T_y-OPERATOR ON INTEGRALS OVER LATTICE POLYTOPES

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Abstract. The Khovanskii-Pukhlikov formula for lattice polytopes is the combinatorial counterpart to the Hirzebruch-Riemann-Roch formula for a toric variety. In this short note, we introduce the T_y-operator for polytopes analogous to the Hirzebruch's Todd class td_y, which directly recovers the generalized Ehrhart polynomial in a purely combinatorial and very elementary way.

1. Introduction

1.1. **Counting lattice points.** *The Euler-MacLaurin formula for lattice polytopes* has extensively been studied in connection with algebraic geometry of toric varieties [9, 5, 3, 4, 11]. In this short note, we introduce the T_y-operator for polytopes in a purely combinatorial context as an analogy to the Hirzebruch's Todd class td_y [7, 8]. We follow notational conventions in [1, §10].

Let $P \subset \mathbb{R}^d$ be d-dimensional unimodular (i.e. smooth or Delzant) convex lattice polytope containing the origin, and let $\mathcal{F}_j = \mathcal{F}_j(P)$ denote the set of j-dimensional face of P. For each facet $F \in \mathcal{F}_{d-1}$, there exists a unique primitive normal vector $\mathbf{u}_F \in \mathbb{R}^d$ and $a_F \in \mathbb{Z}$ so that

$$P = \{ \ \mathbf{x} \in \mathbb{R}^d \mid \mathbf{u}_F \cdot \mathbf{x} + a_F \geq 0 \ \ (\forall F) \ \}.$$

For a sufficiently small vector of real parameters $h = \{h_F\}_{F \in \mathcal{F}_{d-1}}$ indexed by all facets, we define a perturbed polytope

$$P(h) = \{ \ \mathbf{x} \in \mathbb{R}^d \mid \mathbf{u}_F \cdot \mathbf{x} + a_F + h_F \geq 0 \ \ (\forall F) \ \}.$$

Given a polynomial functions φ over $P(h)$, an asymptotic behavior with respect to h of the integral $\int_{P(h)} \varphi(\mathbf{x}) d\mathbf{x}$ is considered as a higher dimensional analogy to the classical Euler-MacLaurin formula of functions in one variable: In [9] Khovanskii-Pukhlikov introduced the *Todd operator* associated to the polytope P:

$$\mathrm{Todd}(h) := \prod_{F \in \mathcal{F}_{d-1}} \frac{\frac{\partial}{\partial h_F}}{1 - e^{-\frac{\partial}{\partial h_F}}},$$

where the factor is a differential operator of infinite order defined by

$$\frac{\frac{\partial}{\partial x}}{1-e^{-\frac{\partial}{\partial x}}} = 1 + \sum_{k\geq 1}(-1)^k \frac{B_k}{k!}\left(\frac{\partial}{\partial x}\right)^k$$

(B_k is the k-th Bernoulli number). It is then proved that

$$\text{Todd}(h)\left(\int_{P(h)} \varphi(\mathbf{x})d\mathbf{x}\right)\Bigg|_{h=0} = \sum_{\mathbf{m}\in P\cap\mathbb{Z}^d} \varphi(\mathbf{m}). \tag{1}$$

In particular, for the volume function $\text{vol}(P(h)) = \int_{P(h)} 1d\mathbf{x}$, the derivation at $h=0$ gives the number of lattice points $|P\cap\mathbb{Z}^d|$. It is particularly notable that the Khovanskii-Pukhlikov formula (1) is obtained as the combinatorial counterpart to the Hirzebruch-Riemann-Roch formula (HRR) for a smooth toric variety X with a very ample divisor D:

$$\int_X td(TX)ch(\mathcal{O}(D)) = \chi(X,\mathcal{O}(D)),$$

were td and ch are the Todd class and the Chern character of vector bundles, respectively [7].

