. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + \cdots \quad -\infty < x < \infty$

2. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots (-1)^{n-1} \frac{x^{2n-2}}{(2n-2)!} + \cdots \quad -\infty < x < \infty$

3. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \cdots \quad -\infty < x < \infty$

4. $\ln|1+x| = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots (-1)^{n-1} \frac{x^n}{n} + \cdots \quad -1 < x \leq 1$

5. $\frac{1}{2}\ln\frac{|1+x|}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots + \frac{x^{2n-1}}{2n-1} + \cdots \quad -1 < x < 1$

6. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + \cdots \quad -1 \leqq x \leqq 1$

7. $(1+x)^p = 1 + px + \frac{P(p-1)}{2!}x^2 + \cdots + \frac{P(p-1)\ldots(p-n+1)}{n!}x^n + \cdots$

This is the *binomial series.*

(a) If p is a positive integer or zero, the series terminates.

(b) If $p > 0$ but is not an integer, the series converges (absolutely) for $-1 \leqq x \leqq 1$.

(c) If $-1 < p < 0$, the series converges for $-1 < x \leqq 1$.

(d) If $p \leqq -1$, the series converges for $-1 < x < 1$.

For all p, the series certainly converges if $-1 < x < 1$.

EXAMPLE. Taylor's theorem applied to the series for e^x enables us to estimate the value of the integral $\int_0^1 e^{x^2}\,dx$. Substituting x^2 for x, we obtain

$$\int_0^1 e^{x^2}\,dx = \int_0^1 \left(1 + x + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \frac{e^{\xi}}{5!}x^{10}\right)dx$$

where

$$p_4(x) = 1 + x + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \frac{1}{4!}x^8$$

and

$$R_4(x) = \frac{e^{\xi}}{5!}x^{10}, \qquad 0 < \xi < x$$

Then

$$\int_0^1 P_4(x)\,dx = 1 + \frac{1}{3} + \frac{1}{5(2!)} + \frac{1}{7(3!)} + \frac{1}{9(4!)} \approx 1.4618$$

$$\left|\int_0^1 R_4(x)\,dx\right| \leq \int_0^1 \left|\frac{e^{\xi}}{5!}x^{10}\right| dx \leq e\int_0^1 \frac{x^{10}}{5!}\,dx = \frac{e}{11.5} < .0021$$

Thus, the maximum error is less than .0021 and the value of the integral is accurate to two decimal places.