The following are some properties of these functions:

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x \qquad \coth^2 x - 1 = \operatorname{csch}^2 x$$

$$\sinh (x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y \qquad \sinh (-x) = -\sinh x$$

$$\cosh (x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y \qquad \cosh (-x) = \cosh x$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y} \qquad \tanh (-x) = -\tanh x$$

6. **Inverse hyperbolic functions.** If $x = \sinh y$, then $y = \sinh^{-1} x$ is the *inverse hyperbolic sine* of x. The following list gives the principal values of the inverse hyperbolic functions in terms of natural logarithms and the domains for which they are real.

(a) $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$, all x (d) $\operatorname{csch}^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{x^2 + 1}}{|x|}\right), x \neq 0$

(b) $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$, $x \geqq 1$ (e) $\operatorname{sech}^{-1} x = \ln\left(\dfrac{1 + \sqrt{1 - x^2}}{x}\right), 0 < x \leqq 1$

(c) $\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right), |x| < 1$ (f) $\coth^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{x+1}{x-1}\right), |x| > 1$

Limits of Functions

Let $f(x)$ be defined and single-valued for all values of x near $x = x_0$ with the possible exception of $x = x_0$ itself (i.e., in a deleted δ neighborhood of x_0). We say that the number l is the *limit of $f(x)$ as x approaches x_0* and write $\lim\limits_{x \to x_0} f(x) = l$ if for any positive number ϵ (however small) we can find some positive number δ (usually depending on ϵ) such that $|f(x) - l| < \epsilon$ whenever $0 < |x - x_0| < \delta$. In such a case we also say that $f(x)$ approaches l as x approaches x_0 and write $f(x) \to l$ as $x \to x_0$.

In words, this means that we can make $f(x)$ arbitrarily close to l by choosing x sufficiently close to x_0.

EXAMPLE. Let $f(x) = \begin{vmatrix} x^2 & \text{if } x \neq 2 \\ 0 & \text{if } x = 2 \end{vmatrix}$. Then as x gets closer to 2 (i.e., x approaches 2), $f(x)$ gets closer to 4. We thus *suspect* that $\lim\limits_{x \to 2} f(x) = 4$. To *prove* this we must see whether the preceding definition of limit (with $l = 4$) is satisfied. For this proof, see Problem 3.10.

Note that $\lim\limits_{x \to 2} f(x) = f(2)$; i.e., the limit of $f(x)$ as $x \to 2$ is not the same as the value of $f(x)$ *at $x = 2$*, since $f(2) = 0$ by definition. The limit would, in fact, be 4 even if $f(x)$ were not defined at $x = 2$.

When the limit of a function exists, it is unique; i.e., it is the only one (see Problem 3.17).

Right- and Left-Hand Limits

In the definition of limit, no restriction was made as to how x should approach x_0. It is sometimes found convenient to restrict this approach. Considering x and x_0 as points on the real axis where x_0 is fixed and x is moving, then x can approach x_0 from the right or from the left. We indicate these respective approaches by writing $x \to x_0 +$ and $x \to x_0 -$.

If $\lim\limits_{x \to x_0 +} f(x) = l_1$ and $\lim\limits_{x \to x_0 -} f(x) = l_2$, we call l_1 and l_{l2}, respectively, the *right- and left-hand limits* of f at x_0 and denote them by $f(x_0 +)$ or $f(x_0 + 0)$ and $f(x_0 -)$ or $f(x_0 - 0)$. The ϵ, δ definitions of limit of $f(x)$ as

$x \to x_0 +$ or $x \to x_0-$ are the same as those for $x \to x_0$ except for the fact that values of x are restricted to $x > x_0$ or $x < x_0$, respectively.

We have $\lim\limits_{x \to x_0} f(x) = l$ if and only if $\lim\limits_{x \to x_0+} f(x) = \lim\limits_{x \to x_0-} fs(x) = l$.

Theorems on Limits

If $\lim\limits_{x \to x_0} f(x) = A$ and $\lim\limits_{x \to x_0} g(x) = B$, then

1. $\lim\limits_{x \to x_0} (f(x) + g(x)) = \lim\limits_{x \to x_0} f(x) + \lim\limits_{x \to x_0} g(x) = A + B$

2. $\lim\limits_{x \to x_0} (f(x) - g(x)) = \lim\limits_{x \to x_0} f(x) - \lim\limits_{x \to x_0} g(x) = A - B$

3. $\lim\limits_{x \to x_0} (f(x)g(x)) = \left(\lim\limits_{x \to x_0} f(x) \right) \left(\lim\limits_{x \to x_0} g(x) \right) = AB$

4. $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to x_0} f(x)}{\lim\limits_{x \to x_0} g(x)} = \dfrac{A}{B}$ if $B \neq 0$

Similar results hold for right- and left-hand limits.

