ON TOPOLOGICAL RADON TRANSFORMATIONS

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0. Introduction

Recently many mathematicians have been working on Radon transformation [3,4,5,7,8,18]. In [3] Brylinski deals with topological Radon transformations of constructible sheaves. Viro in [18] and Ernström in [7,8] deals with topological Radon transformations of constructible functions, in the set-up of the following diagram:

$$\begin{tikzcd} & I_k \arrow[dl, "p"'] \arrow[dr, "q"] & \\ P^N & & Gr_k(P^N) \end{tikzcd}$$

where $Gr_k(P^N)$ is the Grassmannian of k-dimensional planes of the N-dimensional projective space, I_k is the point-k-plane incidence variety of $P^N \times Gr_k(P^N)$ and p and q are the restriction of the projections. In this paper we consider Radon transformations of constructible functions and of homology classes.

The Radon transformation treated in [7,8,18] is the homomorphism

$$\mathcal{F}(P^N) \to \mathcal{F}(Gr_k(P^N))$$

from the abelian group of constructible functions on P^N to that on $Gr_k(P^N)$, defined by the composite $q_* \circ p^*$ of the pull-back $p^* : \mathcal{F}(P^N) \to \mathcal{F}(I_k)$ and the push-forward $q_* : \mathcal{F}(I_k) \to \mathcal{F}(Gr_k(P^N))$. The functor $\mathcal{F}$, assigning to a variety X, the abelian group $\mathcal{F}(X)$ of constructible functions on X, is both covariant and contravariant. Namely, the functor $\mathcal{F}$ is covariant with respect to push-forward, and contravariant with respect to pull-back. The topological Radon transformation $q_* \circ p^*$ mixes both covariant and contravariant natures. In this paper we introduce a category of varieties, whose morphisms are isomorphism classes of pairs of maps $X \leftarrow M \to Y$, and we capture the topological Radon transformation as a covariant functor from this category to the category of abelian groups; i.e., we show the following :

Theorem A. *There is a category Div of compact complex algebraic varieties and there is a covariant functor $\mathcal{F}^{Rad} : Div \to Ab$ from the category Div to the category Ab of abelian groups such that*

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(i) *for a variety* X, $\mathcal{F}^{Rad}(X) = \mathcal{F}(X)$, *is the abelian group of constructible functions on* X,
(ii) *for a morphism* $f \in Hom_{Div}(X,Y)$, $\mathcal{F}^{Rad}(f) : \mathcal{F}(X) \to \mathcal{F}(Y)$ *is the Radon transformation.*

Also, we introduce the notion of a homological Verdier-Radon transformation on smooth varieties, which is closely related to the topological Radon transformation via the Chern-Schwartz-MacPherson transformation C_* ([2,14]). In fact, we show the following result.

Theorem B. *For a divergent diagram* $\alpha : X \overset{p}{\leftarrow} M \overset{q}{\rightarrow} Y$, *where* X, M *and* Y *are smooth and* $p : M \to X$ *is Euler, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{C_*} & H_*(X;\mathbb{Z}) \\ {\scriptstyle \mathcal{F}^{Rad}(\alpha)}\big\downarrow & & \big\downarrow{\scriptstyle H^{V-Rad}(\alpha)} \\ \mathcal{F}(Y) & \xrightarrow[C_*]{} & H_*(Y;\mathbb{Z}) \end{array}$$

Here $H^{V-Rad}(\alpha) : H_*(X;\mathbb{Z}) \to H_*(Y;\mathbb{Z})$ *is the homological Verdier-Radon transformation.*

We use Theorem B to generalize the Plücker formula in [7,8]. This formula can be interpreted as a formula expressing the 0-dimensional component of the Chern-Mather class of the k-dual variety of tangent k-planes of a variety X in P^N, denoted $X^{<k>}$ in $Gr_k(P^N)$, via the 0-dimensional component of the Chern-Mather class of the variety X. Our generalization gives a formula expressing the total Chern-Mather class of the k-dual variety $X^{<k>}$ via the total Chern-Mather class of the source variety X.

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1. The Divergent Category

In this section we introduce a category of algebraic varieties that will be used to generalize the Radon transformations of constructible functions in [7,8]. The construction of such a category is similar to that of the category $Q\underline{\underline{M}}$ defined in Quillen's paper [16, §2].

Definition (1.1). Given varieties X, Y and M, an orderered pair α of maps $p : M \to X$ and $q : M \to Y$

$$\alpha : X \overset{p}{\leftarrow} M \overset{q}{\rightarrow} Y$$

is called *a divergent diagram* from X to Y. Sometimes the notation $\alpha = (p : M : q)$ will be used. If α is a divergent diagram from X to itself and $p = q$, then we call α *a symmetric divergent diagram.*

The term "divergent diagram" is used in the theory of dynamics (see [6]).

