Local properties of analytic sets.

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Introduction.

The aim of these lectures is to describe the local structure of an analytic set.

We suppose everybody familiar with the first notions of holomorphic functions in one variable and also with the notion of analytic manifold. In any case let us recall some definitions: an analytic manifold X is a topological space with a covering of open sets each one homeomorphic to an open set in some $\mathbb{C}^n$ and providing one or more systems of n complex coordinates around a point whose change of coordinates is a holomorphic invertible map between open sets of $\mathbb{C}^n$. In this way we can say whether a function f is holomorphic on X: since the notion has a local nature it will be enough to see that f is holomorphic in a neighboorhood of the point x in an open set giving local coordinates. This definition is well posed and holomorphic functions on an open set U form, in a natural way, a ring $\mathcal{O}(U)$. The local nature of definition leads to consider the ring $\mathcal{O}_x$ of germs of functions, i.e. the $\varinjlim_{x} \mathcal{O}(U) := \mathcal{O}_x$ where the direct limit is on the system of neighbourhoods of a point x. This is the quotient of the direct sum $\coprod \mathcal{O}(U)$ of disjoint rings $\mathcal{O}(U)$ modulo the equivalence relation: (U, f) is equivalent to (V, g) if there exists a neighboorhoud of x $W \subset U \cap V$ such that $f_{|W} = g_{|W}$. In this way the manifold can be viewed also as a ringed space and f is holomorphic if its germ at x belongs to the subring $\mathcal{O}_x$ of the ring of germs of continuous functions.

Nevertheless we need to enlarge the category including some type of singularity. To do this one can proceed changing the local models considered and replacing them by the so called *analytic sets* . An analytic set in an open set $\Omega \subset \mathbb{C}^n$ is a closed set $X \subset \Omega$ described as zero set of finitely many holomorphic functions, i.e. $X = \{x \in \Omega | f_1(x) = \ldots, f_k(x) = 0 \; f_i \in \mathcal{O}(\Omega)\}$; this set can be viewed as ringed space considering at each $x \in X$ the quotient of $\mathcal{O}_x$ by the ideal of germs of holomorphic function vanishing at X. This class of models is the class used by Cartan-Serre; another class used by Behnke-Stein is the class of branching (ramified) covers. The definitions arising from the two classes are equivalent: this was proved by Grauert and Remmert and will be clear during the lectures.

Our main goal will be the proof of the Nullstellensatz for the ring of analytic functions germs. Even if the tools will be some what different, the proof will be very similar to the algebraic one: in this setting the main tool are Weierstrass division and preparation theorems.

We want to add a final remark about the use of the two words "local" and "global": a local property is a property where you are allowed to stretch the open set you are considering: think to the implicit function theorem. Conversely a global property is a property holding in a given open set that you are not allowed to change during the proof.

References

These notes, except for the last section, are basically taken with some details added, from the classic book by

Robert C. Gunning, Hugo Rossi *Analytic Functions of Several Complex Variables* AMS Chelsea Edition

Basic notions of one complex variable can be found in

Henri Cartan *Elementary Theory of Analytic Functions of One or Several Complex Variables* Dover Books on Mathematics

1 The ring of holomorphic function germs at 0.

Let $\mathcal{O}_n$ be the ring of germs of holomorphic functions at the origin of $\mathbb{C}^n$. It is the ring obtained as quotient of the set of couples (U, f), where $0 \in U \subset \mathbb{C}^n$ and f is a holomorphic function defined on U, by the relation (U, f) is equivalent to (V, g) if and only if $f = g$ on an open neighbourhood W of 0, $W \subset U \cap V$.

If $\mathbb{C}\{z_1, \dots, z_n\}$ denotes the ring of convergent power series in n variables, then one has $\mathcal{O}_n = \mathbb{C}\{z_1, \dots, z_n\}$. Indeed if $[(U, f)] = [(V, g)]$, i. e. f and g get the same germ at 0, then the Taylor series $T_0 f$ is the same as $T_0 g$. Conversely a convergent series in $\mathbb{C}\{z_1, \dots, z_n\}$ defines a holomorphic function in a suitable polydisc centered at 0.

As a consequence of Weierstrass Preparation and Division Theorems we will see that $\mathcal{O}_n$ is a factorial noetherian ring. We begin by some definitions.

Definition 1.1. *1. The* total order *of a holomorphic function f at a point z^0 is as follows.*

- *Write $T_{z^0} f$ as a sum of homogeneous polynomials: $T_{z^0} f = P_k(z_1 - z_1^0, \dots, z_n - z_n^0) + P_{k+1}(z_1 - z_1^0, \dots, z_n - z_n^0) + \cdots$.*
- *k is* the total order *of f if $P_k(z_1 - z_1^0, \dots, z_n - z_n^0)$ is not the zero polynomial, while $P_0, \dots, P_{k-1}$ are identically* 0.

2. f is regular *of order k in z_n at the point z^0 if $f(0, \dots, 0, z_n - z_n^0)$ has an (isolated) zero of order k at z_n^0.*

This means that $P_k(z_1 - z_1^0, \dots, z_n - z_n^0)$ is a monic polynomial of degree k in $z_n - z_n^0$, with coefficients in $\mathbb{C}\{z_1 - z_1^0, \dots, z_{n-1} - z_{n-1}^0\}$, up to a constant.

Remark 1.2. *If f has total order k at z^0 there is a linear coordinate change such that in the new coordinates f is regular in z_n of order k.*

Proof: Assume $z^0 = 0$. There exists $a = (a_1, \dots, a_n)$ such that $p_k(a) \neq 0$.

Take the matrix $B = \begin{pmatrix} & a_1 \\ B' & \vdots \\ & a_n \end{pmatrix}$ with $\det B \neq 0$. This is the required change.

□

Regular functions at 0 satisfy the following Proposition which is the base for Preparation Theorem.

Proposition 1.3. *Let f be holomorphic in an open polydisc $\Delta(0, r)$, regular of order k in z_n. Then in a suitable smaller polydisc $\Delta(0, \delta)$, for any $a \in \Delta(0, r) \cap \mathbb{C}^{n-1}$ the function $f(a_1, \dots, a_{n-1}, z_n)$ has exactly k zeros (with multiplicities) in the disc $|z_n| < \delta_n$.*

Proof: For the function $f(0,\dots,0,z_n)$ 0 is a zero of multiplicity k. Then there is a $\delta_n < r_n$ such that $f(0,\dots,0,z_n) \neq 0$ for $0 < z_n \leq \delta_n$. Put $\varepsilon = \inf_{|z_n|=\delta_n} |f(0,\dots,0,z_n)| > 0$.

$f(z)$ is a continous function on the polydisc $|z_1| < r_1, \dots, |z_{n-1}| < r_{n-1}, |z_n| = \delta_n$. Hence there are $\delta_1, \dots, \delta_{n-1}, \delta_i < r_i$ such that

$$|f(z_1,\dots,z_{n-1},z_n) - f(0,\dots,0,z_n)| < \varepsilon \text{ for } |z_j| < \delta_j, |z_n| = \delta_n.$$

Take now $a = (a_1,\dots,a_{n-1})$ with $|a_i| < \delta_i$. Then $f(a_1,\dots,a_{n-1},z_n)$, as a function of z_n, by Rouché Theorem, on the disc $z_1 = a_1, \dots z_{n-1} = a_{n-1}, |z_n| \leq \delta_n$ has the same number k of zeroes as $f(0,\dots,0,z_n)$. Indeed

$$|f(a,z_n) - f(0,z_n)| < \varepsilon \leq |f(0,z_n)|$$

So $f(a,z_n) - f(0,z_n) + f(0,z_n)$ has k zeroes in the open disc as $f(0,z_n)$. □

Last definition

Definition 1.4. *A* Weierstrass polynomial *(or a* distinguished polynomial*) of degree $k > 0$ in z_n is a monic polynomial $h \in \mathcal{O}_{n-1}[z_n]$*

$$h = z_n^k + a_1 z_n^{k-1} + \ldots + a_k$$

where $a_i(z_1,\dots,z_{n-1})$ have total order > 0, that is $a_i(0) = 0$

Theorem 1.5 (Weierstrass Preparation Theorem). *Let $f \in \mathcal{O}_n$ be regular in z_n of order k. Then there is a unique Weierstrass polynomial h of order k such that $f = uh$ where $u \in \mathcal{O}_n$ is a unit .*

Proof: $f \in \mathcal{O}_n$ regular $\Rightarrow$ f is holomorphic on $\Delta(0,r)$ and we can find $\Delta(0,\delta) \subset \Delta(0,r)$ such that the previous proposition applies.

Call $\varphi_1(z_1,\dots,z_{n-1}) \dots, \varphi_k(z_1,\dots,z_{n-1})$ the k roots of $f(z_1,\dots,z_{n-1},z_n)$ for $|z_1| < \delta_1, \dots |z_{n-1}| < \delta_{n-1}$ (with suitable repetitions if needed)

The functions φ_i are not continous, only we know that $\varphi_i(0,\dots,0) = 0$ and that $|\varphi_1(z_1,\dots,z_{n-1})| < \delta_n$.

Put

$$h(z_1,\dots,z_n) = \prod_{i=1}^{k}(z_n - \varphi_i(z_1,\dots,z_{n-1})) = z_n^k + a_1(z_1,\dots,z_{n-1})z_n^{k-1} + \ldots + a_k(z_1,\dots,z_{n-1})$$

$a_1, \dots a_k$ are the simmetric elementary functions of $\varphi_1, \dots, \varphi_k$ and are well defined. Those functions are of course holomorphic in $z_1, \dots, z_{n-1}$. To see that they are holomorphic in the last variable z_n, consider the Euler sums $\sum_{i=1}^{k} \varphi_i(z_1,\dots,z_{n-1})^r$

which generate the ring of symmetric polynomials as elementary simmetrc functions do. By Cauchy formula we get.

$$\sum_{i=1}^{k} \varphi_i(z_1, \dots, z_{n-1})^r = \frac{1}{2\pi i} \int_{|\xi|=\delta_n} \frac{\partial f(z_1, \dots, z_{n-1}, \xi)}{\partial \xi} \cdot \frac{\xi^r}{f(z_1, \dots, z_{n-1}, \xi)} d\xi$$

We are computing the residue of the logarithmic derivative of f times a holomorphic function in ξ. At each 0 of f, that is in $\varphi_i(z_1, \dots, z_n)$ we get a simple (or multiple) pole with residue φ^r times the multiplicity.

