VASSILIEV COMPLEX FOR CONTACT CLASSES OF REAL SMOOTH MAP-GERMS

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ABSTRACT. In this note, we construct the Vassiliev complex for contact singularity classes of real smooth map-germs, and then we discuss on the Thom polynomial theorem which describes relationships between the cohomology group of our complex and characteristic classes associated to contact singularities of smooth mappings.

1. INTRODUCTION

Let N and P be two smooth manifolds of dimension n and p respectively, and $\Sigma \subset J^k(n,p)$ a singularity type (a locally closed submanifold which is invariant under $\mathcal{A}$ (right-left)-equivalence). For a smooth mapping $f : N \to P$, consider the subset $\Sigma(f)$ of N consisting of points at which f is of type Σ. If f is appropriately generic and Σ satisfies a certain good condition, a cohomology class of N dual to $\overline{\Sigma(f)}$ is well defined, and the class constitutes a homotopy invariant of f. In particular, if the dual class does not vanish, any generic map homotopic to f has singularities of type Σ. Thus such dual classes are considered as topological obstructions to the existence of singularities of corresponding types. Furthermore, these classes can be expressed as polynomials of standard characteristic classes of bundles TN and f^*TP, that are usually called *Thom polynomials* (for the detail, see §3). As the condition on Σ, we claim that its topological closure $\overline{\Sigma}$ carries a fundamental class: there exists a unique class of the closed supported homology group $H_m(\overline{\Sigma};\mathbb{Z}_2)$ ($m = \dim \Sigma$) such that for any point $x \in \Sigma$ the image of the class generates $H_m(\overline{\Sigma}, \overline{\Sigma}-\{x\})$. In general, $\overline{\Sigma}$ are semialgebraic sets and contains singular loci with dimension less by one, so the condition is not always satisfied. Hence it is a problem in local geometry of real singularities to determine which kinds of singularity types satisfy the condition and admit Thom polynomial expressions.

In this note, we will treat with this problem in a formal frame work, according to a method introduced by V. A. Vassiliev in [17]. In the case of function-singularities, Vassiliev constructed an abstract cochain complex which represents the combinatorics of adjacency relations between various singularity classes, see [17], [2]. We will carry out a similar construction for contact singularity classes of smooth map-germs.

Consider a stratification of $J^k(n,p)$ whose strata are invariant under contact equivalence. Roughly speaking, associated to the stratification we can define a cochain complex as follows: cochains of the complex are formal sums of strata of the stratification and are graded by the codimension of the coresponding strata; the value of the coboundary operator evaluated on a generator X is given by the formal sum of strata X_i to which X is adjacent (i.e. $X_i \subset \overline{X} - X$) with suitable coefficients. *The universal Vassiliev complex* is defined as a inductive limit of such complexes in a certain sense, see §1, §2. Then for each cocycle of our complex, the closure of its support in $J^k(n,p)$ carries a fundamental class (see Remark (1.6)). Hence, for any generic map $f : N \to P$ we can define a cohomology class of N dual to the singularity set of f corresponding to each cocycle (Lemma (3.2)). In particular, we can know from our complex the *coexistence* of singularities of smooth mappings : the singularity set of generic f corresponding to each *coboundary* cocycle of our complex is always homologous to zero, in other words, its Thom polynomial is trivial.

In the final section §4, we will give some concrete results on some computations of cohomology groups of our Vassiliev complex. This is based on author's master thesis of Tokyo Inst. Tech. in 1990 [10].

Throughout this note, manifolds and maps are assumed to be of class C^∞, and we will consider only coefficients in $\mathbb{Z}_2$ for the simplicity. As usual, we let $\mathcal{K}^k_{n,p}$ (or simply $\mathcal{K}^k$) denote the Lie group of k-jets of contact equivalence acting on $J^k(n,p)$.

2. $\mathcal{K}^k$-classification of $J^k(n,p)$ and Vassiliev complex

This section is devoted to introduce abstract cochain complexes associated to stratifications of $J^k(n,p)$, according to [16].

Definition 2.1. *Let γ be a stratification of $J^k(n,p)$ such that each stratum of γ is a semialgebraic set. γ is said to be a $\mathcal{K}^k$-classification of $J^k(n,p)$ if γ satisfies the following properties.*

1. *Each stratum of γ is $\mathcal{K}^k$-invariant.*
2. *if a stratum of γ has connected components L_1 and L_2, then there are two points $z_i \in L_i (i = 1, 2)$ such that z_1 is $\mathcal{K}^k$-equivalent to z_2.*
3. *γ satisfies the Whitney b-regularity condition.*

Let γ be a $\mathcal{K}^k$-classification of $J^k(n,p)$. Then, it is straightforward from the definition to see the following properties.