1.2. **χ_y-genus.** The Euler-Poincaré characteristic $\chi(X,\mathcal{O}(D))$ of the sheaf cohomology is generalized to the so-called *χ_y-genus*,

$$\chi_y(X,\mathcal{O}(D)) = \sum_{p\geq 0}\chi(X,\mathcal{O}(D)\otimes\Omega^p(X))y^p.$$

On the other hand, a modification of the Todd class td with a parameter y (non-normalized version, cf. [8, p.61]) is introduced by

$$td_y(TX) := td(TX)ch(\lambda_y(T^*X)) = \prod \frac{(1+ye^{-\delta_i})\delta_i}{1-e^{-\delta_i}},$$

where δ_i are the Chern roots of TX and λ_y is the lambda operation, so that it unifies several important characteristic classes:

- $(y=0)$: $td_0(TX) = td(TX)$, the Todd class,
- $(y=-1)$: $td_{-1}(TX) = c_n(TX)$, the top Chern class,
- $(y=1)$: $td_1(TX) = L(TX)$, the L-class.

The so-called *generalized Hirzebruch-Riemann-Roch formula* (gHRR) says that when replacing td in the above HRR by td_y, the corresponding degree expresses the χ_y-genus [7, 8] :

$$\int_X td_y(TX)ch(\mathcal{O}(D)) = \chi_y(X,\mathcal{O}(D)).$$

This is indeed a formal extension of HRR, but it contains hidden information of combinatorics unifying characteristic numbers just mentioned above. So it would be natural to ask what is the combinatorial counterpart to the gHRR formula for toric varieties so that it implies the formula (1) when specializing y to 0. That is just our motivation and we will not use these algebro-geometric facts in the rest of this paper.

1.3. **T_y-operator.** We define the T_y-*operator* associated to P by

$$\mathrm{Todd}_y(h) := (1+y)^{d-n} \cdot \prod_{F \in \mathcal{F}_{d-1}} \frac{(1+ye^{-\frac{\partial}{\partial h_F}})\frac{\partial}{\partial h_F}}{1-e^{-\frac{\partial}{\partial h_F}}}$$

where d is the dimension of P and $n = |\mathcal{F}_{d-1}|$ is the number of facets (=number of variables h_F).

By the same combinatorial proof of (1) with an elementary lemma, we have

Theorem 1.1. *Let P be a d-dimensional unimodular lattice polytope in $\mathbb{R}^d$. For any polynomial function $\varphi : P \to \mathbb{R}$, it holds that*

$$\mathrm{Todd}_y(h)\left(\int_{P(h)} \varphi(\mathbf{x})d\mathbf{x}\right)\Bigg|_{h=0} = \sum_{p=0}^{d} y^p \left(\sum_{j=0}^{p}(-1)^j \binom{d-j}{p-j} \sum_{Q\in\mathcal{F}_{d-j}} \sum_{\mathbf{m}\in Q\cap\mathbb{Z}^d} \varphi(\mathbf{m})\right). \tag{2}$$

Remark 1.2. **(Simple polytope)** Not only unimodular polytopes but also *simple* polytopes can be considered. In algebraic geometry side, a unimodular polytope corresponds to a smooth toric variety, and a simple polytope corresponds to a *simplicial* toric variety, which is a possibly singular variety with quotient singularities. Brion-Vergne [4] generalized the Khovanskii-Pukhlikov formula (1) for a simple lattice polytope P by modifying the definition of the operator $\mathrm{Todd}(h)$, as an analogy to their HRR for simplicial toric varieties [3]. Our formula (2) is also valid for simple polytopes after modifying $\mathrm{Todd}_y(h)$ in a similar way to Brion-Vergne's Todd operator. Then the resulting formula corresponds to a result in the recent work of Maxim-Schürmann [11] on the gHRR formula for simplicial toric varieties in algebraic geometry side.

The right hand side of the above formula is rewritten by

$$\sum_{j=0}^{d} \Psi_j(\varphi)(-y)^j(1+y)^{d-j},$$

where $\Psi_j(\varphi) := \sum_{Q\in\mathcal{F}_{d-j}} \sum_{\mathbf{m}\in Q\cap\mathbb{Z}^d} \varphi(\mathbf{m})$. Specializing the parameter y to particular values, we have formulas for a certain weighted sum of lattice points.

