CHAPTER 4

Derivatives

The Concept and Definition of a Derivative

Concepts that shape the course of mathematics are few and far between. The derivative, the fundamental element of the differential calculus, is such a concept. That branch of mathematics called analysis, of which advanced calculus is a part, is the end result. There were two problems that led to the discovery of the derivative. The older one of defining and representing the tangent line to a curve at one of its points had concerned early Greek philosophers. The other problem of representing the instantaneous velocity of an object whose motion was not constant was much more a problem of the seventeenth century. At the end of that century, these problems and their relationship were resolved. As is usually the case, many mathematicians contributed, but it was Isaac Newton and Gottfried Wilhelm Leibniz who independently put together organized bodies of thought upon which others could build.

The tangent problem provides a visual interpretation of the derivative and can be brought to mind no matter what the complexity of a particular application. It leads to the definition of the derivative as the limit of a difference quotient in the following way. (See Figure 4.1.)

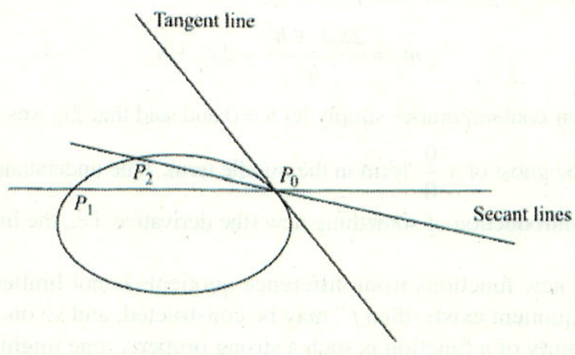

Figure 4.1

Let $P_0\,(x_0)$ be a point on the graph of $y = f(x)$. Let $P(x)$ be a nearby point on this same graph of the function f. Then the line through these two points is called a *secant line*. Its slope, m_s, is the difference quotient

$$m_s = \frac{f(x) - f(x_0)}{x - x_0} = \frac{\Delta y}{\Delta x} \tag{1}$$

where Δx and Δy are called the increments in x and y, respectively. Also this slope may be written

$$m_s = \frac{f(x_0 + h) - f(x_0)}{h} \tag{2}$$

where $h = x - x_0 = \Delta x$. See Figure 4.2.

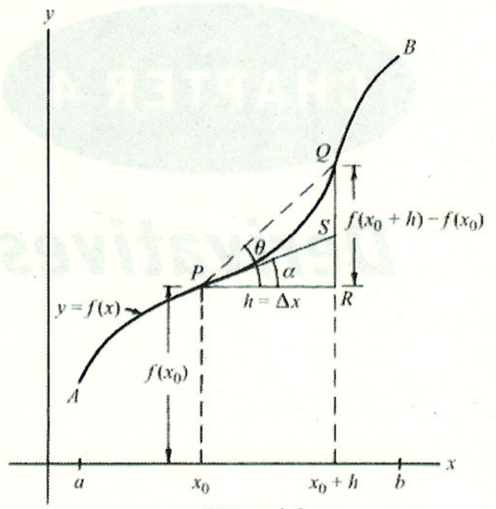

Figure 4.2

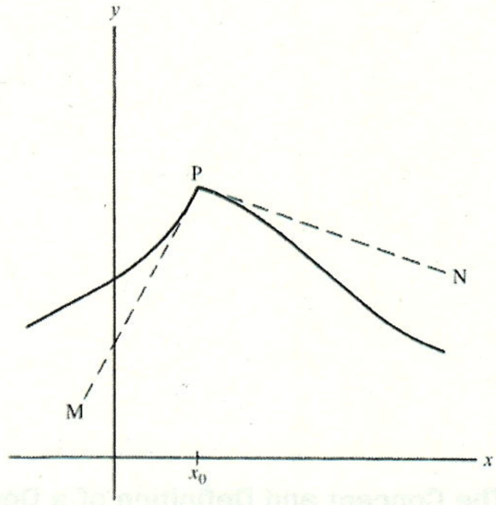

Figure 4.3

We can imagine a sequence of lines formed as $h \to 0$. It is the limiting line of this sequence that is the natural one to be the tangent line to the graph at P_0.