Infinity

It sometimes happens that as $x \to x_0, f(x)$ increases or decreases without bound. In such case it is customary to write $\lim\limits_{x \to x_0} f(x) = +\infty$ or $\lim\limits_{x \to x_0} f(x) = -\infty$, respectively. The symbols $+\infty$ (also written ∞) and $-\infty$ are read "plus infinity" (or "infinity") and "minus infinity," respectively, but it must be emphasized that they are not numbers.

In precise language, we say that $\lim\limits_{x \to x_0} f(x) = \infty$ if for each positive number M we can find a positive number δ (depending on M in general) such that $f(x) > M$ whenever $0 < |x - x_0| < \delta$. Similarly, we say that

$\lim\limits_{x \to x_0} f(x) = -\infty$ if for each positive number M we can find a positive number δ such that $f(x) < -M$ whenever $0 < |x - x_0| < \delta$. Analogous remarks apply in case $x \to x_0 +$ or $x \to x_0 -$.

Frequently we wish to examine the behavior of a function as x increases or decreases without bound. In such cases it is customary to write $x \to +\infty$ (or ∞) or $x \to -\infty$, respectively.

We say that $\lim\limits_{x \to +\infty} f(x) = l$, or $f(x) \to l$ as $x \to +\infty$, if for any positive number ϵ we can find a positive number N (depending on ϵ in general) such that $|f(x) - l| < \epsilon$ whenever $x > N$. A similar definition can be formulated for $\lim\limits_{x \to -\infty} f(x)$.

Special Limits

1. $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x} = 0$

2. $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x} \right)^x = e$ $\lim\limits_{x \to 0+} (1 + x)^{1/x} = e$

3. $\displaystyle\lim_{x\to 0}\frac{e^x-1}{x}=1$ $\displaystyle\lim_{x\to 1}\frac{x-1}{\ln x}=1$

Continuity

Let f be defined for all values of x near $x=x_0$ as well as at $x=x_0$ (i.e., in a δ neighborhood of x_0). The function f is called *continuous* at $x=x_0$ if $\displaystyle\lim_{x\to x_0} f(x)=f(x_0)$. Note that this implies three conditions which must be met in order that $f(x)$ be continuous at $x=x_0$:

1. $\displaystyle\lim_{x\to x_0} f(x)=l$ must exist.
2. $f(x_0)$ must exist; i.e., $f(x)$ is defined *at* x_0.
3. $l=f(x_0)$.

In summary, $\displaystyle\lim_{x\to x_0} f(x)$ is the value suggested for f at $x=x_0$ by the behavior of f in arbitrarily small neighborhoods of x_0. If, in fact, this limit is the actual value, $f(x_0)$, of the function at x_0, then f is continuous there.

Equivalently, if f is continuous at x_0, we can write this in the suggestive form $\displaystyle\lim_{x\to x_0} f(x)=f(\lim_{x\to x_0} x)$.

EXAMPLES. 1.   If $f(x)=\begin{cases} x^2, & x\neq 2 \\ 0, & x=2\end{cases}$ then from the example on Page 45 $\displaystyle\lim_{x\to 2} f(x)=4$. But $f(2)=0$.

Hence, $\displaystyle\lim_{x\to 2} f(x)\neq f(2)$ and the function is not continuous at $x=2$.

2. If $f(x)=x^2$ for all x, then $\displaystyle\lim_{x\to 2} f(x)=f(2)=4$ and $f(x)$ is continuous at $x=2$.

Points where f fails to be continuous are called *discontinuities* of f and f is said to be *discontinuous* at these points.

In constructing a graph of a continuous function, the pencil need never leave the paper, while for a discontinuous function this is not true, since there is generally a jump taking place. This is, of course, merely a characteristic property and not a definition of continuity or discontinuity.

Alternative to the preceding definition of continuity, we can define f as continuous at $x=x_0$ if for any $\epsilon>0$ we can find $\delta>0$ such that $\left|f(x_0)-f(x_0)\right|<\epsilon$ whenever $\left|x-x_0\right|<\delta$. Note that this is simply the definition of limit with $l=f(x_0)$ and removal of the restriction that $x\neq x_0$.

Right- and Left-Hand Continuity

If f is defined only for $x\geq x_0$, the preceding definition does not apply. In such case we call f *continuous (on the right)* at $x=x_0$ if $\displaystyle\lim_{x\to x_{0+}} f(x)=f(x_0)$, i.e., if $f(x_0+)=f(x_0)$. Similarly, f is *continuous (on the left)* at $x=x_0$ if $\displaystyle\lim_{x\to x_{0-}} f(x)=f(x)_0$, i.e., $f(x_0-)=f(x_0)$. Definitions in terms of ϵ and δ can be given.