Corollary 1.3. *The specialization of our formula at $y = 0$ is just the Khovanskii-Pukhlikov formula for φ. The specialization at $y = -1$ gives $\Psi_d(\varphi)$, i.e., the sum of $\varphi(\mathbf{m})$ over all vertices $\mathbf{m} \in \mathcal{F}_0$, and also at $y = 1$, we have $\sum_{j=0}^{d}(-1)^d 2^{d-j}\Psi_j(\varphi)$.*

1.4. **Ehrhart polynomials.** By Theorem 1.1 the coefficient of y^p of

$$\mathrm{Todd}_y(h)\,(\mathrm{vol}(tP(h))|_{h=0}$$

($t \in \mathbb{N}$) is equal to

$$E_P^p(t) := \sum_{j=0}^{p} (-1)^j \binom{d-j}{p-j} \sum_{Q \in \mathcal{F}_{d-j}} |tQ \cap \mathbb{Z}^d|.$$

($E_P^0(t) = |tP \cap \mathbb{Z}^d|$ is the ordinary Ehrhart polynomial). This polynomial $E_P^p(t)$ is known as *the p-Ehrhart polynomial of P* introduced by Materov [10]: Indeed it is proved that

$$\chi(X, \mathcal{O}(D) \otimes \Omega^p(X)) = E_P^p(1)$$

using Ishida's p-th complex of logarithmic differential forms and the Bott vanishing theorem.

As a summary, for a d-dimensional smooth projective toric variety X with a very ample divisor D so that $P = P_D \subset \mathbb{R}^d$, the corresponding polytope, we have the following four equalties:

$$\begin{array}{ccc} \int_X td_y(TX)ch(\mathcal{O}(D)) & \longleftrightarrow & \mathrm{Todd}_y(h)(\mathrm{vol}(P(h))|_{h=0} \\ \text{gHRR}\updownarrow & & \updownarrow\text{Thm 1.1} \\ \chi_y(X, \mathcal{O}(D)) & \longleftrightarrow & \sum_{p=0}^{d} E_P^p(1) y^p \end{array}$$

The vertical equalities are gHRR and Theorem 1.1, the lower horizontal equality is just Materov's theorem, and the upper horizontal equality is verified directly in a standard way [6, 12]. Thus Theorem 1.1 (for $\varphi = 1$) can be obtained passing through the gHRR and Materov's formula, while our short direct proof is purely combinatorial and does not require any algebro-geometric background.

As for a comprehensive reference for viewing the current state of this subject in algebraic geometry side, readers should be referred to Maxim-Schürmann [11].

1.5. **Note.** This note is part of the first and second authors' master-course theses in Hokkaido University. After finishing their theses in January 2013, we were noticed about [11]: we thank L. Maxim and J. Schürmann for letting us know of their paper and pointing out an error in our earlier version. The third author is partly supported by the JSPS grant no.24340007.

2. Proofs

Our proof is an extension of the proof of the Khovanskii-Pukhlikov formula (1) (we follow [1, §10]) with an elementary lemma (Lemma 2.1 below). Perhaps, this is the mostly straightforward way to access to the generalized Ehrhart polynomials.

2.1. **Brion's formula.** Let $P \subset \mathbb{R}^d$ be a *simple* polytope of dimension d, i.e., each vertex $\mathbf{v} \in \mathcal{F}_0$ lies on precisely d edges of P. Equivalently, each vertex $\mathbf{v}$ admits d integral vectors which generate the vertex cone $\mathcal{K}_{\mathbf{v}}$. So, fix a set of such vectors for each $\mathbf{v}$, and denote them by $\mathbf{w}_{1,v}, \cdots, \mathbf{w}_{d,v} \in \mathbb{Z}^d$. The fundamental parallelepiped of $\mathcal{K}_{\mathbf{v}}$ is defined by

$$\Pi_{\mathbf{v}} := \{ \ \lambda_1 \mathbf{w}_{1,v} + \cdots + \lambda_d \mathbf{w}_{d,v}, \ 0 \le \lambda_i < 1 \ (\forall i) \ \}.$$