To make this mode of reasoning precise, the limit (when it exists), is formed as follows:

$$f'(x) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{3a}$$

As indicated, this limit is given the name $f'(x_0)$. It is called the *derivative* of the function f at its domain value x_0. If this limit can be formed at each point of a subdomain of the domain of f, then f is said to be *differentiable* on that subdomain and a new function f' has been constructed.

This limit concept was not understood until the middle of the nineteenth century. A simple example illustrates the conceptual problem that faced mathematicians from 1700 until that time. Let the graph of f be the parabola $y = x^2$; then a little algebraic manipulation yields

$$m_s = \frac{2x_0 h + h^2}{h} = 2x_0 + h \tag{3b}$$

Newton, Leibniz, and their contemporaries simply let $h = 0$ and said that $2x_0$ was the slope of the tangent line at P_0. However, this raises the ghost of a $\frac{0}{0}$ form in the middle term. True understanding of the calculus is in the comprehension of how the introduction of something new (the derivative, i.e., the limit of a difference quotient) resolves this dilemma.

Note 1: The creation of new functions from difference quotients is not limited to f'. If, starting with f', the limit of the difference quotient exists, then f'' may be constructed, and so on.

Note 2: Since the continuity of a function is such a strong property, one might think that differentiability followed. This is not necessarily true, as is illustrated in Figure 4.3.

The following theorem puts the matter in proper perspective.

Theorem: If f is differentiable at a domain value, then it is continuous at that value.

As indicated, the converse of this theorem is not true.

Right- and Left-Hand Derivatives

The status of the derivative at endpoints of the domain of f, and in other special circumstances, is clarified by the following definitions.

or

$$dy - \varepsilon\Delta x < \Delta y < dy + \varepsilon\Delta x$$

From this relation we see that dy is an approximation to Δy in small neighborhoods of x, dy is called the *principal part* of Δy.

The representation of f' by $\frac{dy}{dx}$ has an algebraic suggestiveness that is very appealing and appears in much of what follows. In fact, this notation was introduced by Leibniz (without the justification provided by knowledge of the limit idea) and was the primary reason his approach to the calculus, rather than Newton's, was followed.

The Differentiation of Composite Functions

Many functions are a composition of simpler ones. For example, if f and g have the rules of correspondence $u = x^3$ and $y = \sin u$, respectively, then $y = \sin x^3$ is the rule for a composite function $F = g(f)$. The domain of F is that subset of the domain of F whose corresponding range values are in the domain of g. The rule of composite function differentiation is called the *chain rule* and is represented by $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}[F'(x) = g'(u)f'(x)]$.

In the example,

$$\frac{dy}{dx} \equiv \frac{d(\sin x^3)}{dx} = \cos x^3(3x^2dx)$$

The importance of the chain rule cannot be too greatly stressed. Its proper application is essential in the differentiation of functions, and it plays a fundamental role in changing the variable of integration, as well as in changing variables in mathematical models involving differential equations.

Implicit Differentiation

The rule of correspondence for a function may not be explicit. For example, the rule $y = f(x)$ is *implicit* to the equation $x^2 + 4xy^5 + 7xy + 8 = 0$. Furthermore, there is no reason to believe that this equation can be solved for y in terms of x. However, assuming a common domain (described by the independent variable x), the left-hand member of the equation can be construed as a composition of functions and differentiated accordingly. (The rules for differentiation are listed here for your review.)

In this example, differentiation with respect to x yields

$$2x + 4\left(y^5 + 5xy^4\frac{dy}{dx}\right) + 7\left(y + x\frac{dy}{dx}\right) = 0$$

Observe that this equation can be solved for $\frac{dy}{dx}$ as a function of x and y (but not of x alone).

Rules for Differentiation

If f, g, and h are differentiable functions, the following differentiation rules are valid.