Since f is not 0 for $|z_j| < \delta_j, |z_n| = \delta_n$, the result is holomorphic in $z_1, \dots, z_{n-1}$. Hence the simmetric elementary functions are also holomorphic. They vanish at 0 so h is Weierstrass.

We are left with the proof that $u = \frac{f}{h}$ is a holomorphic unit. Now u is meromorphic and holomorphic outside the zeroset of h. Also it does not vanish there.

Let

$$M = \max_{z \in \overline{\Delta}(0,\delta)} |f| \quad m = \min_{|z_j|<\delta_j, |z_n|=\delta_n} |h(z)| > 0$$

The Maximum principle in 1 variable implies $|u(z_n)| \leq \frac{M}{m}$ in $\overline{\Delta}(0, \delta)$. Since u is holomorphic outside $h = 0$ and it is bounded, it is holomorphic on $\Delta(0, \delta)$. Finally it is a unit, since for any fixed $a = (a_1, \dots, a_{n-1})$ the holomorphic functions f and h have the same zeroes with the same multiplicity.

Uniqueness: $h(z)$ has the same zeroes as f and there is only one monic polynomial with this property. □

Theorem 1.6 (Division theorem.). *Let h be a Weierstrass polynomial of order k in z_n. Then $\forall f \in \mathcal{O}_n$ there are $g \in \mathcal{O}_n$ and $r \in \mathcal{O}_{n-1}[z_n]$ of degree $\leq k-1$ such that $f = gh + r$. If $f \in \mathcal{O}_{n-1}[z_n]$, also g does.*

Proof: (Sketch.) Choose a polydisc $\Delta(0, \delta)$ such that $h(z) \neq 0$ for $|z_j| < \delta_j, |z_n| = \delta_n$. Put $g(z_1, \dots, z_n) = \frac{1}{2\pi i} \int_{|\xi|=\delta_n} \frac{f(z_1, \dots, z_{n-1}, \xi) d\xi}{h(z_1, \dots, z_{n-1}, \xi)(\xi - z_n)}$ and $r(z) = f(z) - g(z)h(z)$.

g is holomorphic (Cauchy integral) and one can see, by direct calculation, that r is a polynomial of degree $\leq k-1$. If $f \in \mathcal{O}_{n-1}[z_n]$ from the formula one gets that g is also a polynomial.

The uniqueness is an exercise. □

Weierstrass Theorems have several important consequences.

Theorem 1.7. *$\mathcal{O}_n$ is an integral domain, a UFD and a noetherian ring.*

The strategy is induction on n and the inclusions $\mathcal{O}_{n-1} \hookrightarrow \mathcal{O}_{n-1}[z_n] \hookrightarrow \mathcal{O}_{n-1}$

Before proving the theorem let us compare irreducibility in $\mathcal{O}_{n-1}[z_n]$ and in $\mathcal{O}_n$.

Definition 1.8. *$f \in \mathcal{O}_n$ is* reducible *if $f = g_1 \cdot g_2$, with g_i not unity.*

$f \in \mathcal{O}_{n-1}[z_n]$ is reducible *if $f = g_1 \cdot g_2$, with $g_i \in \mathcal{O}_{n-1}[z_n]$ not unity.*

Lemma 1.9. *Let $h \in \mathcal{O}_{n-1}[z_n]$ be a Weierstrass polynomial. Then*

$$h \text{ is reducible in } \mathcal{O}_n \iff \text{it is reducible in } \mathcal{O}_{n-1}[z_n]$$

Proof: Let $h = g_1 \cdot g_2$, $g_i \in \mathcal{O}_n$ not units. h is regular, then each g_i is regular so $g_i = u_i \cdot h_i$ with h_i Weierstrass and u_i units in $\mathcal{O}_n$. Then $h = (u_1 \cdot u_2) \cdot h_1 \cdot h_2$.

Also $h_1 \cdot h_2$ is a Weierstrass polynomial and the uniqueness of Preparation Theorem implies $u_1 \cdot u_2 = 1$. Then $h = h_1 \cdot h_2$ is reducible in $\mathcal{O}_{n-1}[z_n]$.

Conversely assume $h = h_1 \cdot h_2$ in $\mathcal{O}_{n-1}[z_n]$: we have to prove that h_i is not a unit in $\mathcal{O}_n$.

If h_1 were a unit, then $h_2 = h_1^{-1} \cdot h$. Weierstrass Preparation Theorem implies that h is the Weierstrass polynomial with the same zeroes as h_2. But h_2 has smaller degree. Contradiction. □

Proof of the theorem.

- $\mathcal{O}_n$ *is an integral domain.* $fg = 0, f \neq 0 \Rightarrow g \equiv 0$ because $f = 0$ is a thin set $\Rightarrow g$ is zero in a connected open set $\Rightarrow g$ is the zero function.
- $\mathcal{O}_n$ *is a UFD.* By induction.
  - $\mathcal{O}_0 = \mathbb{C}$ is a field.
  - $\mathcal{O}_n$ UFD $\Rightarrow \mathcal{O}_{n-1}[z_n]$ UFD.
  - Now take $f \in \mathcal{O}_n$. By a linear change of cordinates it becomes regular $\Rightarrow f = uh$, $h \in \mathcal{O}_{n-1}[z_n]$ with h Weierstrass. Up to units, h is the product of irreducible polynomials and this implies the unique factorization of f.
- $\mathcal{O}_n$ *is noetherian.*By induction.
  - $\mathcal{O}_0 = \mathbb{C}$ is a field.
  - $\mathcal{O}_{n-1}$ noetherian $\Rightarrow \mathcal{O}_{n-1}[z_n]$ noetherian.
  - Let $\mathfrak{a} \subset \mathcal{O}_n$ be an ideal and pick $g \in \mathfrak{a}$. We can assume g to be regular. $\Rightarrow$ up to a unity g is Weierstrass. Hence $g \in \mathfrak{a} \cap \mathcal{O}_{n-1}[z_n]$ which is finitely generated by induction say, $\mathfrak{a} \cap \mathcal{O}_{n-1}[z_n] = (g_1, \ldots, g_k)$.

CLAIM: $\mathfrak{a} = (g, g_1, \dots, g_k)$.

In fact

$$f \in \mathfrak{a} \Rightarrow f = g'g + r,\ g' \in \mathcal{O}_n \Rightarrow r \in \mathfrak{a} \cap \mathcal{O}_{n-1}[z_n] \Rightarrow$$

$$\Rightarrow r = \sum_{i=1}^{k} a_i g_i \Rightarrow f = g'g + \sum_{i=1}^{k} a_i g_i$$

□

Note that in the literature Division Theorem is a bit stronger. One can prove (cfr Ruiz) that for any f of order k and any $g \in \mathcal{O}_n$ one has $g = \alpha f + r$ where $\alpha \in \mathcal{O}_n$ and $r \in \mathcal{O}_{n-1}[z_n]$ of degree $< k$. (Elegant use of fixed point theorem).

If f is regular, divide by z_n. Then $z_n^k = \beta f + r, r \in \mathcal{O}_{n-1}[z_n]$.

Since f is regular of the same degree you prove easily that $z_n^k - r$ is Weierstrass and β is a unit. Hence

$$\beta f = z_n^k - r \text{ and } f = \beta^{-1}(z_n^k - r)$$

that is Preparation Theorem.

2 Germs of analytic sets and ideals in $\mathcal{O}_n$

Definition 2.1. *Let Ω be an open subset of $\mathbb{C}^n$ and X be a closed set in Ω. X is an* analytic set *if for any $z \in \Omega$ there is an open set U_z containing z and finitely many functions $f_1, \dots, f_k$ holomorphic in U_z such that $X \cap U_z = \{x \in U_z | f_1(z) = \dots = f_k(z) = 0\}$.*

As for functions, we can speak about *analytic set germs* . Let X_1, X_2 be analytic sets resp. in Ω_1, Ω_2 and assume $0 \in \Omega_1 \cap \Omega_2$. Then we say that have the same germ at 0 if they are the same set in an open neighbourhood $U \ni 0$, that is there exists an open neighbourhood U of 0 such that $X_1 \cap U = X_2 \cap U$.

If X is an analytic set germ at 0 and $f \in \mathcal{O}_n$ it makes sense to say that f vanishes on X. In fact f is the sum of a convergent power series in a polydisc $\Delta(0, r)$ and if r is sufficiently small X is the germ of an analytic set in Δ. Hence we can say $X \subset \mathcal{V}(f)$ where $\mathcal{V}(f)$ denote the zero set of f.

Remark 2.2. *Finite intersections and unions of analytic set germs are still well defined germs of analytic sets.*

Let X be an analytic set germ. The set $\{f \in \mathcal{O}_n | X \subset \mathcal{V}(f)\}$ is an ideal in $\mathcal{O}_n$ and since $\mathcal{O}_n$ is noetherian $\Rightarrow X = \mathcal{V}(f_1, \dots, f_r)$.

Now if $Y = \mathcal{V}(g_1, \dots, g_k)$ we have

$$X \cap Y = \mathcal{V}(f_1, \dots, f_r, g_1, \dots, g_k)$$

and

$$X \cup Y = \mathcal{V}(f_i \cdot g_j) i = 1, \dots, r, j = 1, \dots, k$$

The ideal vanishing on a set X will be indicated by $\mathcal{I}(X)$ and the zero set of an ideal $\mathfrak{a}$ by $\mathcal{V}(\mathfrak{a})$. It is well defined.

An ideal $\mathfrak{a}$ is *radical* if $\mathfrak{a} = \sqrt{\mathfrak{a}}$ where

$$\sqrt{\mathfrak{a}} = \{f \in \mathcal{O}_n | f^k \in \mathfrak{a} \text{ for some } k\}$$

Note that $\mathcal{I}(\mathcal{V}(\mathfrak{a}))$ is a radical ideal.

Next proposition summarizes the relations between germs of analytic sets and ideals in $\mathcal{O}_n$. The proof is an easy exercise.

Proposition 2.3.