Broin's theorem for a simple polytope [2] is described in the following discrete and continuous exponential forms [1, Thm.3.5, 10.4]:

$$\sum_{\mathbf{m} \in P \cap \mathbb{Z}^d} e^{\mathbf{m}\cdot\mathbf{z}} = \sum_{\mathbf{v} \in \mathcal{F}_0} e^{\mathbf{v}\cdot\mathbf{z}} \frac{\sum_{\mathbf{m} \in \Pi_{\mathbf{v}} \cap \mathbb{Z}^d} e^{\mathbf{m}\cdot\mathbf{z}}}{\prod_{k \in J}(1 - e^{\mathbf{w}_{k,v}\cdot\mathbf{z}})}, \tag{3}$$

$$\int_P e^{\mathbf{x}\cdot\mathbf{z}} d\mathbf{x} = (-1)^d \sum_{\mathbf{v} \in \mathcal{F}_0} e^{\mathbf{v}\cdot\mathbf{z}} \frac{|\det[\mathbf{w}_{1,v} \cdots \mathbf{w}_{d,v}]|}{\prod_{k=1}^d \mathbf{w}_{k,v} \cdot \mathbf{z}}. \tag{4}$$

Here $\mathbf{z} = (z_1, \cdots, z_d)$ are formal variables of integral-point transforms (multivariate generating functions). Note that the continuous form (4) is valid for simple rational polytopes.

2.2. **Unimodular (smooth) polytope.** We prove Theorem 1.1. Assume that P is unimodular: For each vertex $\mathbf{v}$, $\{\mathbf{w}_{k,v}\}_{1 \le k \le d}$ is a basis of $\mathbb{Z}^d$. In particular $|\Pi_{\mathbf{v}} \cap \mathbb{Z}^d| = |\det[\mathbf{w}_1 \cdots \mathbf{w}_d]| = 1$.

Now a perturbed polytope $P(h)$ has vertices $\mathbf{v} - \sum_{k=1}^d h_{k,v}\mathbf{w}_{k,v}$ for some $h_{k,v}$. We may assume that h is a rational vector small enough so that $P(h)$ is unimodular and rational. We apply (4) to $P(h)$:

$$\int_{P(h)} e^{\mathbf{x}\cdot\mathbf{z}} d\mathbf{x} = (-1)^d \sum_{\mathbf{v} \in \mathcal{F}_0} e^{(\mathbf{v} - \sum h_{k,v}\mathbf{w}_{k,v})\cdot\mathbf{z}} \frac{1}{\prod_{k=1}^d \mathbf{w}_{k,v} \cdot \mathbf{z}}. \tag{5}$$

We operate $\mathrm{Todd}_y(h)$ to this integral over $P(h)$ at $h = 0$. First, put

$$\widehat{t_y}(x) := \frac{(1 + ye^{-\frac{\partial}{\partial x}})\frac{\partial}{\partial x}}{1 - e^{-\frac{\partial}{\partial x}}} = 1 + y + \frac{\partial}{\partial x}(\cdots).$$

Note that $\widehat{t_y}(x)(1) = 1 + y$ and

$$\widehat{t_y}(x)(e^{xz}) = \frac{(1 + ye^{-\frac{\partial}{\partial x}})\frac{\partial}{\partial x}}{1 - e^{-\frac{\partial}{\partial x}}} e^{xz} = \frac{(1 + ye^{-z})z}{1 - e^{-z}} e^{xz}.$$