1. $\frac{d}{dx}\{f(x) + g(x)\} = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) = f'(x) + g'(x)$ (Addition rule)

2. $\frac{d}{dx}\{f(x) - g(x)\} = \frac{d}{dx}f(x) - \frac{d}{dx}g(x) = f'(x) - g'(x)$

3. $\frac{d}{dx}\{Cf(x)\} = C\frac{d}{dx}f(x) = Cf'(x)$ where C is any constant

4. $\frac{d}{dx}\{f(x)g(x)\} = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x) = f(x)g'(x) + g(x)f'(x)$ (Product rule)

5. $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$ if $g(x) \neq 0$ (Quotient rule)

6. If $y = f(u)$ where $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = f'(u)\frac{du}{dx} = f'\{g(x)\}g'(x) \tag{12}$$

Similarly, if $y = f(u)$ where $u = g(v)$ and $v = h(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx} \tag{13}$$

The results (*12*) and (*13*) are often called *chain rules* for differentiation of composite functions. These rules probably are the most misused (or perhaps unused) rules in the application of the calculus.

7. If $y = f(x)$ and $x = f^{-1}(y)$, then dy/dx and dx/dy are related by

$$\frac{dy}{dx} = \frac{1}{dx/dy} \tag{14}$$

8. If $x = f(t)$ and $y = g(t)$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)} \tag{15}$$

Similar rules can be formulated for differentials. For example,

$$d\{f(x) + g(x)\} = df(x) + dg(x) = f'(x)dx + g'(x)dx = \{f'(x) + g'(x)\}dx$$

$$d\{f(x)g(x)\} = f(x)dg(x) + df(x) = \{f(x)g'(x) + g(x)f'(x)\}dx$$

Derivatives of Elementary Functions

In the following we assume that u is a differentiable function of x; if $u = x$, $du/dx = 1$. The inverse functions are defined according to the principal values given in Chapter 3.

1. $\frac{d}{dx}(C) = 0$
2. $\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$
3. $\frac{d}{dx}\sin u = \cos u\frac{du}{dx}$
4. $\frac{d}{dx}\cos u = -\sin u\frac{du}{dx}$
5. $\frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx}$
6. $\frac{d}{dx}\cot u = -\csc^2 u\frac{du}{dx}$
7. $\frac{d}{dx}\sec u = \sec u\tan u\frac{du}{dx}$

16. $\frac{d}{dx}\cot^{-1} u = -\frac{1}{1+u^2}\frac{du}{dx}$
17. $\frac{d}{dx}\sec^{-1} u = \pm\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}\begin{cases} + \text{ if } u > 1 \\ - \text{ if } u < -1 \end{cases}$
18. $\frac{d}{dx}\csc^{-1} u = \mp\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}\begin{cases} - \text{ if } u > 1 \\ + \text{ if } u < -1 \end{cases}$
19. $\frac{d}{dx}\sinh u = \cosh u\frac{du}{dx}$
20. $\frac{d}{dx}\cosh u = \sinh u\frac{du}{dx}$
21. $\frac{d}{dx}\tanh u = \operatorname{sec}h^2 u\frac{du}{dx}$
22. $\frac{d}{dx}\coth u = -\operatorname{csch}^2 u\frac{du}{dx}$

8. $\frac{d}{dx}\csc u = -\csc u \cot u \frac{du}{dx}$

9. $\frac{d}{dx}\log_a u = \frac{\log_a e}{u}\frac{du}{dx} \quad a>0, a\neq 1$

10. $\frac{d}{dx}\log_e u = \frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$

11. $\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}$

12. $\frac{d}{dx}e^u = e^u \frac{du}{dx}$

13. $\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

14. $\frac{d}{dx}\cos^{-1} u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

15. $\frac{d}{dx}\tan^{-1} u = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$

23. $\frac{d}{dx}\text{sech } u = -\text{sech } u \tanh u \frac{du}{dx}$

24. $\frac{d}{dx}\text{csch } u = -\text{csch } u \coth u \frac{du}{dx}$

25. $\frac{d}{dx}\sinh^{-1} u = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$

26. $\frac{d}{dx}\cosh^{-1} u = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$

27. $\frac{d}{dx}\tanh^{-1} u = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u|<1$

28. $\frac{d}{dx}\coth^{-1} u = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u|>1$

29. $\frac{d}{dx}\text{sech}^{-1} u = \frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}$

30. $\frac{d}{dx}\text{csch}^{-1} u = -\frac{1}{u\sqrt{u^2+1}}\frac{du}{dx}$

Higher-Order Derivatives

If $f(x)$ is differentiable in an interval, its derivative is given by $f'(x)$, y' or dy/dx, where $y = f(x)$. If $f'(x)$ is also differentiable in the interval, its derivative is denoted by $f''(x)$, y'' or $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$. Similarly, the nth derivative of $f(x)$, if it exists, is denoted by $f^{(n)}(x)$, $y^{(n)}$ or $\frac{d^ny}{dx^n}$, where n is called the order of the derivative. Thus, derivatives of the first, second, third, . . . orders are given by $f'(x), f''(x), f'''(x), \ldots$.