1. $V_1 \supset V_2 \Rightarrow \mathcal{I}(V_1) \subset \mathcal{I}(V_2)$
2. $V_1 \neq V_2 \Rightarrow \mathcal{I}(V_1) \neq \mathcal{I}(V_2)$
3. $\mathcal{I}(V) = \sqrt{\mathcal{I}(V)}$
4. $\mathfrak{a}_1 \supset \mathfrak{a}_2 \Rightarrow \mathcal{V}(\mathfrak{a}_1) \subset \mathcal{V}(\mathfrak{a}_2)$
5. $\mathcal{V}(\mathcal{I}(X)) = X$
6. $\mathcal{V}(\mathfrak{a}) = \mathcal{V}(\sqrt{\mathfrak{a}})$
7. $\mathcal{I}(\mathcal{V}(\mathfrak{a})) \supset \sqrt{\mathfrak{a}}$

Definition 2.4. *A germ of analytic set is* reducible *if* $X = X_1 \cup X_2$ *with* X_i *analytic set germs and* $X_i \neq X$ *for* $i = 1, 2$.

X *is* irreducible *if* $X = X_1 \cup X_2$ *implies either* $X_1 = X$ *or* $X_2 = X$.

Theorem 2.5. *X is irreducible $\iff$ $\mathcal{I}(X)$ is a prime ideal.*

Proof: Assume X reducible, $X = X_1 \cup X_2$, $X \neq X_i \Rightarrow \mathcal{I}(X) \neq \mathcal{I}(X_i)$ $i = 1, 2$. But $\mathcal{I}(X) \subset \mathcal{I}(X_i)$ $i = 1, 2$. Take $f_1 \in \mathcal{I}(X_1) \setminus \mathcal{I}(X)$, $f_2 \in \mathcal{I}(X_2) \setminus \mathcal{I}(X)$. Then $f_1 f_2 \in \mathcal{I}(X)$ but neither f_1 nor f_2 belong to $\mathcal{I}(X)$. $\Rightarrow \mathcal{I}(X)$ is not a prime ideal.

Conversely assume $\mathcal{I}(X)$ not prime $\Rightarrow$ there are $f_1, f_2 \notin \mathcal{I}(X)$ such that $f_1 f_2 \in \mathcal{I}(X)$ $\Rightarrow$ each f_i is not identically 0 on $X \Rightarrow X \cap \mathcal{V}(f_i) \neq X \Rightarrow X = (X \cap \mathcal{V}(f_1)) \cup (X \cap \mathcal{V}(f_1))$. Then X is reducible. □

Corollary 2.6. *$\mathcal{V}(f)$ is irreducible $\iff$ $f = p^n$ with p irreducible.*

Theorem 2.7. *Let X be an analytic set germ. Then there are finitely many irreducible analytic set germs X_i $i = 1, \dots, k$ such that*

$$X = \bigcup_i X_i$$

and $X_i \not\subseteq X_j$ for $i \neq j$.

Those germs are uniquely determined up to order.

Proof: Note that any descending chain of analytic set germs $X_1 \supset X_2 \supset \dots \supset X_n \supset \dots$ must stabilize since their ideals $\mathcal{I}(X_i)$ form an ascending chain in a noetherian ring. Next assume than the theorem is not true.

Hence there is a minimal analytic set germ X which cannot be written as a union of finitely many irreducible analytic set germs. Hence X is reducible $X = X_1 \cup X_2$. Since X is minimal both X_1 and X_2 have a decomposition into irreducible germs. Hence X itself is is a finite union of irreducible germs.

This proves the existence. For the uniqueness assume $X = X_1 \cup \dots \cup X_k = Y_1 \cup \dots \cup Y_s$ with X_i and Y_j irreducible, $X_i \not\subset X_j$ $\;Y_l \not\subset Y_t$.

$Y_i \subset X_1 \cup \dots \cup X_k$. Being irreducible we have $Y_i = X_j \cap Y_i$ for some $j = j(i)$ i.e. $Y_i \subseteq X_{j(i)}$. Reversing the roles $X_{j(i)} \subseteq Y_{k(j(i))} \Rightarrow Y_i \subseteq Y_{k(j(i))} \Rightarrow Y_i = Y_{k(j(i))} \Rightarrow k(j(i)) = i$.

In the same way $i(k(j)) = j$ which means there is a bijection between the set $\{1, \dots, k\}$ and the set $\{1, \dots, s\} \Rightarrow s = k$ and

$$Y_i \subset X_{j(i)} \subset Y_i \Rightarrow X_{j(i)} = Y_i$$

and the two decompositions are the same. □

Corollary 2.8. *$X = X_1 \cup \dots \cup X_k$, X_i irreducible components. Then for all i $X_i \not\subset \bigcup_{j \neq i} X_j$.*

3 Nullstellensatz for ideals in $\mathcal{O}_n$

We have seen for an ideal $\mathfrak{a} \subset \mathcal{O}_n$ that $\mathcal{IV}(\mathfrak{a}) \supset \sqrt{\mathfrak{a}}$. We would like to prove they are equal. Indeed this is true and is a result by Rückert (Math.Annalen 1932).

The proof is simpler when is a principal ideal, generated by g. Let us see this case. We need a Bézout like lemma.

Lemma 3.1. *Assume $f, g \in \mathcal{O}_n$ are relatively prime, i.e. they have no common irreducible factor. Then there are $a, b \in \mathcal{O}_n$ and $p \in \mathcal{O}_{n-1}$, $p \neq 0$ such that $af + bg = p$.*

Proof: By Weierstrass preparation, after a linear change of coordinates, we can assume f, g regular, hence

$$f = uP \qquad\qquad g = vQ$$

with P, Q Weierstrass polynomial of appropriate degree.

Let F be the quotient field of $\mathcal{O}_{n-1}$: Bézout applied to $F[z_n]$ gives us $h, k \in F[z_n]$ such that

$$hP + kQ = 1$$

h, k are polynomials with fractional coefficients. Reduce to a common denominator and get $h = \dfrac{c}{d}, k = \dfrac{c'}{d'}$ with $c, c' \in \mathcal{O}_{n-1}[z_n]$, $c, d \in \mathcal{O}_{n-1}$ and $c, d \neq 0$.

Hence $\tilde{a}P + \tilde{b}Q = cd$. Put $a = \tilde{a}u^{-1}, b = \tilde{b}v^{-1}$. □

Theorem 3.2. *Let g be an irreducible function germ in $\mathcal{O}_n$. Then $\mathcal{I}(\mathcal{V}(g)) = (g)$.*

Proof: We can assume $g \neq 0$ and regular. Hence $g = uP$ with P Weierstrass polynomial and $(g) = (P)$.

Take $f \in \mathcal{I}(\mathcal{V}(P))$. Since P is irreducible or P is a factor of f or f and P are relatively prime.

Hence there are a, b such that $af + bP = c$ with $c \in \mathcal{O}_{n-1}$ not 0.

All this is true in a suitable polydisc $\Delta(0, r) = \Delta' \times \{|z_n| < r_n\}$ and we know that for any $z^0 \in \Delta'$ there is at least one z_n^0 such that $P(z^0, z_n^0) = 0$, that is $(z^0, z_n^0) \in \mathcal{V}(P)$ $\Rightarrow f(z^0, z_n^0) = 0 \Rightarrow c(z^0) = 0 \Rightarrow c$ vanishes on Δ'. Contradiction. Hence P is a factor of f and we are done. □

Corollary 3.3. *Let $f \in \mathcal{O}_n$ and let $f = \prod p_i^{n_i}$ be its decomposition into irreducible germs. Then $\mathcal{V}(f) = \cup \mathcal{V}(p_i)$ and the irreducible components of the germ $\mathcal{V}(f)$ are the germs $\mathcal{V}(p_i)$.*

Proof: $\mathcal{V}(p_i)$ is irreducible since (p_i) is prime. If $\mathcal{V}(p_i) \subset \mathcal{V}(p_j)$ then $(p_i) \supset (p_j)$, hence $p_i | p_j$: impossible.

So they are precisely the irreducible components □

Before proving Rückert Nullstellensatz (NSS for short) we make some considerations.

First of all it is enough to prove the NSS for prime ideals.

Indeed $\mathcal{V}(\mathfrak{a}) = \mathcal{V}(\sqrt{\mathfrak{a}})$ and in the noetherian ring $\mathcal{O}_n$ the ideal $\sqrt{\mathfrak{a}}$ is finite intersection of prime ideals $\mathfrak{p}_i$. Then $\mathcal{V}(\mathfrak{a}) = \cup_i \mathcal{V}(\mathfrak{p}_i)$ and $\mathcal{I}(\mathcal{V}(\mathfrak{a})) = \cap_i \mathcal{I}(\mathcal{V}(\mathfrak{p}_i))$.

If we know $\mathcal{I}(\mathcal{V}(\mathfrak{p}_i)) = \mathfrak{p}_i$ we get $\mathcal{I}(\mathcal{V}(\sqrt{\mathfrak{a}})) = \cap \mathfrak{p}_i = \sqrt{\mathfrak{a}}$

So let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_n$.

We have two special case

1. $\mathfrak{p} = (0) \Rightarrow \mathcal{V}(\mathfrak{p}) = (\mathbb{C}^n)_0$ (the germ of $\mathbb{C}^n$ at 0) and $\mathcal{I}(\mathcal{V}(\mathfrak{p})) = (0)$
2. $\mathfrak{p} = \mathcal{O}_n = (1)\mathcal{O}_n$. Hence $\mathcal{V}(\mathfrak{p}) = \emptyset \Rightarrow \mathcal{I}(\mathcal{V}(\mathfrak{p})) = (1)$

Exclude these two cases. If $\mathfrak{p}$ is not a trivial prime ideal $\dfrac{\mathcal{O}_n}{\mathfrak{p}}$ is an integral domain.

We want to find a suitable linear change of coordinates, in order to get an integer $k < n$ such that $\mathfrak{p} \cap \mathcal{O}_l \neq (0)$ for $l > k$ and $\mathfrak{p} \cap \mathcal{O}_k = (0)$, so that $\mathcal{O}_k \hookrightarrow \dfrac{\mathcal{O}_k}{\mathfrak{p}}$. More over we want $\dfrac{\mathcal{O}_n}{\mathfrak{p}}$ to be integral over $\mathcal{O}_k$.

This leads to the following definition of a regular system of coordinates.