Continuity in an Interval

A function f is said to be *continuous in an interval* if it is continuous at all points of the interval. In particular, if f is defined in the closed interval $a\leq x\leq b$ or $[a,b]$, then f is continuous in the interval if and only if $\displaystyle\lim_{x\to x_0} f(x)=f(x_0)$ for $a<x_0<b$, $\displaystyle\lim_{x\to x_{a+}} f(x)=f(a)$, and $\displaystyle\lim_{x\to b-} f(x)=f(b)$.

Theorems on Continuity

Theorem 1   If f and g are continuous at $x = x_0$, so also are the functions whose image values satisfy the relations $f(x) + g(x), f(x) - g(x), f(x)g(x)$, and $\dfrac{f(x)}{g(x)}$, the last only if $g(x_0) \neq 0$. Similar results hold for continuity in an interval.

Theorem 2   Functions described as follows are continuous in every finite interval: (*a*) all polynomials; (*b*) $\sin x$ and $\cos x$; and (*c*) a^x, $a > 0$.

Theorem 3   Let the function f be continuous at the domain value $x = x_0$. Also suppose that a function g, represented by $z = g(y)$, is continuous at y_0, where $y = f(x)$ (i.e., the range value of f corresponding to x_0 is a domain value of g). Then a new function, called a *composite function*, $f(g)$, represented by $z = g[f(x)]$, may be created which is continuous at its domain point $x = x_0$. (One says that *a continuous function of a continuous function is continuous*.)

Theorem 4   If $f(x)$ is continuous in a closed interval, it is bounded in the interval.

Theorem 5   If $f(x)$ is continuous at $x = x_0$ and $f(x_0) > 0$ [or $f(x_0) < 0$], there exists an interval about $x = x_0$ in which $f(x) > 0$ [or $f(x) < 0$].

Theorem 6   If a function $f(x)$ is continuous in an interval and either strictly increasing or strictly decreasing, the inverse function $f^{-1}(x)$ is single-valued, continuous, and either strictly increasing or strictly decreasing.

Theorem 7   If $f(x)$ is continuous in $[a, b]$ and if $f(a) = A$ and $f(b) = B$, then corresponding to any number C between A and B there exists at least one number c in $[a, b]$ such that $f(c) = C$. This is sometimes called the *intermediate value theorem*.

Theorem 8   If $f(x)$ is continuous in $[a, b]$ and if $f(a)$ and $f(b)$ have opposite signs, there is at least one number c for which $f(c) = 0$ where $a < c < b$. This is related to Theorem 7.

Theorem 9   If $f(x)$ is continuous in a closed interval, then $f(x)$ has a maximum value M for at least one value of x in the interval and a minimum value m for at least one value of x in the interval. Furthermore, $f(x)$ assumes all values between m and M for one or more values of x in the interval.

Theorem 10   If $f(x)$ is continuous in a closed interval and if M and m are, respectively, the least upper bound (l.u.b.) and greatest lower bound (g.l.b.) of $f(x)$, there exists at least one value of x in the interval for which $f(x) = M$ or $f(x) = m$. This is related to Theorem 9.

Piecewise Continuity

A function is called *piecewise continuous* in an interval $a \leq x \leq b$ if the interval can be subdivided into a finite number of intervals in each of which the function is continuous and has finite right- and left-hand limits. Such a function has only a finite number of discontinuities. An example of a function which is piecewise continuous in $a \leq x \leq b$ is shown graphically in Figure 3.4. This function has discontinuities at x_1, x_2, x_3, and x_4.
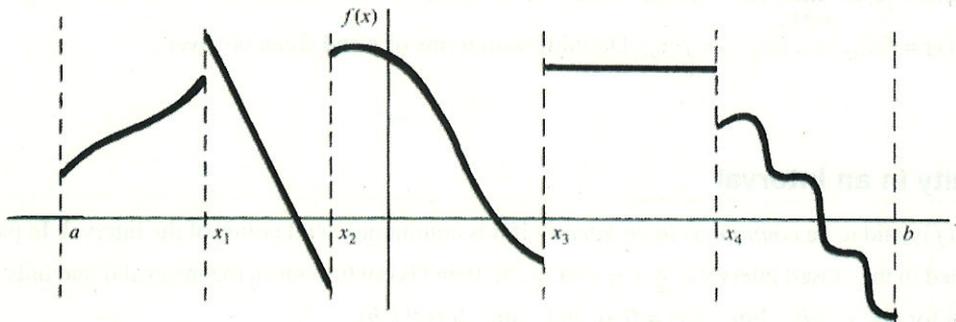

Figure 3.4