Lemma 2.2. *It holds that*

1. *γ is a finite set.*
2. *γ satisfies the frontier condition, i.e., if $X, Y \in \gamma$ and $\overline{X} \cap Y \neq \emptyset$, then $Y \subset \overline{X}$.*
3. *If $X, Y \in \gamma$ and $\overline{X} \cap Y \neq \emptyset$, then X is locally topologically trivial along Y.*

Proof: (1) It follows from the locally finiteness of γ and that the closure of each stratum of γ contains the k-jet of constant map-germ 0. (2) Since we know the fact that all connected components of a stratification with Whitney b-regularity condition form a Whitney stratification satisfying the frontier condition (see [3], p.61, (5.6) and (5.7)), the assertion of (2) follows from the definition 2.1. (3) From (3) of Definition 2.1, we can see by using Thom's isotopy lemma (cf. [3]) that X is locally topologically trivial along each connected component of Y. On the other hand, for any two connected components L_1 and L_2 of Y, it follows from (2) of (1.1) that there are an element $H \in \mathcal{K}^k$ sending a point of L_1 to a point of L_2. Since H induces a local diffeomorphism which preserves X and Y, X is locally topologically trivial of X along $L_1 \cup L_2$. Thus we have the assertion (3). □

Proposition 2.3. *Let η be a locally finite partition of $J^k(n,p)$ into semialgebraic $\mathcal{K}^k$-invariant subsets. Then, there is a $\mathcal{K}^k$-classification of $J^k(n,p)$, any stratum of which belongs to some element of η.*

In fact, for any locally finite partition into semialgebraic subsets, there is a canonical Whitney stratification which refines the partition (cf. Gibson et al. [3]). This is proved by using set theoretical operations on semialgebraic sets: Boolean operations, taking the topological closure, partition into families of connected components, and removing the singular locus and bad point sets. These operations can be used also in our equivariant situation.
(1.4). Assume that we are given a $\mathcal{K}^k$-classification γ of $J^k(n,p)$. Associated with the classification γ we introduce a cochain complex as follows. First, we set in a formal manner

$$C^s(\gamma) := \mathbb{Z}_2\text{-module generated by elements of } \gamma \text{ with codim } = s \quad (s \geq 0),$$
$$C(\gamma) := \oplus_{s\geq 0} C^s(\gamma), \text{ i.e., } \mathbb{Z}_2\text{-module generated by all elements of } \gamma.$$

Second, let us define the boundary operator $\delta_\gamma : C^s(\gamma) \to C^{s+1}(\gamma)$ as follows. Let X be a strata of γ with codim s. From the frontier condition of γ ((2) fo Lemma (1.2)), there is a filtration $\{V_i\}_{i\geq 0}$ of the topological closure $\overline{X}$ where V_i is the union of the strata included in $\overline{X}$ with codimension $\geq s+i$ (here $V_0 = \overline{X}$). Set $m = \dim J^k(n,p)$, and let

$$\partial : H_{m-s}(V_0, V_1; \mathbb{Z}_2) \to H_{m-s-1}(V_1, V_2; \mathbb{Z}_2)$$

denote the connection homomorphism of relative homology groups with closed supports. Let μ_X denote the fundamental class of $H_{m-s}(V_0, V_1)$, i.e., for any point $x \in V_0 - V_1$, the image of μ_X generates $H_{m-s-1}(V_0, V_0 - x)$. Choose any stratum Y of γ contained in $V_1 - V_2$ and any point $y \in Y$. Then we define $[X;Y]$ by the value of $j_* \circ \partial(\mu_X)$ where $j_* : H_{m-s-1}(V_1, V_2; \mathbb{Z}_2) \to H_{m-s-1}(V_1, V_1 - y; \mathbb{Z}_2) \simeq \mathbb{Z}_2$. Note that the value $[X;Y]$ dose not depend on the choice of y, since X is topologically trivial along Y ((3) of Lemma 2.2). For any $Y \in \gamma$ such that $Y \not\subset V_1 - V_2$, we set $[X;Y] := 0$. Now we define $\delta_\gamma(X) := \sum_{Y\in\gamma}[X;Y]Y \in C^{s+1}(\gamma)$.

Lemma 2.4. $\delta_\gamma \circ \delta_\gamma = 0$.

Proof: It suffices to see the value on the above $X \in \gamma$. Let V_i be the filtration as above. By definition, $[X, Y]$ is the coefficient of $\partial\mu_X$ on the component of Y in $H_{m-s-1}(V_1, V_2)$. Considering the exact sequence

$$H_{m-s}(V_0, V_1) \xrightarrow{\partial} H_{m-s-1}(V_1, V_2) \xrightarrow{\partial} H_{m-s-2}(V_2, V_3),$$

it is easy to see that $\delta_\gamma \circ \delta_\gamma(X) = 0$. □

We will call the complex $(C(\gamma), \delta_\gamma)$ *the Vassiliev complex for* γ .