Definition 3.4. *A* regular system of coordinates $z_1, \ldots, z_n$ *for* $\mathfrak{p}$ *is such that*

1. $\mathfrak{p} \cap \mathcal{O}_k = (0)$
2. $\dfrac{\mathcal{O}_n}{\mathfrak{p}}$ *is integral over* $\mathcal{O}_k$
3. *the quotient field* $\dfrac{\mathcal{O}_n}{\mathfrak{p}}$ *is a finite algebraic extension of the quotient field* $\mathcal{F}_k$ *of* $\mathcal{O}_k$*, generated by the image of* z_{k+1} *in* $\dfrac{\mathcal{O}_n}{\mathfrak{p}}$*. If* $\pi : \mathcal{O}_n \to \dfrac{\mathcal{O}_n}{\mathfrak{p}}$ *call* $\eta_{k+1} = \pi(z_{k+1})$

Also the number k should be indipendent on the regular system we choose (there are a lot of regular systems).

Proposition 3.5. *Any prime ideal* $\mathfrak{p} \subset \mathcal{O}_n$ *has a regular system of coordinates.*

Proof: The proof is by induction on n. Note that 3) means that $\dfrac{\mathcal{O}_n}{\mathfrak{p}}$ is a finite integral extension of $\mathcal{O}_k$ generated by some elements in $\dfrac{\mathcal{O}_n}{\mathfrak{p}}$. Hence $\mathcal{F}$ is an algebraic finite extension. There is a primitive element that can be choosen to be η_{k+1}.

Again we have trivial cases. If $\mathfrak{p} = (0)$ then $k = n$ and there is nothing to prove. If $\mathfrak{p} = \mathcal{O}_n$ then $k = 0$ and again nothing to prove.

Also the case $n = 0$ is trivial. So we can aasume the result is true for $\mathcal{O}_{n-1}$.

Let $\mathfrak{p}$ be not trivial and $f \in \mathfrak{p}$. There is a linear change of coordinates such that f becames regular with respect to z_n, hence $f = uh$ and h is a Weierstrass polynomial in $\mathcal{O}_{n-1}[z_n]$. Note that $h \in \mathfrak{p}$ since u is a unit.

$$h = z_n^r + \sum_{i<r} a_i(z_1, \dots, z_{n-1}) z_n^i.$$

$\mathfrak{p}$ is a prime ideal, hence $\mathfrak{p} \cap \mathcal{O}_{n-1}$ is also prime. By induction hypothesis there is a k such that $(\mathfrak{p} \cap \mathcal{O}_{n-1}) \cap \mathcal{O}_k = (0)$ and $\dfrac{\mathcal{O}_{n-1}}{\mathfrak{p} \cap \mathcal{O}_{n-1}}$ is integral over $\mathcal{O}_k$ and $\mathcal{F}' = \mathrm{Frac}\dfrac{\mathcal{O}_{n-1}}{\mathfrak{p} \cap \mathcal{O}_{n-1}}$ is a finite algebraic extension of $\mathcal{F}_k$ generated by the images $\eta_j = \pi(z_j), j = k+1, \dots, n-1$.

The first condition implies $p \cap \mathcal{O}_{n-1} = \{0\}$. To prove the second condition we need only to prove that $\dfrac{\mathcal{O}_n}{\mathfrak{p}}$ is integral over $\dfrac{\mathcal{O}_{n-1}}{\mathfrak{p} \cap \mathcal{O}_{n-1}}$.

Take $g \in \mathcal{O}_n$ and divide by h

$$g = Hh + \sum b_i z_n^i \quad b_i \in \mathcal{O}_{n-1}.$$

The image of g in $\dfrac{\mathcal{O}_n}{\mathfrak{p}}$ is

$$\pi(g) = \sum \pi(b_i)\pi(z_n)^i.$$

Hence the elements of $\dfrac{\mathcal{O}_n}{\mathfrak{p}}$ are polynomial of bounded degree in $\eta_n = \pi(z_n)$ with coefficients in $\dfrac{\mathcal{O}_{n-1}}{\mathfrak{p} \cap \mathcal{O}_{n-1}}$. Hence $\pi(b_i)$ is integral over $\mathcal{O}_k$ by the induction hypothesis and η_n is integral over $\dfrac{\mathcal{O}_{n-1}}{\mathfrak{p} \cap \mathcal{O}_{n-1}}$ since it verifies h ($h \in \mathfrak{p}$ means $h(\eta_1, \dots, \eta_{n-1}, \eta_n) = 0$ and this is an equation of integral dependence of η_n over $\dfrac{\mathcal{O}_{n-1}}{\mathfrak{p} \cap \mathcal{O}_{n-1}}$).

So we have that $\dfrac{\mathcal{O}_n}{\mathfrak{p}}$ is integral over $\mathcal{O}_k$, Also since $\dfrac{\mathcal{O}_n}{\mathfrak{p}}$ in integrally generated by η_n over $\dfrac{\mathcal{O}_{n-1}}{\mathfrak{p} \cap \mathcal{O}_{n-1}}$, which is integrally generated by $\{\eta_j = \pi(z_j)\}, k+1 \leq j \leq n-1$ over $\mathcal{O}_k$ we get $\dfrac{\mathcal{O}_n}{\mathfrak{p}}$ integrally generated over $\mathcal{O}_k$ by $\{\eta_j = \pi(z_j)\}, k+1 \leq j \leq n-1$.

Now the same elements $\{\eta_{k+1}, \dots, \eta_n\}$ generate $\mathcal{F}$ over $\mathcal{F}_k$. By the theorem on the primitive element there is $\eta = \sum_{j=k+1}^n c_j \eta_j$ $c_j \in \mathbb{C}$ such that $\mathcal{F} = \mathcal{F}_k[\eta]$.

Now take a linear change of coordinates putting $z'_{k+1} = \sum c_j z_j$ and then choosing among $z_{k+1}, \dots, z_n$ new variables $z'_{k+2}, \dots, z'_n$. This way the proposition is proved □

Now we have got a regular system of coordinates for p: $(z_1, \dots, z_k, z_{k+1}, \dots, z_n)$ and we put as before $\eta_j = \pi(z_j) = [z_j]_{\text{mod } p}$.

For all $j = k+1, \dots, n$ there is a polynomial $q_j \in \mathcal{O}_k[X]$ which is the minimal polynomial of η_j over . The polynomial q_j is monic and let r_j be its degree.

Now since η_j generates $\mathcal{F}$ over $\mathcal{F}_k$ any element in $\dfrac{\mathcal{O}_n}{\mathfrak{p}}$ is a polynomial in $\mathcal{F}[X]$ evaluated in η_{k+1} of degree $\leq r_{k+1} - 1$.

Cleaning denominators we can obtain a polynomial in $\mathcal{O}_k[X]$. But we want to be sure that there is a universal denominator that will be exactly the discriminant of q_{k+1}. This is a consequence of the following Lemma.

Lemma 3.6. *Put* $A = \mathcal{O}_k$, $B = \mathrm{qf}\,\dfrac{\mathcal{O}_n}{\mathfrak{p}}$. *We know* B *is integral over* A *and there is* $\eta \in B$ *primitive element of the algebraic extension* $F = \mathrm{qf}\ B$ *of* $E = \mathrm{qf}\ A$.

Let q *be the minimal polynomial of* η, $q \in A[X]$, $\deg q = r$. $d = \mathrm{discr}\ q$, *which is the Greatest Common Divisor between* q *and its derivative* q' *or equivalently the Resultant* Res *(q,q')*$\in A$. *Then* $dB \subset A[X]$

Proof: Take $y \in B$, we can write y as a polynomial in η.

$$y = b_0 + b_1\eta + \cdots + b_{r-1}\eta^{r-1} \quad b_i \in E.$$

We have to prove that $db_j \in A$.

Let L be the splitting field of q, i.e. the minimal field were q gets all its roots. In L the roots of q are conjugate to η with respect to the Galois group of q. They are, say, $\eta_1 = \eta, \eta_2, \dots, \eta_r$. The Galois group is transitive on the roots, so there are automorphisms σ_i of $L_{|E}$ such that $\sigma_1 = id, \sigma_i(\eta) = \eta_i$. Apply all these automorphisms to the equation above and note that b_i does not change since it is in E. We get a linear system whose matrix V is the Vandermonde matrix of $\eta_1, \dots, \eta_r$.

$$\begin{cases} y & = b_0 + b_1\eta_1 + \cdots + b_{r-1}\eta_1^{r-1} \\ \sigma_2(y) & = b_0 + b_1\eta_2 + \cdots + b_{r-1}\eta_2^{r-1} \\ \vdots & \\ \sigma_r(y) & = b_0 + b_1\eta_r + \cdots + b_{r-1}\eta_r^{r-1} \end{cases}$$

The determinant of V is $\det V = \prod_{i<j}(\eta_i - \eta_j)$. It is not symmetric with respect to $\eta_1, \dots, \eta_r$, while $d = \prod_{i<j}(\eta_i - \eta_j)^2 = (\det V)^2$ is symmetric with respect to the roots.

Now we apply Cramer rule to solve the system

$$b_j = \frac{\delta(y, \sigma_2(y), \dots, \sigma_r(y), \eta_1, \dots, \eta_r)}{\det V}.$$

So $db_j = \det V\delta$. The determinant δ is a polynomial in its $2r$ variables with integer coefficients and all its variables are integral aver A. Indeed $\eta \in B$ is integral over A and the polynomial q provides a relation of integral dependence over A for all roots.

Also $y \in B$ and its images $\sigma_j(y)$ are integral over A. But A is integrally closed in its quotient field, hence $\delta(y, \sigma_2(y), \ldots, \sigma_r(y), \eta_1, \ldots, \eta_r) \in A$ and $db_j \in A$ as wanted. □

Let us come back to $\mathfrak{p}$.

Lemma 3.7. *The polynomials $q_j(z_j)$ are Weierstrass polinomials and belong to $\mathfrak{p}$*

Proof: The polynomials $q_j(z_j)$ are in $\mathfrak{p}$ because $q_j(z_j)_{\text{mod } \mathfrak{p}} = q_j(\eta_j) = 0$. Also they are monic, hence regular with respect to z_j. Hence

$$q_j(z_j) = uq_j'(z_j)$$

with $q_j'(z_j)$ Weierstrass. But they have the same degree and both are monic, so $u = 1$ and $q_j = q_j'$. □

Note that Lemma 3.6 gives an expression for η_j in terms of η_{k+1}, namely

$$d(z_1, \ldots, z_{k-1})\eta_j = T_j(\eta_{k+1}).$$

This implies $d \cdot z_j - T_j(z_{k+1}) \in \mathfrak{p}$.