Let us see how $\widehat{t_y}(h_F)$ operates the right hand side of (5): If a vertex $\mathbf{v}$ is contained in a facet F, say $h_F = h_{k,v}$, then

$$\widehat{t_y}(h_{k,v}) \left(e^{(\mathbf{v} - \sum h_{k,v}\mathbf{w}_{k,v})\cdot\mathbf{z}} \right)\Big|_{h=0} = e^{\mathbf{v}\cdot\mathbf{z}} \frac{(1 + ye^{\mathbf{w}_{k,v}\cdot\mathbf{z}})(-\mathbf{w}_{k,v}\cdot\mathbf{z})}{1 - e^{\mathbf{w}_{k,v}\cdot\mathbf{z}}}.$$

If $\mathbf{v}$ is not contained in F, then the unnormalized constant appears:

$$\widehat{t_y}(h_F) \left(e^{(\mathbf{v} - \sum h_{k,v}\mathbf{w}_{k,v})\cdot\mathbf{z}} \right)\Big|_{h=0} = (1 + y)e^{\mathbf{v}\cdot\mathbf{z}}.$$

The number of F not containing $\mathbf{v}$ is just $n-d$, where $n=|\mathcal{F}_{d-1}|$, hence $\prod_{F\in\mathcal{F}_{d-1}}\widehat{t}_y(h_F)$ operated on the right hand side of (5) at $h=0$ yields the factor $(1+y)^{n-d}$, but this factor cancels out the correction term in the definition of $\mathrm{Todd}_y(h)$. Thus

$$\begin{aligned}
&\mathrm{Todd}_y(h)\left(\int_{P(h)} e^{\mathbf{x}\cdot\mathbf{z}}d\mathbf{x}\right)\Bigg|_{h=0}\\
&=(-1)^d\sum_{\mathbf{v}\in\mathcal{F}_0}\mathrm{Todd}_y(h)\left(e^{(\mathbf{v}-\sum h_{k,v}\mathbf{w}_{k,v})\cdot\mathbf{z}}\frac{1}{\prod_{k=1}^d\mathbf{w}_{k,v}\cdot\mathbf{z}}\right)\Bigg|_{h=0}\\
&=(-1)^d\sum_{\mathbf{v}\in\mathcal{F}_0}\frac{e^{\mathbf{v}\cdot\mathbf{z}}}{\prod_{k=1}^d\mathbf{w}_{k,v}\cdot\mathbf{z}}\prod_{k=1}^d\frac{(1+ye^{\mathbf{w}_{k,v}\cdot\mathbf{z}})(-\mathbf{w}_{k,v}\cdot\mathbf{z})}{1-e^{\mathbf{w}_{k,v}\cdot\mathbf{z}}}\\
&=\sum_{\mathbf{v}\in\mathcal{F}_0}e^{\mathbf{v}\cdot\mathbf{z}}\cdot\prod_{k=1}^d\frac{1+ye^{\mathbf{w}_{k,v}\cdot\mathbf{z}}}{1-e^{\mathbf{w}_{k,v}\cdot\mathbf{z}}}.
\end{aligned}$$

Next we expand the product in the last formula. Let $[d]=\{1,\cdots,d\}$, and the number of elements of a subset $J\subset[d]$ is denoted by $|J|$.

Lemma 2.1. *Given a sequence* $a_1,a_2,\cdots$ $(a_k\neq 1)$, *it holds that*

$$\prod_{k=1}^d\frac{1+a_ky}{1-a_k}=\sum_{p=0}^d y^p\left(\sum_{j=0}^p(-1)^j\binom{d-j}{p-j}\sum_{|J|=d-j}\frac{1}{\prod_{k\in J}(1-a_k)}\right).$$

Proof : The right hand side of the formula multiplied by $\prod_{k=1}^d(1-a_k)$ is equal to

$$B_d:=\sum_{j=0}^d A_{j,d}\,(-y)^j(1+y)^{d-j},$$

where $A_{0,d}=1$, $A_{j,d}=0$ for $j>d$, and

$$A_{j,d}:=\sum_{|J|=d-j}\prod_{k\in[d]-J}(1-a_k)$$

for $1\le j\le d$. Obviously, $B_1=1+a_1y$. Since

$$A_{j,d+1}=A_{j,d}+(1-a_{d+1})A_{j-1,d},$$

we see $B_{d+1}=(1+y)B_d-(1-a_{d+1})yB_d=(1+a_{d+1}y)B_d$. By induction, we have $B_d=\prod_{k=1}^d(1+a_ky)$. □