Remark 2.5. *Let $c = \sum X_i$ be a cochain of $C^s(\gamma)$ and Σ_c the union of X_i. Then the topological closure $\overline{\Sigma_c}$ is a Whitney stratified closed subset of $J^k(n,p)$ which is invariant under the $\mathcal{K}^k$ action. Note that $H_{m-s}(\overline{\Sigma_c}, \overline{\Sigma_c} - \Sigma_c) \simeq \oplus H_{m-s}(\overline{X_i}, \overline{X_i} - X_i)$. If c is cocycle, i.e. $\delta_\gamma(c) = 0$, then $\sum \partial\mu_{X_i} = 0$ and hence there is a unique lift of the class $\sum \mu_{X_i}$ via the exactness of the following*

$$H_{m-s}(\overline{\Sigma_c}) \to H_{m-s}(\overline{\Sigma_c}, \overline{\Sigma_c} - \Sigma_c) \xrightarrow{\partial} H_{m-s-1}(\overline{\Sigma_c} - \Sigma_c).$$

The lift is the fundamental class of $H_{m-s}(\overline{\Sigma_c})$. According to the terminology of R. M. Goresky [5], *$\overline{\Sigma_c}$ is a Whitney stratified $(m-s)$-cycle in $J^k(n,p)$.*

(1.7). Let Γ denote the set of $\mathcal{K}^k$-classifications of $J^k(n,p)$. For γ, γ' in Γ, we define $\gamma \prec \gamma'$ if any stratum of γ' is contained in some strata of γ. For $\gamma, \gamma' \in \Gamma$, set $\gamma \cap \gamma' = \{X \cap X' | X \in \gamma, X' \in \gamma'\}$, which is a locally finite partition of $J^k(n,p)$ whose elements are $\mathcal{K}^k$ invariant semialgebraic sets, and hence through the procedure in (1.2) we can obtain a $\mathcal{K}^k$-classification γ'' such that $\gamma \prec \gamma''$ and $\gamma' \prec \gamma''$. Thus $(\Gamma, \prec)$ is a directed set.

If $\gamma \prec \gamma'$, then there is a natural homomorphism $(\rho^\gamma_{\gamma'}) : C(\gamma) \to C(\gamma')$ defined by assigning $X \in \gamma$ to the linear combination $\sum X_i$ of all $X_i \in \gamma'$ with $X_i \subset X$. It is easy to see that $(\rho^\gamma_{\gamma'})$ commutes with δ_γ and $\delta_{\gamma'}$, and hence $(\{C(\gamma)\}, \{(\rho^\gamma_{\gamma'})\})_{\gamma\in\Gamma}$ form an inductive system of cochain complices.

Definition 2.6. *A cochain complex $C(\mathcal{K}^k_{n,p})$ is defined by $\varinjlim C(\gamma)$.*

The map $(\rho^\gamma_{\gamma'})$ induces a homomorphism $H^*(C(\gamma); \mathbb{Z}_2) \to H^*(C(\gamma'); \mathbb{Z}_2)$, and it is easy to see $H^*(C(\mathcal{K}^k_{n,p}); \mathbb{Z}_2) \simeq \lim_{\to} H^*(C(\gamma); \mathbb{Z}_2)$.

3. The universal Vassiliev complex

The complex $C(\mathcal{K}^k_{n,p})$ defined in the previous section depends on positive integers n, p and k. In this section, we are going to construct an "universal" cochain complex depending only on an integer l, as the inductive limit of $\{C(\mathcal{K}^k_{n,p})\}_{l=p-n}$. In what follows in this section, we fix an integer l , and a positive integer n is always assumed $n + l > 0$.

$J^k(n, n+l)$ is simply denoted by J^k_n. For integers m, n such that $m \geq n$, let id_{m-n} be the identity-germ of $\mathbb{R}^{m-n}$ at the origin, and $i^n_m : J^k_n \to J^k_m$ a natural inclusion defined by $j^k f \mapsto j^k(f \times id_{m-n})$. For each $z = j^k f \in J^k_n$, set $\mathrm{corank}(z) := \min(n, n+l) - \mathrm{rank}\, df$. For a subset X of J^k_n, we also set

$\mathrm{corank}(X) := \min\{\mathrm{corank}\,(z), z \in X\}$, and we define a subset $X(m)$ of J^k_m to be $\mathcal{K}^k(i^n_m(X))$ ($= \{Hi^n_m(z) \in J^k_m | z \in X, H \in \mathcal{K}^k_{m,m+l}\}$).

Lemma 3.1. *The following properties hold:*

1. *The map i^n_m is transverse to every $\mathcal{K}^k$-orbit in J^k_m.*
2. *Let X be a semialgebraic smooth submanifold of J^k_n invariant under the $\mathcal{K}^k$-action, then $X(m)$ is a $\mathcal{K}^k$-invariant semialgebraic submanifold of J^k_m such that* $\mathrm{codim}X(m) = \mathrm{codim}X$ *and* $\mathrm{corank}X(m) = \mathrm{corank}X$.
3. *Let Y be a subset of J^k_m invariant under the $\mathcal{K}^k$- action. Then, $(i^n_m)^{-1}Y = \emptyset$ if and only if* $\mathrm{corank}(Y)$ *is greater than* $\min(n, n+l)$.
4. *Let Y be a semialgebraic subset of J^k_m. If* $\mathrm{corank}(Y)$ *is greater than* $\min(n, n+l)$, *then* $\mathrm{codim}Y \geq (n+1)(n+l+1)$.