We ave got several elements of $\mathfrak{p}$, namely

$$q_{k+1}(z_{k+1}), \ldots, q_n(z_n), dz_{k+2} - T_{k+2}(z_{k+1}), \ldots, dz_n - T_n(z_{k+1}).$$

Let $\mathcal{I}_1 \subset \mathcal{O}_n$ be the ideal generated by all these elements and $\mathcal{I}_2$ be the one generated by $q_{k+1}(z_{k+1})$ and $dz_{k+2} - T_{k+2}(z_{k+1}), \ldots, dz_n - T_n(z_{k+1})$.

We have $\mathcal{I}_2 \subset \mathcal{I}_1 \subset \mathfrak{p}$. Hence $V_2 = \mathcal{V}(\mathcal{I}_2) \supset V_1 = \mathcal{V}(\mathcal{I}_1) \supset V = \mathcal{V}(\mathfrak{p})$.

We want to prove that these 3 germs of analytic sets in $\mathbb{C}^n_0$ are the same outside the zeroset of $d \in \mathcal{O}_k \subset \mathcal{O}_n$. (The zeroset of d in $\mathbb{C}^n_0$ is a cylinder over the zeroset of d in $\mathbb{C}^k_0$). This is done in several steps.

Step 1 $V_1 \setminus \mathcal{V}(d) = V_2 \setminus \mathcal{V}(d)$.

We have only to show

$$V_2 \setminus \mathcal{V}(d) \subseteq V_1 \setminus \mathcal{V}(d)$$

because the converse is clear.

Since the only generators in $\mathcal{I}_1 \setminus \mathcal{I}_2$ are $q_j(z_j), k+1 < j \leq n$, we have only to prove that they vanish on $V_2 \setminus \mathcal{V}(d)$.

Consider $h_j(X) = d^{r_j} q_j\left(\dfrac{T_j(X)}{d}\right) \in \mathcal{O}_k[X]$ where r_j is the degree of q_j.

This polynomial vanishes if evaluated in η_{k+1} because it gives $d^{r_j} q_j(\eta_j) = 0$.

Hence $h_j(X)$ is divisible by $q_{k+1}(X)$, that is $h_j(X) = Q_j(X)q_{k+1}(X)$ with $Q_j(X) \in \mathcal{O}_k[X]$.

Next take a polydisc $\Delta(0,\rho) = \Delta_1 \times \Delta_2$ where $\Delta \subset \mathbb{C}^k$ such that everythink we considered is defined and holds true in Δ.

Call a_0 the elements in Δ_1. Take $a = (a_0, a_{k+1}, \dots, a_n) \in V_2 \setminus \mathcal{V}(d)$.

Then we get

$$\begin{aligned}
&q_{k+1}(a_0, a_{k+1}) = 0 \\
&d(a_0)a_j = T_j(a_{k+1}) \\
&h_j(a_0, a_{k+1}) = 0 = d^{r_j}(a_0) q_j \left(a_0, \frac{T_j(a_0.a_{k+1})}{d(a_0)} \right) = d^{r_j}(a_0) q_j(a_0, a_j) \\
&d(a_0) \neq 0 \Rightarrow q_j(a_0, a_j) = 0 \text{ for } j = k+2, \dots, n
\end{aligned}$$

as wanted.

Step 2 Now we prove a technical lemma which shows that for all $f \in \mathcal{O}_n$ there is a power α of d and a polynomial $R \in \mathcal{O}_k[z_{k+1}]$ such that $d^\alpha f - R \in p$.

Lemma 3.8. *Let f be a germ in $\mathcal{O}_n$ and $\alpha = \sum_{k+2}^{n} (r_j - 1)$. Then there is a polynomial $R \in \mathcal{O}_k[z_{k+1}]$,* deg $R < r_{k+1}$ *such that $d^\alpha f - R \in \mathcal{I}_1 \subset \mathfrak{p}$*

Proof: Divide f by q_n. We get

$$f = A_n q_n + R_n = A_n q_n + \sum_{i=0}^{r_n - 1} A_{i,n} z_n^i, \; A_{i,n} \in \mathcal{O}_{n-1}.$$

Divide all $A_{i,n}$ by q_{n-1} in $\mathcal{O}_{n-1}$. We get

$$f = A_n q_n + A_{n-1} q_{n-1} + R_{n-1}, \quad R_{n-1} \in \mathcal{O}_{n-2}[z_{n-1}, z_n].$$

Go on dividing the coefficients of R_{n-1} by q_{n-2} and inductively the coefficients of R_j by q_j in $\mathcal{O}_j$. We get

$$f = \sum_{j=k+1}^{n} A_j q_j(z_1, \dots, z_k, z_j) + R$$

where $R \in \mathcal{O}_k[z_{k+1}, \dots, z_n]$ and deg $R \leq \sum (r_j - 1)$ since R_n has degree $\leq r_n - 1$, R_{n-1} has degree $\leq r_{n-1} - 1$ in z_{n-1} and $\leq r_n - 1$ in z_n and so on.

Next multiply by d^α, $\alpha = \sum_{j=k+2}^{n} (r_j - 1)$. We get

$$d^\alpha f = \sum A'_j q_j(z_j) + R'.$$

The exponent of d is big enough to have in R' each variable multiplied by d, that is R' is a polynomial in $z_{k+1}, dz_{k+2}, \dots, dz_n$.

Replace dz_j by $dz_j - T_j(z_{k+1}) + T_j(z_{k+1})$ and expand. We get

$$d^\alpha f = \sum A'_j q_j + R''(z_{k+1}, dz_j - T_j(z_{k+1})) + R'''(z_{k+1}, T_j(z_{k+1})).$$

Hence $R''' \in \mathcal{O}_k[z_{k+1}]$ and we can divide it by q_{k+1} getting $R''' = Qq_{k+1} + \tilde{R}$ where $\tilde{R} \in \mathcal{O}_k[z_{k+1}]$ and has degree at most $r_{k+1} - 1$.

Now $\sum A'_j q_j \in \mathcal{I}_1$, $R'' \in I_1$ because each one of its monomials is a multiple of some $dz_j - T_j(z_{k+1})$ and $R''' - \tilde{R} \in \mathcal{I}_1$. Hence $d^\alpha f - \tilde{R} \in \mathcal{I}_1$ as wanted □

Step 3 $V \setminus \mathcal{V}(d) = V_1 \setminus \mathcal{V}(d)$

As before $V \setminus \mathcal{V}(d) \subset V_1 \setminus \mathcal{V}(d)$. To prove the opposite inclusion we have to prove that any $f \in \mathfrak{p}$ vanishes on $V_1 \setminus \mathcal{V}(d)$. This comes from *Step 2*. We got $d^\alpha f - \tilde{R} \in \mathcal{I}_1 \subset \mathfrak{p}$. But $d^\alpha f \in \mathfrak{p}$, hence $\tilde{R} \in \mathfrak{p}$. This implies $\tilde{R}(z_{k+1}) = 0$ hence q_{k+1} divides $\tilde{R}$ and this is not possible unless $\tilde{R}$ is the zero polynomial because of its degree. So $d^\alpha f \in \mathcal{I}_1$ and f vanishes on $V_1 \setminus \mathcal{V}(d)$.

We are ready to prove Rückert Nullstellensatz.

Theorem 3.9. *Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_n$. Then $\mathcal{I}(\mathcal{V}(\mathfrak{p})) = \mathfrak{p}$.*

Proof: Take $f \in \mathcal{I}(\mathcal{V}(\mathfrak{p}))$. By *Step 2* above we have $d^\alpha f = Q + \tilde{R}(z_{k+1})$ where $Q \in \mathcal{I}_1 \subset \mathfrak{p}$ and $\tilde{R} \in \mathcal{O}_k[z_{k+1}]$ and has degree less than r_{k+1}. Hence $\tilde{R}$ vanishes on V and by the previous steps it vanishes on $V_2 \setminus \mathcal{V}(d)$.

Choose a polidisc $\Delta(0, \rho)$ so small that everything happens in $\Delta(0, \rho) = \Delta_1 \times \Delta_2, \Delta_1 \subset \mathbb{C}^k$.

Take $a_0 \in \Delta_1$ such that $d(a_0) \neq 0$. Hence there is at least one $a_{k+1} \in \mathbb{C}$ such that $q_{k+1}(a_0, a_{k+1}) = 0$. For such a_{k+1} consider the point

$$a = (a_0, \ldots, a_n) = \left(a_0, a_{k+1}, \frac{T_{k+2}(a_{k+1})}{d(a_0)}, \ldots, \frac{T_n(a_{k+1})}{d(a_0)}\right)$$

Then $q_j(a_0, a_j) = 0$ hence $|a_j| < \rho_j$ and $a \in \Delta$ or better $a \in V_2 \setminus \mathcal{V}(d)$. Hence $\tilde{R}(a) = \tilde{R}(a_0, a_{k+1}) = 0$. Now a_{k+1} was one among the r_{k+1} distinct roots of $q_{k+1}(a_0, X)$ and this implies that $\tilde{R}$ has r_{k+1} distinct roots. This is a contradiction since $\tilde{R}$ has smaller degree. So $\tilde{R}$ is the zero polynomial and $d^\alpha f = Q \in \mathfrak{p}$. But $d \notin \mathfrak{p}$ and $\mathfrak{p}$ is prime. Hence $f \in \mathfrak{p}$. □

4 Germs of analytic sets as ramified coverings

We can derive some nice consequences from Theorem 3.9.

We have attached to a prime ideal $\mathfrak{p} \in O_n$ several objects.

- a system of regular coordinates $(z_1, \ldots, z_k, z_{k+1}, \ldots, z_n)$
- Weierstrass polynomials $q_{k+1}, \ldots, q_n$ with $q_j \in \mathcal{O}_k[z_j]$
- the discriminant of q_{k+1} $d \in \mathcal{O}_k$
- polynomials $T_j \in \mathcal{O}_k[X]$ such that $dz_j - T_j(z_{k+1}) = 0 \mod \mathfrak{p}$

We will say that a polydisc $\Delta(0, \rho) = \Delta_1 \times \Delta_2$ provides an *admissible representation* for a prime ideal $\mathfrak{p}$ if

- There are representative of $q_{k+1}, \ldots, q_n, d, T_j$ defined on Δ
- $\forall a \in \Delta_1$, $q_j(a, b_j) = 0 \Rightarrow |b_j| < \rho_j \forall j = k+1, \ldots, n$

Such polydiscs exist and can be arbitrarily small. This is enough to give the following geometric description of the zeroset of $\mathfrak{p}$.

