Linear Algebra Notes

Chapter 5

REFLECTION MATRICES

Let us find the matrix R_ℓ of reflection about a given line ℓ through the origin in the plane. This matrix R_ℓ should fix every vector on ℓ, and should send any vector not on ℓ to its mirror image about ℓ.

If ℓ is the x-axis, this is easy. We must have

$$R_{x-axis}\ \mathbf{e}_1 = \mathbf{e}_1, \qquad R_{x-axis}\ \mathbf{e}_2 = -\mathbf{e}_2,$$

so

$$R_{x-axis} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

It is similarly easy to find R_{y-axis}. If ℓ is not the x- or y-axis, it less straightforward to find R_ℓ.

Let θ be the counterclockwise angle between ℓ and the x-axis and consider the rotation matrix

$$B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

The matrix B rotates the x-axis onto ℓ. Let

$$\mathbf{u} = B\mathbf{e}_1, \qquad \mathbf{v} = B\mathbf{e}_2.$$

Thus, $\mathbf{u}$ is the first column of B; it lies on ℓ. And $\mathbf{v}$ is the second column of B; it is perpendicular to ℓ, and points 90 degrees counterclockwise from $\mathbf{u}$.

Since $\mathbf{u}$ lies on ℓ, we have $R_\ell\mathbf{u} = \mathbf{u}$. Since $\mathbf{v}$ is perpendicular to ℓ, we have $R_\ell\mathbf{v} = -\mathbf{v}$. Since $B\mathbf{e}_1 = \mathbf{u}$ and $B\mathbf{e}_2 = \mathbf{v}$, we get

$$R_\ell B\mathbf{e}_1 = B\mathbf{e}_1 \quad \text{and} \quad R_\ell B\mathbf{e}_2 = -B\mathbf{e}_2.$$

Applying B^{-1}, we have

$$B^{-1}R_\ell B\mathbf{e}_1 = \mathbf{e}_1 \quad \text{and} \quad B^{-1}R_\ell B\mathbf{e}_2 = -\mathbf{e}_2.$$

This means that

$$B^{-1}R_\ell B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{4b}$$

Now multiply on the left by B, and on the right by B^{-1}, and get

$$\begin{aligned} R_\ell &= B\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}B^{-1} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}. \end{aligned}$$

In summary, we have shown that the matrix R_ℓ reflecting about the line ℓ having counterclockwise angle θ with respect to the x-axis is given by

$$R_\ell = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

Note that you could write R_ℓ as a product of a rotation times R_{x-axis}:

$$R_\ell = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

This says that reflection about the line with angle θ is the same as reflection about the x-axis followed by rotation by 2θ.

The key step in the computation of R_ℓ was equation (4b). The idea in this computation is that R_ℓ behaves simply with respect to $\mathbf{u}, \mathbf{v}$, but not simply with respect to $\mathbf{e}_1, \mathbf{e}_2$. So we take the matrix B sending $\mathbf{e}_1, \mathbf{e}_2$ to $\mathbf{u}, \mathbf{v}$, then R_ℓ acts simply, and then we go back to $\mathbf{e}_1, \mathbf{e}_2$ by means of B^{-1}. This means the matrix $B^{-1}AB$ will be simple (that is equation (4b)), and then we multiply by B on the left, B^{-1} on the right, to extract R_ℓ.

The remarkable thing is that the same idea works for almost any matrix, even if it is not a reflection. That is, almost any matrix has a favorite pair of vectors $\mathbf{u}, \mathbf{v}$, called **eigenvectors**, on which the matrix behaves simply. The eigenvectors are usually hidden from view, but there is a way to find them, and then use them to compute and analyze the matrix. This is an important theme in the course and is the topic of the next few chapters.

Exercise5.1. *A hexagon has six vertices, starting at* $(1, 0)$ *and rotating by multiples of* $\pi/3$.

(a) *Find the coordinates of the remaining five vertices.*
(b) *There are six reflections that map the hexagon to itself. Draw the reflecting lines of these reflections and find their matrices.*

Exercise 5.2. *Suppose* A *is reflection matrix about a line with angle* θ*, as above, and* A' *is a reflection about a line with angle* ϕ*. Then* $A'A$ *is a rotation matrix. What is the angle of rotation of* $A'A$*? Check your answer by taking* A, A' *to be two reflections of the hexagon, as in exercise 5.1b.*

Exercise 5.3. *Find two reflection matrices that do not commute with each other.*

Exercise 5.4. *Suppose* A *and* A' *are two distinct reflections that commute with each other. What is the relation between their reflecting lines? (Hint: Compare* AA' *and* $A'A$*, which were computed in exercise 5.2.)*