Proof: (1): As usual, let $\theta(f)$ denote the $\mathcal{E}_m$-module of C^∞ vector field-germs along map-germs $f : \mathbb{R}^m, 0 \to \mathbb{R}^{m+l}, 0$. Let $z = j^k f$ in J^k_m. The tangent spaces at z of the jet space J^k_m and the $\mathcal{K}^k$-orbit of z are written as

$$\begin{aligned} T_z J^k_m &= \mathrm{m}_m\theta(f)/\mathrm{m}_m^{k+1}\theta(f), \\ T_z\mathcal{K}^k z &= \{tf(\mathrm{m}_m\theta(id_m) + f^*(\mathrm{m}_{m+l})\theta(f)\}/\mathrm{m}_m^{k+1}\theta(f) \end{aligned}$$

where $tf : \theta(id_m) \to \theta(f)$ is defined by the differential of f $tf(v) := Tf \circ v$ (see e.g. [3], [8]).

Now assume that f is written as $g \times id_{m-n}$ for some $g : \mathbb{R}^n, 0 \to \mathbb{R}^{n+l}, 0$. Set $w = j^k g \in J^k_n$. We can naturally identify $\theta(f)$ with the direct sum $\theta(g\circ p_1)\oplus\theta(p_2)$ where p_1 and p_2 denote the projections from $\mathbb{R}^m = \mathbb{R}^n\times\mathbb{R}^{m-n}$ to the first and second factors respectively. It is easy to see that the subspace

$$(i^n_m)_* T_w J^k_n = \mathrm{m}_m\theta(g \circ p_1)/\{p_2^*(\mathrm{m}_{m-n}) + \mathrm{m}_n^{k+1}\}\theta(g \circ p_1).$$

Furthermore, $p_2^*(\mathrm{m}_{m-n})\theta(g\circ p_1) \subset f^*(\mathrm{m}_{m+l})\theta(f)$ and $\mathrm{m}_m\theta(p_2) \subset tf(\mathrm{m}_m\theta(id_m))$. It hence follows that $T_z J^k_m = (i^n_m)_* T_w J^k_n + T_z\mathcal{K}^k z$.
(2): Let τ denote the map $\mathcal{K}^k_{m,m+l} \times J^k_n \to J^k_m$ defined by $\tau(H, z) = H.i^n_m(z)$, and then $X(m)$ is the image of τ. Since τ is an algebraic map, $X(m)$ is semialgebraic by using Tarski-Seidenberg Theorem. It follows from (1) that the map τ is submersive, hence $X(m)$ is a smooth submanifold with the same codim and corank as X.
(3) and (4): Let Σ^{n+1} be the set of J^k_m consisting of jets of kernel rank $n+1$. Note that $\mathrm{corank}\Sigma^{n+1} > \min(n, n+l)$ and $\mathrm{codim}\Sigma^{n+1} = (n+1)(n+l+1)$, and if Y is $\mathcal{K}^k$-invariant, then $(i^n_m)^{-1}Y = \emptyset \Leftrightarrow Y \subset \overline{\Sigma^{n+1}}$. These yield (3) and (4). □

(2.2). Let γ be a $\mathcal{K}^k$-classification of J^k_n and let $m \geq n$. We will construct a $\mathcal{K}^k$-classification of J^k_m induced from γ via i^n_m as follows. Set $A := \{z \in J^k_m | \mathrm{corank}(z) \leq \min(n, n+l)\}$. Then $J^k_m = A \cup \overline{\Sigma^{n+1}}$(disjoint). From (2), (3) in Lemma 3.1, it follows that J^k_m has a $\mathcal{K}^k$-invariant semialgebraic partition consisting of $\overline{\Sigma^{n+1}}$ and all $X(m)$ for $X \in \gamma$. Using Lemma (1.3), we obtain a

$\mathcal{K}^k$-classification $(i^n_m)_*\gamma$ subordinate the semialgebraic partition. Note that in this process, we need only to decompose the subset $\overline{\Sigma^{n+1}}$ by set theoretical operations, since all $X(m)$ form a $\mathcal{K}^k$-classification of the set A. Since the codimension of X is the same as of $X(m)$, we can define a cochain map $C^s(\gamma) \to C^s((i^n_m)_*\gamma)$ by $X \mapsto X(m)$. When we take the inductive limit of such cochain maps over all $\mathcal{K}^k$-classifications γ of J^k_n, we obtain a cochain map $(i^n_m)_\sharp : C(\mathcal{K}^k_{n,n+l}) \to C(\mathcal{K}^k_{m,m+l})$.
(2.3). Let γ^k_n be a $\mathcal{K}^k$-classification of J^k_n, and $\pi^r_k : J^r_n \to J^k_n (k \leq r)$ the natural projection. Then, J^r_n has a $\mathcal{K}^r$-classification which consists of all $(\pi^r_k)^{-1}X$ where $X \in \gamma^k_n$, which is denoted by $(\pi^r_k)^*\gamma^k_n$. A cochain map $C(\gamma^k_n) \to C((\pi^r_k)^*\gamma^k_n)$ is defined by $X \mapsto (\pi^r_k)^{-1}X$, and hence we obtain $(\pi^r_k)^\sharp : C(\mathcal{K}^k_{n,n+l}) \to C(\mathcal{K}^r_{n,n+l})$.