Theorem 4.1. *Let $\Delta = \Delta_1 \times \Delta_2$, q_j, T_j, d be an admissible representation for a prime ideal $\mathfrak{p} \subset \mathcal{O}_n$. Put $s = degq_{k+1}$. Then*

1. *$V \setminus \mathcal{V}(d)$ is a complex manifold in Δ and $\pi : V \setminus \mathcal{V}(d) \to \Delta_1 \setminus \mathcal{V}(d)$ is an s-sheeted covering map.*
2. *$\pi : \overline{V} \to \Delta_1$ is a proper map.*
3. *$V \setminus \mathcal{V}(d)$ is connected and $\overline{V} = \mathcal{V}(\mathfrak{p})$*

Proof: Remark that $\mathcal{V}(d)$ is a closed nowhere dense subset in $\subset \Delta_1$ because d is holomorphic and not identically 0.

1. Take $a = (a_1, \ldots, a_k) \in \Delta_1$ such that $d(a) \neq 0$. Then $q_{k+1}(a, z_{k+1})$ has s distincts roots $b^1_{k+1}, \ldots, b^s_{k+1}$. Define $b^i_j = \dfrac{T_j(a, b^i_{k+1})}{d(a)}$. We get a point $b^i = (a, b^i_{k+1}, b^i_{k+2}, \ldots, b^i_n)$ for each i and we see that $\pi^{-1}(a) \cap V = \{b^1, \ldots, b^s\}$. Moreover the derivative of q_{k+1} with the respect to z_{k+1} is different from 0 in all these points. By the implicit function theorem there is a unique holomorphic function $h^i(z_1, \ldots, z_k)$ on an open neighbourhood U^i of a such that $q_{k+1}(z_1, \ldots, z_k, h^i(z_1, \ldots, z_k)) = 0$ and $h(a) = b^i$.
   Put $U = U^1 \cap U^2 \cap \ldots \cap U^s$. Then $V \cap \pi^{-1}(U) = \cup_{i=1}^s W_i$ where $W_i = \left\{ z \in \pi^{-1}(U) \,\middle|\, z_{k+1} = h^i(z_1, \ldots, z_k), z_j = \dfrac{T_j(h^i)}{d(z_1, \ldots z_k)} \right\}$.

If U is sufficiently small we can get $W_i \cap W_l = \emptyset$ for $i \neq l$. This implies $\pi : V \setminus \mathcal{V}(d) \to \Delta_1 \setminus \mathcal{V}(d)$ is a covering map and π is a local biholomorphism. U is a trivializing neighbourhood of a since $\pi^{-1}(U)$ is a disjoint union of open sets W_i such that $\pi_{|W_i}$ is a biholomorphism whose inverse is h^i.

In particular $V \setminus \mathcal{V}(d)$ is a complex manifold of dimension $k = \dim \Delta_1$.

2. $q_{k+1}, \dots, q_n$ vanish on V and $V_0 = \mathcal{V}(q_{k+1}, \dots, q_n)$ is closed in Δ, so $\overline{V} \subset V_0$. Hence it is enough to prove that $\pi_{|V_0}$ is proper.

   Let K be compact set in Δ_1 and let $\{b_n\}$ a sequence in $\pi^{-1}(K) \cap V_0$. Denote $a_n = \pi(b_n) \in K$. The sequence $\{b_n\}$ lies in $\overline{\Delta}$ which is compact and so it has a limit point $d \in \overline{\Delta}$ that we may assume to be the limit of the sequence up to passing to the subsequence. Then $a_n = \pi(b_n)$ converges to $a = \pi(b)$, but K is compact hence $a \in K$. Also $q_j(b_n) = 0 = q_j(a_n, (b_n)_j)$ hence $|(b_n)_j| < r_j$ for $j = k+1, \dots, n$. This means that $b_n \in \Delta$ because $a \in \Delta_1$. By continuity q_j vanishes also on the limit point $b = (a, b_{k+1}, \dots, b_n)$. Since $a \in \Delta_1$ we get also $|b_j| < r_j$ and so $b \in \Delta$. Summing up $b \in \pi^{-1}(K) \cap V_0$. Hence this set is compact.

3. Let W be a connected component of $V \setminus \mathcal{V}(d)$. Then $\pi_{|W}$ is a t-sheeted covering map and $t \leq s$. Let $\overline{W}$ be the closure of W in Δ. Then π maps $\overline{W}$ properly on Δ_1.

   We want to prove that the closure of W in Δ is an analytic subset of Δ.

   Suppose proved the statement above. Then if $W_1, \dots, W_l$ are the connected components of $V \setminus \mathcal{V}(d)$ we get that $\overline{W}_j$ is a closed analytic set in Δ for all j.

   Now
$$\mathcal{V}(\mathfrak{p}) \subset V \setminus \mathcal{V}(d) \cup \mathcal{V}(d) \subset \cup_j \overline{W}_j \cup \mathcal{V}(d)$$
   while
$$\mathcal{V}(\mathfrak{p}) \supset V \setminus \mathcal{V}(d) = \cup_j W_j.$$
   Since $\mathcal{V}(\mathfrak{p})$ is closed and contains W_j it must contain its closure i.e. $\mathcal{V}(\mathfrak{p}) \supset \cup_j \overline{W}_j$, hence
$$\mathcal{V}(\mathfrak{p}) = \cup_j \overline{W}_j \cup (\mathcal{V}(d) \cap \mathcal{V}(\mathfrak{p}))$$
   .

   Since $\mathcal{V}(\mathfrak{p})$ is irreducible and cannot be contained in $\mathcal{V}(d)$, there is an index j_0 such that $\mathcal{V}(\mathfrak{p}) = \overline{W}_{j_0}$. So W_{j_0} is dense in $\mathcal{V}(\mathfrak{p})$ and $V \setminus \mathcal{V}(d) = W_{j_0}$ is connected.

   So we are left with the proof that $\overline{W}$ is an analytic set in Δ. The strategy is as follows. For any point $x_0 \in \Delta \setminus \overline{W}$ we want to find a holomorphic function F_0 on Δ such that $F_0(x_0) \neq 0$ and F_0 vanishes on $\overline{W}$ so $F_0 \in \mathcal{I}(\overline{W})$. As a consequence we will get $\overline{W} = \mathcal{V}(\mathcal{I}(\overline{W}))$, that is $\overline{W}$ is an analytic set.

   So take $x_0 \in \Delta \setminus \overline{W}$, and let a_0 be $\pi(x_0)$. Now $\pi^{-1}(a_0) \cap \overline{W}$ is a finite set of at most t points and we can find a polynomial h such that $h(x_0) = 0$ and $h(z) = 1$ for all $z \in \pi^{-1}(a_0) \cap \overline{W}$.

Next we proceed to the construction of the function we need. If $d(a) \neq 0$ there is a neighbourhood U_a of a such that $\pi^{-1}(U_a) \cap W = U^1 \cup \ldots \cup U^t$ and π induces a biholomorphism between U^i and U_a whose inverse $U_a \to U^i$ we call φ_a^i. Note that if b is another such point and $U_a \cap U_b \neq \emptyset$ the functions $\{\varphi_a^i\}$ and $\{\varphi_b^i\}$ are the same on $U_a \cap U_b$ up to a permutation.

Next define $h_a^i = h_{|U^i} \circ \varphi_a^i$. Again if $U_a \cap U_b \neq \emptyset$ we get $\{h_a^i\} = \{h_b^j\}$ up to a permutation. Hence if S is a symmetric polynomial in t variables

$$S_a = S(h_a^1, \ldots, h_a^t) = S_b = S(h_b^1, \ldots, h_b^t)$$

on $U_a \cap U_b$.

So all these functions glue together to a holomorphic function $\tilde{S}$ on $\Delta_1 \setminus \mathcal{V}(d)$. Also $\tilde{S}$ is bounded on $\Delta_1 \setminus \mathcal{V}(d)$ because depends only on the values of the polinomial h on $\overline{\Delta}$. So $\tilde{S}$ extends to a holomorphic function on Δ_1.

Consider now the polinomial $R(X) = \prod_{i=1}^{t}(X - h_a^i)$. It can be written as $R(X) = X^t + \sum_{i=0}^{t-1} S_i X^i$ where S_i is the $t-i$-th elementary symmetric function in t variables evaluated in $(h_a^1, \ldots, h_a^t)$. Then $R(X)$ is a polynomial holomorphic on Δ_1, and $R(X) \in \mathcal{O}_k[X]$.

Note that for $a \in \Delta_1 \setminus \mathcal{V}(d)$ the roots of $R(X)$ are precisely the values that h takes on $\pi^{-1}(a) \cap W$.

What happens if $d(a) = 0$? In this case $\pi^{-1}(a) \cap \overline{W}$ will get less than t points. But the above property is still true. Take $a \in \mathcal{V}(d)$ and consider a sequence $\{a_n\}$ converging to a with $a_n \in \Delta_1 \setminus \mathcal{V}(d)$. Assume $\pi^{-1}(a) \cap \overline{W} = \{x^1, \ldots, x^p\}$. Each x^i will be the limit point of a sequence of points lying over the sequence $\{a_n\}$. This means that we can write

$$\pi^{-1}(a_n) \cap W = \bigcup_{j=1}^{p} \{x_n^{j,1}, \ldots, x_n^{j,k_j}\}$$

where $\lim_{n\to\infty} x_n^{j,l} = x^j$.

Now $R(a_n)(X) = \prod(X - h(x_n^{j,l})$ hence $\lim_{n\to\infty} R(a_n)(X) = \prod(X - h(x^j))^{k_j} = R(a)(X)$ that is again the roots of $R(a)$ are the values of h at the points in $\pi^{-1}(a) \cap \overline{W}$.

Finally put $F_0(z) = h(z)^t + \sum_{i=1}^{t-1} S_i(\pi(z))h(z)^i = R(\pi(z))(h)$. Then F_0 vanishes on W because for $z \in W$ R vanishes on $h(z)$. But since the only root of $R(a_0)(z)$ is 1, F_0 does not vanish at x_0 because h does not take the value 1 at x_0.

Hence the set $A = \{F \in \mathcal{O}(\Delta) | F^{-1}(0) \supset W\}$ has precisely $\overline{W}$ as zeroset. Since $A \subset \mathcal{I}(\overline{W})$, $\overline{W}$ is an analytic set

□

What can be said in the general case when $\mathfrak{q} = \mathcal{I}(V)$ is not prime?