In the above lemma, we set $a_k=e^{\mathbf{w}_{k,v}\cdot\mathbf{z}}$ $(1\le i\le d)$. Let $\mathbf{v}\in\mathcal{F}_0$. Any face $Q\in\mathcal{F}_{d-j}$ having $\mathbf{v}$ as its vertex is uniquely determined by some $J=J(Q)\subset[d]$ with $|J|=d-j$ so that the cone $\mathcal{K}(Q)_{\mathbf{v}}$ has the generators

$\mathbf{w}_{k,v}$ $(k \in J)$. Hence we have

$$\begin{aligned}
&\sum_{\mathbf{v}\in\mathcal{F}_0} e^{\mathbf{v}\cdot\mathbf{z}} \prod_{k=1}^{d} \frac{1+ye^{\mathbf{w}_{k,v}\cdot\mathbf{z}}}{1-e^{\mathbf{w}_{k,v}\cdot\mathbf{z}}} \\
&= \sum_{p=0}^{d} y^p \left(\sum_{j=0}^{p} (-1)^j \binom{d-j}{p-j} \sum_{\mathbf{v}\in\mathcal{F}_0} \sum_{|J|=d-j} \frac{e^{\mathbf{v}\cdot\mathbf{z}}}{\prod_{k\in J}(1-e^{\mathbf{w}_{k,v}\cdot\mathbf{z}})} \right) \\
&= \sum_{p=0}^{d} y^p \left(\sum_{j=0}^{p} (-1)^j \binom{d-j}{p-j} \sum_{Q\in\mathcal{F}_{d-j}} \sum_{\mathbf{v}\in\mathcal{F}_0(Q)} \frac{e^{\mathbf{v}\cdot\mathbf{z}}}{\prod_{k\in J}(1-e^{\mathbf{w}_{k,v}\cdot\mathbf{z}})} \right).
\end{aligned}$$

Note that each $Q \in \mathcal{F}_{d-j}$ is an unimodular lattice polytope in an affine subspace isomorphic to $\mathbb{R}^{d-i} \times \{0\} \subset \mathbb{R}^d$ by some unimodular linear transform and an integral parallel transition. Thus Brion's formula (3) can be applied to Q, and we get

$$\sum_{\mathbf{v}\in\mathcal{F}_0(Q)} \frac{e^{\mathbf{v}\cdot\mathbf{z}}}{\prod_{k\in J}(1-e^{\mathbf{w}_{k,v}\cdot\mathbf{z}})} = \sum_{\mathbf{m}\in Q\cap\mathbb{Z}^d} e^{\mathbf{m}\cdot\mathbf{z}}.$$

Thus we have

$$\begin{aligned}
&\mathrm{Todd}_y(h) \left(\int_{P(h)} e^{\mathbf{x}\cdot\mathbf{z}} d\mathbf{x} \right) \Bigg|_{h=0} \\
&\quad = \sum_{p=0}^{d} y^p \left(\sum_{j=0}^{p} (-1)^j \binom{d-j}{p-j} \sum_{Q\in\mathcal{F}_{d-j}} \sum_{\mathbf{m}\in Q\cap\mathbb{Z}^d} e^{\mathbf{m}\cdot\mathbf{z}} \right).
\end{aligned}$$

Finally, expanding $e^{\mathbf{x}\cdot\mathbf{z}}$ and $e^{\mathbf{m}\cdot\mathbf{z}}$ into power series, we obtain our formula for homogeneous polynomials $\varphi = (\mathbf{c}\cdot\mathbf{x})^l$ with arbitrary values $\mathbf{z} = \mathbf{c}$ and $l = 0, 1, \cdots$. Any polynomial function is a sum of such powers, thus our theorem is proved.

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