Lemma 3.2. *$(\pi^r_k)^\sharp$ commutes with $(i^n_m)_\sharp$.*

This can be easily verified from the constructions in (2.2) and (2.3).

Definition 3.3. *For an integer l, a cochain complex $C(\mathcal{K}(l))$ is defined by the inductive limit of $C(\mathcal{K}^k_{n,n+l})$ tending $n, k \to \infty$, which is called the universal Vassiliev complex for contact classes with difference dimension l.*

Proposition 3.4. *For an arbitrary integer $t > 0$, there are two integers $k = k(t), n = n(t)$ such that the natural homomorphism $H^s(C(\mathcal{K}^k_{n,n+l})) \to H^s(C(\mathcal{K}(l)))$ is isomorphic for $0 \leq s \leq t$.*

Proof: The proof is divided by two steps. We first claim that
(i) For any integers t and k, there exists an integer n such that for $\forall m > n$, $\forall s \leq t$, $(i^n_m)_\sharp : C^s(\mathcal{K}^k_{n,n+l}) \to C^s(\mathcal{K}^k_{m,m+l})$ is a cochain isomorphism.
We take an integer n satisfying $(n+1)(n+l+1) > t$, and in what follows we write i^n_m by i simply. Now let γ be a $\mathcal{K}^k$-classification of J^k_m. Since i is transverse to each element of γ, the pull buck of γ via i becomes a $\mathcal{K}^k$-classification of J^k_n, which we will write by $i^*\gamma$. If X is a strata of γ with codim $= s$ less than t, then by using Lemma (2.1) corank$X \leq \min(n, n+l)$, and $X' := i^{-1}X \neq \emptyset$. Then $X'(m)$ coincides with X off the semialgebraic proper subset $X \cap \overline{\Sigma^{n+1}}$. Thus we have $C^s(i^*\gamma) \simeq C^s(i_*i^*\gamma) \simeq C^s(\gamma)$ for $s < t$. Throughout formal arguments, we have (i).
Next we claim that
(ii) For any integers t and n, there exists an integer k such that for $\forall r > k, \forall s \leq t$ $(\pi^r_k)_\sharp : C^s(\mathcal{K}^k_{n,n+l}) \to C^s(\mathcal{K}^r_{n,n+l})$ is a cochain isomorphism.
Set W^k_n to be the set of all k-jets $j^kf \in J^k_n$ such that f is not $k - \mathcal{K}$-determined. Namely, for any r-jet z ($r > k$) such that $\pi(z)$ is not in W^k_n, (here π denotes π^r_k), it holds that $\pi^{-1}\pi(z) \subset \mathcal{K}^r z$. It is known (e.g., [3]) that W^k_n is a semialgebraic set and codimW^k_n tends to ∞ where $k \to \infty$. Thus, for given t we take an integer k to satisfy codimW^k_n greater than t. Let γ be a $\mathcal{K}^r$-classification of J^r_n which refines the partition consisting of two elements $\pi^{-1}W^k_n$ and $J^r_n - \pi^{-1}W^k_n$. Then for each strata X of γ with

codimension less than t, it holds that $\pi^{-1}\pi X = X$, hence πX becomes smooth and $\mathcal{K}^k$-invariant. Let γ' be a $\mathcal{K}^k$-classification of J^k_n subordinate to the semialgebraic partition consisting of all πX and W^k_n, then it holds that $C^s(\gamma') \simeq C^s(\pi^*\gamma') \simeq C^s(\gamma)$ for $s < t$. Taking the inductive limits, (ii) follows. This completes the proof. □

4. Thom polynomials

In this section, we shall describe relations between the abstract Vassiliev complex constructed in the previous sections and Thom polynomials of contact singularities.

(3.1). For any cocyle $c \in C^s(\mathcal{K}(l))$, i.e., $\delta_{\mathcal{K}} c = 0$, we take integers n and k which satisfy Proposition 3.4, and then there is some $\mathcal{K}^k$-classification γ of $J^k(n, n+l)$ and $\{X_i\} \subset \gamma$ whose linear combination represents c. Set Σ_c to be the union of X_i. Given smooth manifolds N and P of dimension n and $n+l$ respectively, we have the subbundle $\Sigma_c(N,P)$ of $J^k(N,P)$ with fibre Σ_c. Since $\delta_{\mathcal{K}} c = 0$, we can see that $\overline{\Sigma_c(N,P)}$ is a Whitney stratified cycle in $J^k(N,P)$ as well $\overline{\Sigma_c}$ in $J^k(n, n+l)$, see Remark 2.5.