First of all we know that $\mathfrak{q}$ is radical, hence $\mathfrak{q} = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_s$ where $\mathfrak{p}_i$ is a prime ideal and $\mathfrak{p}_i \not\subset \mathfrak{p}_j$ for $i \neq j$.

Also we know from Theorem 4.1 that

1. $\mathcal{V}(\mathfrak{p}_1), \ldots, V(\mathfrak{p}_s)$ are the irreducible components of the analytic set germ V.
2. $\forall i$ there is an open dense set in $\mathcal{V}(\mathfrak{p}_i) = V_i$ where V_i is a complex manifold of dimension k_i and k_i is such that $\mathfrak{p}_i \cap \mathcal{O}_{k_i} = 0$.

We define the *dimension of* V as $k = \max_i k_i$. We give a provisional definition of *regular point.* Fix a suitable polydisc for V. A point $z \in V$ is *regular of dimension* m if there is a neighbourhood U of z in V such that U is a complex manifold of complex manifold of dimension m.

This way , we can remark that the set of regular points is open and dense in V. Indeed since $V_i \cap V_j$ is an analytic set whose dimension is strictly less than dim V_i and dim V_j, the set of regular points in V_i defined as covering of $\Delta_{k_i} \setminus \mathcal{V}(d_i) \subset \mathbb{C}^{k_i}$ which are not points of V_j for some $j \neq i$ is still open and dense in V_i.

A point $z \in V$ is *singular* if z is not regular of some dimension .

Denote by $\mathcal{R}(V)$ the set of regular points and by $\mathcal{S}(V)$ the set of singular points. Till now we can only state that $\mathcal{R}(V)$ is open and dense in V and $\mathcal{S}(V)$ is contained in the analytic set germ $\cup_i \mathcal{V}(d_i)$. We will see that $\mathcal{S}(V)$ is actually an analytic subset of V.

Next we caracterize analytic set germs of codimension 1.

Theorem 4.2. *a) Take $f \in \mathcal{O}_n$ such that $f(0) = 0, f \not\equiv 0$. Then all irreducible components of $\mathcal{V}(f)$ have dimension $n-1$.*

b) If V is an analytic set germ whose irreducible components have all dimension $n-1$ then $\mathcal{I}(V)$ is a principal ideal.

Proof:

a) $f = u \cdot \prod_i p_i^{s_i}$, a finite product. All p_i are irreducible and u is a unit. Then $\mathcal{V}(f) = \cup_i \mathcal{V}(p_i)$ and we know from the Nullstellensatz for principal ideals that $(p_i) \cap \mathcal{O}_{n-1} = \{0\}$. Hence dim $\mathcal{V}(p_i) = n-1$.

b) Let $V = V_1 \cup \ldots \cap V_s$ be the decomposition of V into irreducible components. For each i we can find a regular system of coordinates $(z_1, \ldots, z_{n-1}; z_n)$ for V_i. If q_n^i is the Weierstrass polynomial of z_n as in the proof of Nullstellensatz q_n^i is irreducible and $\mathcal{I}(\mathcal{V}(q_n^i)) = (q_n^i)$. Also $\mathcal{V}(q_n^i) = V_i$ because in this case there is only one variable involved. Hence, with the notations as in the proof of Nullstellensatz $\mathcal{I}_1 = \mathcal{I}_2 = (q_n^i)$.

Hence $V_i \setminus \mathcal{V}(d_i) = \mathcal{V}(q_n^i) \setminus \mathcal{V}(d_i)$ and so $V_i = \overline{V(q_n^i) \setminus \mathcal{V}(d_i)} = \mathcal{V}(q_n^i)$.

Now if f vanishes on V it vanishes on V_i, hence q_n^i divides f, then $\prod_i q_n^i | f$ and so $\mathcal{I}(V) = (\prod_i q_n^i)$ is principal.

□

Next theorem is a nice consequence.

Theorem 4.3. *Let V be an analytic set germ of dimension k, let $f \in \mathcal{O}_n$ be such that $f(0) = 0$. Then* $\dim V \cap \mathcal{V}(f) \geq k - 1$. *If V is irreducible and $f \notin \mathcal{I}(V)$ then* $\dim V \cap \mathcal{V}(f) = k - 1$.

Proof: Take an irreducible component V_i of V of dimension k. If $f \in \mathcal{I}(V_i)$ then $\dim \mathcal{V}(f) \cap V = k \geq k - 1$. Assume next $f \notin \mathcal{I}(V_i)$ for all i and take an admissible polydisc $\Delta = \Delta_1 \times \Delta_2$, $\Delta \subset \mathbb{C}^k$. Then $V_i \setminus \mathcal{V}(d)$ is a connected manifold of dimension k and since $f \notin \mathcal{I}(V_i)$, by the theorem above $\mathcal{V}(f) \cap (V_i \setminus \mathcal{V}(d))$ has exactly dimension $k - 1$. Thus if $f \notin \mathcal{I}(V_i)$ for all i such that $\dim V_i = k$ then $\dim \mathcal{V}(f) \cap V = k - 1$

□

Corollary 4.4.

1. *Let V be irreducible and Δ, q_j, T_j, d be an admissible system for $\mathcal{I}(V)$, then* $\dim V \cap \mathcal{V}(d) = \dim V - 1$.
2. *The set of singular points of an irreducible analytic set germ V is contained in an analytic subset of dimension* $\dim V - 1$.

5 Regular points of an analytic set

Let $X \subset \Omega \subset \mathbb{C}^n$ be an analytic set.

This means

1. X is closed in Ω .
2. $\forall z \in \Omega$ there is an open set $U_z \ni z$ such that $X \cap U_z = \mathcal{V}(f_1, \ldots, f_l)$ where each f_i is holomorphic on U_z.

Now we can consider X as the collection of its germs, $X = \bigcup_{x \in X} X_x$. Hence at each $x \in X$ we can attach an ideal $\mathcal{J}_x = \mathcal{I}(X_x) \subset \mathcal{O}_{n,x}$.

This ideal is finitely generated by, say, $\{f_1, \ldots, f_l\}$. The *rank* r_x at x is defined as $r_x = \text{rk} \, \dfrac{\partial(f_1, \ldots, f_l)}{\partial z_1, \ldots, \partial z_n} \leq \min(l, n)$.

It is not difficult to prove that r_x does not depend on the choosen generators. Here comes an important fact.

Theorem 5.1 (Oka-Cartan). *If $\mathcal{I}_x = (f_1, \dots, f_l)\mathcal{O}_{n,x}$ there is an open polydisc Δ centered at x such that*

- *$f_1, \dots, f_l$ are holomorphic on Δ.*
- *$\forall y \in \Delta \;\; \mathcal{I}_y = (f_1, \dots, f_l)\mathcal{O}_{n,y}$.*

Assume now X_x to be irreducible . Then it should be clear that $r_y \geq r_x$ for y close to x, let us say $y \in \Delta$.

Put $r = \sup_{X \cap \Delta} r_y$.

Definition 5.2. *$y \in X \cap \Delta$ is* regular *if $r_y = r$. It is* singular *if it is not regular.*

Theorem 5.3. *Let $X \subset \Omega \subset \mathbb{C}^n$ be an analytic set. Then the set $\mathcal{S}(X)$ of singular points of X is a proper closed analytic subset of X.*

Proof: Take $x \in X$ and assume for a moment X_x to be irreducible. Let $f_1, \dots, f_l$ be generators of $\mathcal{I}_x$ and let Δ_x be an open polydisc such that $f_1, \dots, f_l$ generate $\mathcal{I}_y$ for all $y \in X \cap \Delta_x$. Then $\mathcal{S}(X) \cap \Delta_x$ is defined by the vanishing of all $r \times r$ minors of the jacobian matrix of $f_1, \dots, f_l$.

If $X_x = \bigcup_{i=1}^{t} Y_i$ repeat the argument for each component. Then

$$\mathcal{S}(X) \cap \Delta_x = \bigcup_{i=1}^{t} \mathcal{S}(Y_i) \cup \bigcup_{i \neq j} (Y_i \cap Y_j)$$

So $\mathcal{S}(X)$ is locally a zeroset of finitely many holomorphic functions as wanted. $\square$

Proposition 5.4. *Let $x_0 \in \mathcal{R}(X)$ be a regular point. Then there is an open neighbourhood U of x_0 such that $X \cap U$ is a complex manifold of dimension $n - r$ where $r = r_{x_0}$.*

Proof: Fix $q_1, \dots, q_r \in \mathcal{I}_{x_0}$ such that their jacobian has rank r at any point of U where U is a suitable neighbourhood of x_0. Of course $X_0 = \{q_1 = 0, \dots, q_r = 0\}$ is a complex manifold in U of dimension $n - r$. We have to prove that in a possibly smaller neighbourhood W of x_0, $X_0 \cap W = X \cap W$. It is enough to show that any germ $h \in \mathcal{I}_{x_0}$ vanishes on the germ $(X_0)_{x_0} \supset X_{x_0}$.

Since the rank of the jacobian matrix of $q_1, \dots, q_r$ is r, we can complete this list with linear functions $l_{r+1}, \dots, l_n$ in such a way that

$$\begin{cases} y_1 = q_1(z_1, \dots, z_n) \\ y_2 = q_2(z_1, \dots, z_n) \\ \vdots \\ y_r = q_r(z_1, \dots, z_n) \\ y_{r+1} = l_{r+1}(z_1, \dots, z_n) \\ \vdots \\ y_n = l_n(z_1, \dots, z_n) \end{cases}$$

is a holomorphic coordinate system in a neighbourhood of $x_0 \in \mathbb{C}^n$.

In these coordinates the germ $h \in \mathcal{I}_{x_0}$ verifies

$$h_{|X_0} = h(0, \dots, 0, y_{r+1}, \dots, y_n)$$

and so $h_{|X_0}$ is a holomorphic function of the last $n - r$ coordinates. What we have to show is that for any multiindex $\alpha = (\alpha_1, \dots, \alpha_{n-r})$ the derivative

$$\frac{\partial^\alpha h}{\partial y_{r+1}^{\alpha_1} \dots \partial y_n^{\alpha_{n-r}}}(x_0) = 0$$

This would imply the Taylor series of $h_{|X_0}$ to be the zero series and hence as wanted $h_{|X_0} = 0$.