Computation of higher-order derivatives follows by repeated application of the differentiation rules given here.

Mean Value Theorems

These theorems are fundamental to the rigorous establishment of numerous theorems and formulas. (See Figure 4.5.)

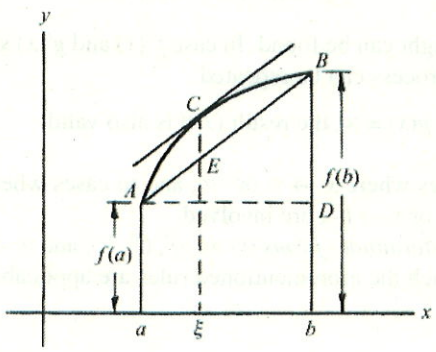

Figure 4.5

1. **Rolle's theorem.** If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b) and if $f(a) = f(b) = 0$, then there exists a point ξ in (a, b) such that $f'(\xi) = 0$.

Rolle's theorem is employed in the proof of the mean value theorem. It then becomes a special case of that theorem.

2. **The mean value theorem.** If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b), then there exists a point ξ in (a, b) such that

$$\frac{f(b)-f(a)}{b-a} = f'(\xi) \quad a < \xi < b \tag{16}$$

Rolle's theorem is the special case of this where $f(a) = f(b) = 0$.

The result (16) can be written in various alternative forms; for example, if x and x_0 are in (a, b), then

$$f(x) = f(x_0) + f'(\xi)(x - x_0) \quad \xi \text{ between } x_0 \text{ and } x \tag{17}$$

We can also write result (16) with $b = a + h$, in which case $\xi = a + \theta h$, where $0 < \theta < 1$.

The mean value theorem is also called the *law of the mean*.

3. **Cauchy's generalized mean value theorem.** If $f(x)$ and $g(x)$ are continuous in $[a, b]$ and differentiable in (a, b), then there exists a point ξ in (a, b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)} \quad a < \xi < b \tag{18}$$

where we assume $g(a) \neq g(b)$ and $f'(x)$, $g'(x)$ are not simultaneously zero. Note that the special case $g(x) = x$ yields (16).

L'Hospital's Rules

If $\lim_{x \to x_0} f(x) = A$ and $\lim_{x \to x_0} g(x) = B$, where A and B are either both zero or both infinite, $\lim_{x \to x_0} \frac{f(x)}{g(x)}$ is often called an *indeterminate* of the form 0/0 or ∞/∞, respectively, although such terminology is somewhat misleading since there is usually nothing indeterminate involved. The following theorems, called *L'Hospital's rules*, facilitate evaluation of such limits.

1. If $f(x)$ and $g(x)$ are differentiable in the interval (a, b) except possibly at a point x_0 in this interval, and if $g'(x) \neq 0$ for $x \neq x_0$, then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \tag{19}$$

whenever the limit on the right can be found. In case $f'(x)$ and $g'(x)$ satisfy the same conditions as $f(x)$ and $g(x)$ given above, the process can be repeated.

2. If $\lim_{x \to x_0} f(x) = \infty$ and $\lim_{x \to x_0} g(x) = \infty$, the result (*19*) is also valid.

These can be extended to cases where $x \to \infty$ or $-\infty$, and to cases where $x_0 = a$ or $x_0 = b$ in which only one-sided limits, such as $x \to a+$ or $x \to b-$, are involved.

Limits represented by the *indeterminate forms* $0 \cdot \infty$, ∞^0, 0^0, 1^∞, and $\infty - \infty$ can be evaluated on replacing them by equivalent limits for which the aforementioned rules are applicable (see Problem 4.29).