Lemma 4.1. cf., [6], [5]. *If the extension $j^k f$ is transverse to $\overline{\Sigma_c(N,P)}$, then $\overline{\Sigma_c(f)}$ also becomes a Whitney stratified cycle in N and $Dual[\overline{\Sigma_c(f)}] = (j^k f)^* Dual[\overline{\Sigma_c(N,P)}] \in H^s(N; \mathbb{Z}_2)$ (here Dual means the Poincaré dual).*

These classes constitute homotopy invariants of f. Furthermore these classes are universally represented by standard topological invariants of N, P and f, which we will explain bellow.

(3.3). Let $G_n(\mathbb{R}^{n+q})$ be the Grassmanian of n-dimensional subspaces in $\mathbb{R}^{n+q}$, and $Bo(n)$ the classifying space of real n-bundles ($= \lim_{q\to\infty} G_n(\mathbb{R}^{n+q})$). Recall that there is the principal $O(n)$-bundle (the Stiefel bundle) over $G_n(\mathbb{R}^{n+q})$ (cf., [[9]). The Lie group $L^k(n)$ of invertible k-jets in $J^k(n,n)$ is isomorphic to the product of $O(n)$ and some affine spaces, hence considering the action of $L^k(n)$ on $J^k(n, n+l)$ (the action of right-equivalence), we can define the associated bundle $p : H_n \to G_n(\mathbb{R}^{n+q})$ with fibre $J^k(n, n+l)$. Note that H_n is homotopy equivalent to the space $G_n(\mathbb{R}^{n+q})$, because the fibre is contractible. Each class c of $H^*(C(\mathcal{K}^k_{n,n+l}))$ determines a stratified cycle $\overline{\Sigma_c(H_n)}$ in H_n in a similar way as above (3.1), and then taking $q \to \infty$, we can define a degree 0 map $P_n : H^*(C(\mathcal{K}^k_{n,n+l})) \to H^*(Bo(n); \mathbb{Z}_2)$ by asigning c to $\mathrm{Dual}\, p_*[\overline{\Sigma_c(H_n)}]$.

Recall that for $m > n$, there are natural inclusions $\phi^n_m : G_n(\mathbb{R}^{n+q}) \to G_m(\mathbb{R}^{m+q})$ and $i^n_m : J^k(n, n+l) \to J^k(m, m+l)$ (see §2). They induce a inclusion $\bar{\phi}^n_m : H_n \to H_m$:

$$\begin{array}{ccc} H_n & \stackrel{\phi}{\to} & H_m \\ p \downarrow & & \downarrow p \\ G_n(\mathbb{R}^{n+q}) & \stackrel{\phi}{\to} & G_m(\mathbb{R}^{m+q}) \end{array}$$

Assume that m, n and s are as in Proposition 3.4. Then $\bar{\phi}^n_m$ is transverse to the stratified set $\overline{\Sigma_c(H_m)}$, and $\overline{\Sigma_c(H_n)} = (\bar{\phi}^n_m)^{-1}\overline{\Sigma_c(H_m)}$ (see (1) of Lemma 3.1 and (2.2)), hence by the similar arguments of Lemma 4.1 we have

$$\mathrm{Dual}\,[\overline{\Sigma_c(H_n)}] = (\bar{\phi}^n_m)^*\mathrm{Dual}\,[\overline{\Sigma_c(H_m)}].$$

Since p induces isomorphisms of (co)homologies, it follows that $P_n(c) = (\phi^n_m)^* P_m(c)$. Consequently, taking $n, k \to \infty$, we have a graded group homomorphism of degree 0

$$P : H^*(C(\mathcal{K}(l))) \to H^*(\mathbf{Bo}; \mathbb{Z}_2).$$

Here $\mathbf{Bo}$ is the inductive limit of $Bo(n)$, and we note that for any $s > 0$, $H^s(\mathbf{Bo}; \mathbb{Z}_2)$ is generated by homogeneous polynomials of $w_1, \cdots, w_s$ (cf. [[9]]).

Definition 4.2. cf. [15], [12]. *For $[c] \in H^s(C(\mathcal{K}(l)))$, we will call the image P_c of the above homomorphism the Thom polynomial for singularity type c. (In particular, for each coboundary $c \in C(\mathcal{K}(l))$, the Thom polynomial for c is zero.)*

More generally, consider the direct sum of $H^*(C(\mathcal{K}(l)))$ over all l, and let $\mathcal{H}^*$ be the reduced group defined by $\mathcal{H}^0 := H^0(C(\mathcal{K}(0)))$ and $\mathcal{H}^s := \oplus_{l\in\mathbb{Z}} H^s(C(\mathcal{K}(l)))$ for $s > 0$. Then we extend P as a homomorphism $\mathcal{H}^* \to H^*(\mathbf{Bo}; \mathbb{Z}_2)$.