Note that for $i > r$ one has

$$\frac{\partial h}{\partial y_i} = \det \begin{pmatrix} & & & 0 \\ & I_r & & \vdots \\ & & & 0 \\ \frac{\partial h}{\partial y_1} & \dots & \frac{\partial h}{\partial y_r} & \frac{\partial h}{\partial y_i} \end{pmatrix} = \det \frac{\partial(q_1, \dots, q_r, h)}{\partial y_1, \dots, \partial y_r, \partial y_i}$$

Next consider for $s = (s_1, \dots, s_{r+1}) \in \{1, \dots, n\}^{r+1}$ the $r + 1 \times r + 1$ minor $D_s h$ of the jacobian matrix of $(q_1, \dots, q_r, h)$

$$D_s h = \frac{\partial(q_1, \dots, q_r, h)}{\partial z_{s_1}, \dots, \partial z_{s_{r+1}}}$$

$D_s h$ vanishes on the regular points of X in U since there the rank is r so it vanishes on X_0 and finally $D_s h \in \mathcal{I}_{x_0}$ because $X_0 \supset X$.

We have $\dfrac{\partial q_l}{\partial y_j} = \displaystyle\sum_{i=1}^{n} \frac{\partial q_l}{\partial z_i} \cdot \frac{\partial z_i}{\partial y_j}$. Hence

$$\frac{\partial h}{\partial y_j} = \sum_{s_1 < s_2 < \dots < s_{r+1}} D_s h \cdot \det \frac{\partial(z_{s_1}, \dots, z_{s_{r+1}})}{\partial y_1 \dots, \partial y_r, \partial y_j}.$$

This implies

$$h \in \mathcal{I}_{x_0} \Rightarrow \frac{\partial h}{\partial y_j} \in \mathcal{I}_{x_0} \forall j > r.$$

Now we can apply recursively the same trick to the first derivatives of h to get $\dfrac{\partial^2 h}{\partial y_j \partial y_k} \in \mathcal{I}_{x_0}$ and so on. This way we proved that all derivatives of $h_{|X_0}$ of all orders vanish at x_0. So $h_{|X_0}$ is the zero function and we are done. □

Next notion to be introduced is the one of Zariski tangent space.

Definition 5.5. *Let $x \in X$ be any point and let $f_1, \dots, f_l$ be generators of $\mathcal{I}_x$. The* Zariski tangent space *of X at x is $T_xX = \ker J(f_1, \dots, f_l)$ where J is the jacobian matrix of $f_1, \dots, f_l$ at the point x*

There is a caracterization of T_xX in terms of $\dfrac{\mathcal{O}_{n,x}}{\mathcal{I}_x} := \mathcal{O}_{X,x}$

Lemma 5.6. *Let $\mathfrak{m}_x$ be the maximal ideal of the local ring $\mathcal{O}_{X,x}$. Then the $\dfrac{\mathcal{O}_{X,x}}{\mathfrak{m}_x}$ vector space $\dfrac{\mathfrak{m}_x}{\mathfrak{m}_x^2}$ is isomorphic to T_xX.*

Proof: Denote by $\mathcal{M}_x$ the maximal ideal of $\mathcal{O}_{n,x}$. Since $\dfrac{\mathcal{O}_{X,x}}{\mathfrak{m}_x} = \dfrac{\frac{\mathcal{O}_{n,x}}{\mathcal{I}_x}}{\mathfrak{m}_x} = \dfrac{\mathcal{O}_{n,x}}{\mathcal{M}_x} = \mathbb{C}$ and T_xX is the kernel of a complex matrix we have only to show that they have the same dimension.

Let $(\mathbb{C}^n)^*$ be the dual space of $\mathbb{C}^n$, that is $(\mathbb{C}^n)^* = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C})$. We have a linear surjective map $L : \mathcal{M}_x \to (\mathbb{C}^n)^*$ defined by $L(f) = \sum_{i=1}^{n} \dfrac{\partial f}{\partial z_i}(x) \cdot u_i$.

L induces an isomorphism

$$L' : \frac{\mathcal{M}_x}{\ker L} = \frac{\mathcal{M}_x}{\mathcal{M}_x^2} \to (\mathbb{C}^n)^*.$$

Let now $f_1, \dots, f_l$ be generators of $\mathcal{I}_x$. Then $T_xX = \ker(L(f_1), \dots, L(f_l))$, so we can identify $(T_xX)^*$ with the quotient $\dfrac{(\mathbb{C}^n)^*}{< L(f_1), \dots, L(f_l) >}$ where $< L(f_1), \dots, L(f_l) >$ denotes the subspace of $(\mathbb{C}^n)^*$ generated by $L(f_1), \dots, L(f_l)$.

Now $L^{-1}(< L(f_1), \dots, L(f_l) >)$ contains the ideal $\mathcal{I}_x$ and it is precisely $\mathcal{M}_x^2 + \mathcal{I}_x$. Hence

$$(T_xX)^* = \frac{\mathcal{M}_x}{\mathcal{M}_x^2 + \mathcal{I}_x} = \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2}.$$

□

Now we caracterize regular points in terms of algebraic properties of the ring $\mathcal{O}_{X,x}$.

Definition 5.7. *A local noetherian ring* $(A, \mathfrak{m})$ *is called* regular *if dim* $A =$ *dim* $\frac{\mathfrak{m}}{\mathfrak{m}^2}$

Remember that dim $A = d$ means that for any sequences of strict inclusions of prime ideals in A

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_k \subset A$$

one has $k \leq d$ and $d = \sup k$.

Remark 5.8. *Consider the local noetherian ring* $A = \mathcal{O}_{X,x} = \dfrac{\mathcal{O}_{n,x}}{\mathcal{J}_x}$ *and assume* $\mathcal{J}_x$ *to be a prime ideal. Then we know that up to a linear change of coordinates* A *is an integral extension of the noetherian local ring* $\mathcal{O}_k$*. Hence dim* $A = k$*. Note that* $k = n - r$ *where* r *is the rank of* $\mathcal{J}_y$ *for* $y \in X \cap (\Delta \setminus \mathcal{V}(d))$*. Indeed we know that the germ* X_x *can be realized in a polydisc* Δ *centered at* x *in such a way that an open dense set* U *is a covering of an open set in* $\mathbb{C}^k$*. Also* U *is defined as the zeroset of* $n - k$ *elements of* $\mathcal{J}_x$*, namely* $q_{k+1}, dz_j - T_j(z_{k+1})$ $\;j = k+2, \ldots, n$ *as a manifold of dimension* k*. Hence* $U \subset \mathcal{R}(X)$ *and* $k = n - r$ *since there are not manifolds of bigger dimension inside* X *in a neighbourhood of* x*. So* $k \leq n - r_x = \dim T_x X \leq n$*.*

Finally we are able to get an algebraic description of regular points.

Proposition 5.9. *Let* X *be an analytic set. Then a point* $x \in X$ *is regular if and only if* $\mathcal{O}_{X,x}$ *is a regular ring.*

Proof: x regular implies $r_x = r$ implies dim $T_x X = n - r =$ dim $\mathcal{O}_{X,x}$ implies $\mathcal{O}_{X,x}$ is regular. Conversely $\mathcal{O}_{X,x}$ regular implies dim $\mathcal{O}_{X,x} =$ dim $T_x X = n - r$ implies $r_x = r$ implies x is regular. □

Next we show by an example that complex singularities cannot be topologically smooth.

Exemple 5.10. Consider in $\mathbb{C}^3$ the zero set Z of the complex polynomial $z_3^2 - z_1 z_2$ From the classification of complex quadrics we know that Z is a cone with vertex in $0 = (0,0,0)$. Also we know from the rank theorem that $Z \setminus 0$ is a complex manifold of dimension 2. We want to prove that the germ Z_0 is not even a topological manifold. This show that a complex isolated singularity is singular even from a topological point of view.

Note that this is not true when we deal with real polynomials. For instance take the real polynomial $y^2 - x^3$. The zeroset is a cusp, so it is singular at $(0,0)$ as an algebraic variety, but it is homeomorphic to a line.

Come back to our cone. If it were topologically smooth at the origin, a suitable neighbourhood W of $0 \in Z$ would be topologically a 4 ball with the sphere S^3 as a boundary. Note that S^3 is simply connected and $W \setminus 0$ retracts on S^3. But no neighbourhood of 0 can be simply connected. Indeed there is a map $\varphi : \mathbb{C}^2 \setminus (0,0) \to Z\setminus(0,0,0)$ which is a not trivial double covering map, namely $\varphi(t_1,t_2) = (t_1^2, t_2^2, t_1 t_2)$.

Note that φ maps $\mathbb{C}^2$ on Z and the open ball $|t_1|^2 + |t_2|^2 < \varepsilon$ into an open ball of $\mathbb{C}^3$. Indeed $|t_1^2|^2 + |t_2^2|^2 + |t_1 t_2|^2 = (|t_1|^2 + |t_2|^2)^2 < \varepsilon^2$

So the open ball intersected with Z cannot be simply connected.

A Covering maps

In this appendix we recall shortly the notion of *covering map.*

Definition A.1. *Let X, E be topological spaces locally connected by arcs.*

$p : E \to X$ is a covering map *if is continous, srjective and for each $x \in X$ there is an open neighbourhood U such that $p^{-1}(U)$ a disjoint union of open sets V_i each of which is mapped homeomorphically onto U by p.*

The open set U is called a *trivializing open neighbourhood* of x. Of course if $x \in U' \subset U$, U' is also a trivializing neighbourhood for x.

Exemples A.2.

1. $p : \mathbb{R} \to S^1 \quad p(t) = e^{2\pi i t} = \cos 2\pi t + i \sin 2\pi t$
2. $p : \mathbb{C} \to \mathbb{C}^* = \mathbb{C} \setminus \{0\} \quad p(z) = e^z$
3. $p : \mathbb{C}^* \to \mathbb{C}^* \quad p(z) = z^k \quad k \in \mathbb{N}$
4. $p : S^n \to \mathbb{P}_n(\mathbb{R}) \quad p((x_1, \ldots, x_{n+1})) = [x_1, \ldots, x_{n+1}]$

Let $p : E \to X$ be a covering map. If X is a complex manifold, $E \subset \mathbb{C}^n$ and p is the restriction of a holomorphic map then also E is a complex manifold of the same dimension as X.

When both E and X are connected is rather easy to prove that the fundamental group $\pi_1(E, e_0)$ embeds as a subgroup into $\pi_1(X, p(e_0))$ and its index equals the cardinality of the fiber. In particular when X is simply connected it cannot get non trivial coverings.