Theorem 4.3. (The universal Thom polynomial theorem, e.g., [6], [15]). *Let $[c] \in H^s(C(\mathcal{K}(l)))$. For any n and k satisfying Proposition 3.4, and for any smooth map $f : N^n \to P^{n+l}$ satisfying the transversality as just described in (3.1), the Poincaré dual to $[\overline{\Sigma_c(f)}]$ is expressed by the polynomial P_c replaced generators w_i by the Stiefel-Whitney classes of the difference bundle $TN - f^*TP$:*

$$Dual\,[\overline{\Sigma_c(f)}] = P_c(w_i(TN - f^*TP)) \in H^s(N; \mathbb{Z}_2).$$

Proof: First, by using Whitney's immersion theorem (e.g., see [4]), we choose a immersion $P^{n+l} \to \mathbb{R}^{m+l}$ for some large m. Let us consider the orthonormal bundle ν of the immersion and its pull buck bundle $f^*\nu$ via f. The total space of $f^*\nu$ is denoted by M, and we let $i_N : N \to M$ be the natural inclusion to the zero section. Then M becomes a smooth m-manifold and there is a smooth map F from M to $\mathbb{R}^{m+l}$ given by the composition of f and the exponential map associated to the normal vectors. Then for each point $x \in N$, the germ of F at $i(x)$ is written as suspension of the germ of f at x (here the fibre ν_x is the suspension parameter space), hence $\overline{\Sigma_c(f)}$ is the transeverse intersection $N \cap \overline{\Sigma_c(F)}$. In particular, we may assume that over M, j^kF is transverse to the Whitney stratified cycle $\overline{\Sigma_c(M, \mathbb{R}^{m+l})}$ (otherwise, we take a sufficiently small neighborhood of $i_N(N)$ instead of M). Consider the classifying map of the m-bundle TM into $G_n(\mathbb{R}^{n+q})$ for sufficiently large q and the following diagram:

$$\begin{array}{ccc} J^k(M,\mathbb{R}^{m+l}) & \xrightarrow{i} & H_m \\ j^kF \uparrow & & \downarrow p \\ M & \xrightarrow{\rho} & G_m(\mathbb{R}^{m+q}) \end{array}$$

Then it follows that

$$\begin{aligned} \mathrm{Dual}\,[\overline{\Sigma_c(F)}] &= (j^kF)^*\mathrm{Dual}\,[\overline{\Sigma_c(M,\mathbb{R}^{m+l})}] \\ &= (j^kF)^* i^*\mathrm{Dual}\,[\overline{\Sigma_c(H_m)}] = \rho^* P_c(w_i) = P_c(w_i(TM)). \end{aligned}$$

Since $i_N^*TM = TN \oplus f^*\nu$ and $\nu \oplus TP = T\mathbb{R}^{m+l}$, we have

$$\begin{aligned} \mathrm{Dual}\,[\overline{\Sigma_c(f)}] &= i_N^*\mathrm{Dual}\,[\overline{\Sigma_c(F)}] \\ &= P_c(i_N^* w_i(TM)) = P_c(w_i(TN \oplus f^*\nu)) = P_c(w_i(TN - f^*TP)). \end{aligned}$$

This completes the proof. □

Definition 4.4. *We shall consider $\mathcal{H}^*$ as the set of all definable Thom polynomials. The author expects that $\mathcal{H}^*$ would admit some hidden algebraic structures, as V.I.Arnol'd mentioned in* [1], *§5.2. Also see* [10].

5. Calculation of $C(\mathcal{K}(0))$

In this section, we consider the case of $l = 0$ (this is the equidimensional case), and we give the initial part of $H^*(C(\mathcal{K}(0)))$ without the detail.

From Mather [8], we have the following proposition.

Proposition 5.1. *Let k be sufficient large ($k \geq 9$) and $n \geq 2$. Then there exists $\mathcal{K}^k$-invariant semialgebraic subset Δ_n^k of $J^k(n,n)$ which satisfies the following properties:*

1. *codim$\Delta_n^k = 9$*
2. *$J^k(n,n) - \Delta_n^k$ contains finitely many $\mathcal{K}^k$-orbits with the associated algebras $Q_k \simeq \mathcal{E}_{x,y}/I + \mathrm{m}^{k+1}$ listed in Table 1 below.*

Table 1

$\mathcal{K}$-class	Ideal I	Restriction	TB-symbol	codim
A_q	$< x^{q+1}, y >$	$0 \leq q \leq 8$	Σ^1	q
$I_{a,b}$	$< x^a + y^b, xy >$	$2 \leq a \leq b, \quad a+b \leq 8$	$\Sigma^{2,0}$	$a+b$
$II_{a,b}$	$< x^a - y^b, xy >$	$a \leq b, \quad a+b \leq 8; \quad a,b:$ even		$a+b$
IV_3	$< x^2 + y^2, x^3 >$	$-$		6
I_7	$< x^2, y^3 >$	$-$	$\Sigma^{2,1}$	7
I_8	$< x^2 + y^3, xy^2 >$	$-$		8

Thus we have a partition η of $J^k(2,2)$ where elements are Δ_2^k and $\mathcal{K}^k$-orbits in $J^k(2,2)$ listed above. Let γ_2^k be the $\mathcal{K}^k$-classification obtained from

η by (1.3) and (2.6) we see

$$C^s(\gamma_2^k) \simeq C^s(\mathcal{K}_{n,n}^k) \simeq C^s(\mathcal{K}(0)) \quad \text{for} \quad s \leq 8, n \geq 2.$$

In the following theorem, we determine the value of the differential δ on generators of $C^s(\mathcal{K}(0))$ $(s \leq 8)$.

Theorem 5.2. *The differential operator of* $(C^s(\mathcal{K}(0)), \delta)$ *for* $s \leq 8$ *are described by the following formulae:*

$$\begin{aligned}
&(1) \quad \delta A_s = 0 \quad (0 \leq s \leq 8), \quad (2)\delta I_{2,2} = \delta II_{2,2} = I_{2,3},\\
&(3)\delta I_{2,3} = 0, \quad (4)\delta I_{2,4} = \delta II_{2,4} = I_{2,5} + I_{3,4},\\
&(5)\delta I_{3,3} = \delta IV_3 = I_7, (6)\delta I_{2,5} = \delta I_{3,4} = 0,\\
&(7)\delta I_7 = 0.
\end{aligned}$$

This result is obtained from direct computations of $[X;Y]$ using normal forms calculus (see Ohmoto [11] and Lander [7]).

Corollary 5.3. *Cohomology groups* $H^s(C(\mathcal{K}(0)); \mathbb{Z}_2)(s \leq 7)$ *are given in Table 2 bellow. In particular, coboundaries of* $C^s(\mathcal{K}(0))$ *for* $s \leq 8$ *are* $I_{2,3}(s = 5)$*,* $I_{2,5} + I_{3,4}(s = 7)$ *and* $I_7(s = 7)$*.*

Table 2

\ dim	1	2	3	4	5	6	7
$H^s(C(\mathcal{K}(0)))$	$\mathbf{Z_2}$	$\mathbf{Z_2}$	$\mathbf{Z_2}$	$(\mathbf{Z_2})^2$	$\mathbf{Z_2}$	$(\mathbf{Z_2})^3$	$(\mathbf{Z_2})^2$
generators	A_1	A_2	A_3	A_4	A_5	A_6	A_7
				$I_{2,2} + II_{2,2}$		$I_{2,4} + II_{2,4}$	$I_{2,5}(= I_{3,4})$
						$I_{3,3} + IV_3$	
coboundaries					$I_{2,3}$		$I_{2,5} + I_{3,4}$
							I_7

(4.4). If $c \in C(\mathcal{K}(l))$ is a coboundary, then the Thom polynomial $P_{[c]}$ is always trivial. Hence it follows immediately from the Table 2 in Corollary 5.3 that

Proposition 5.4. *In the case of* $n = p$*,*

(1) *the Thom polynomials of type* $I_{2,3}$ *and* I_7 *are trivial.*
(2) *the Thom polynomial of type* $I_{2,5}$ *coincides with one of type* $I_{3,4}$*.*

Remark 5.5. *For generators of* $H^i(C(\mathcal{K}(0)))$ *listed in Table 2, we have not well known concrete forms of corresponding Thom polynomials in the case of dimension* $i \geq 6$*. Let* $\bar{w}_i$ *be the element of* $H^i(\mathbf{Bo}\ , \mathbb{Z}_2)$ *satisfying that*

$$(1 + w_1 + w_2 + \cdots)(1 + \bar{w}_1 + \bar{w}_2 + \cdots) = 1$$

in the ring of formal series $H^{\Pi}(\mathbf{Bo}\ ,\mathbb{Z}_2)$*, see [*[9]*]. Known results are only as follows (*[12], [2], [10], [6], [13]*):*

$$P(A_1) = \bar{w}_1, \quad P(A_2) = \bar{w}_1^2 + \bar{w}_2, \quad P(A_3) = \bar{w}_1^3 + \bar{w}_1 w_2,$$
$$P(A_4) = \bar{w}_1^4 + \bar{w}_1\bar{w}_3,$$
$$P(A_5) = \bar{w}_1^5 + \bar{w}_1^2\bar{w}_3,$$
$$P(I_{2,2} + II_{2,2}) = \bar{w}_2^2 + \bar{w}_1\bar{w}_3,$$
$$P(I_{2,4} + II_{2,4} + I_{3,3} + IV_3) = (\bar{w}_1^2 + \bar{w}_2)(\bar{w}_2^2 + \bar{w}_1\bar{w}_2) + \bar{w}_3^2 + \bar{w}_2\bar{w}_4$$

For Thom polynomials for Boardman singularities Σ^i *and* $\Sigma^{i,j}$*, the readers are referred to* [12], [14].

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