Remark (1.2). The reverse of the divergent diagram $\alpha := (p : M : q)$ is

$$\alpha^* : Y \overset{q}{\leftarrow} M \overset{p}{\rightarrow} X.$$

The diagram $\alpha^* := (q : M : p)$ is a divergent diagram from Y to X. It is different from the divergent diagram α.

Definition (1.3). Let $\alpha = (p : M : q)$ and $\alpha' = (p' : M' : q')$ be two divergent diagrams from X to Y. We say that the divergent diagram α is isomorphic to the divergent diagram α', denoted by $\alpha \sim \alpha'$, if there exists an isomorphism $h : M \to M'$ such that the following diagram commutes:

$$\begin{tikzcd} & M \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, "h"] & \\ X & & Y \\ & M' \arrow[ul, "p'"] \arrow[ur, "q'"'] & \end{tikzcd}$$

i.e., $p = p' \circ h$ and $q = q' \circ h$.

Note that the relation $\sim$ is an equivalence relation. We denote the isomorphism class of a divergent diagram α by $[\alpha]$.

Definition (1.4). The composite $\beta \circ \alpha$ of two divergent diagrams

$$\alpha = (p : M : q) : X \leftarrow M \to Y \text{ and } \beta = (r : N : s) : Y \leftarrow N \to Z$$

is defined using the following diagram:

$$\begin{tikzcd} & & M \times_Y N \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & & \\ & M \arrow[dl, "p"'] \arrow[dr, "q"] & & N \arrow[dl, "r"'] \arrow[dr, "s"] & \\ X & & Y & & Z \end{tikzcd}$$

where $M \times_Y N$ is the fiber product of $q : M \to Y$ and $r : N \to Y$, and pr_1 and pr_2 are the projections. Then the composite $\beta \circ \alpha$ is defined by:

$$\beta \circ \alpha := (p \circ pr_1 : M \times_Y N : s \circ pr_2)$$

Now we are ready to define the divergent category Div of compact complex algebraic varieties.

Proposition (1.5). *There is a category Div defined by the following data:*
(i) *the objects $Obj(Div)$ consist of all compact complex algebraic varieties,*
(ii) *for two objects X and Y, the morphisms $Hom_{Div}(X,Y)$ consist of the isomorphism classes of divergent diagrams from X to Y. For $[\alpha] \in Hom_{Div}(X,Y)$ and $[\beta] \in Hom_{Div}(Y,Z)$; we define the composite $[\beta] \circ [\alpha]$ by*

$$[\beta] \circ [\alpha] := [\beta \circ \alpha] \in Hom_{Div}(X,Z).$$

We shall call this category the divergent category of algebraic varieties.

Proof. We have to show the following three statements: (the proofs are straightforward using the properties of a fiber product and therefore omitted).
(1) The composite $[\beta] \circ [\alpha]$ is well-defined; i.e., it is independent of the choice of the representatives in the isomorphism classes $[\alpha]$ and $[\beta]$; i.e., if $\alpha \sim \alpha'$ and $\beta \sim \beta'$ then $\beta \circ \alpha \sim \beta' \circ \alpha'$.
(2) The composite operation is associative, i.e., $[\gamma] \circ ([\beta] \circ [\alpha]) = ([\gamma] \circ [\beta]) \circ [\alpha]$ for any divergent diagrams $\alpha : X \overset{p}{\leftarrow} M \overset{q}{\rightarrow} Y$, $\beta : Y \overset{r}{\leftarrow} N \overset{s}{\rightarrow} Z$ and $\gamma : Z \overset{t}{\leftarrow} L \overset{u}{\rightarrow} W$.
(3) The composition law has units: let $1_X := [e_X]$ be the isomorphism class of the "identity" divergent diagram $e_X : X \overset{id}{\leftarrow} X \overset{id}{\rightarrow} X$. Then, for any morphism $[\alpha] \in \mathrm{Hom}_{Div}(X,Y)$, and for any morphism $[\beta] \in \mathrm{Hom}_{Div}(W,X)$, we have that $[\alpha] \circ 1_X = [\alpha]$ and $1_X \circ [\beta] = [\beta]$; i.e., the equivalences $\alpha \circ 1_X \sim \alpha$ and $1_X \circ \beta \sim \beta$ hold. □

Remark (1.6). Correspondingly, for categories of topological spaces, real analytic varieties etc., we can define divergent categories, provided that the fiber product is closed in the category; i.e., if three objects X, Y and Z belong to the category then the fiber product $X \times_Z Y$ also belongs to the category.

Definition (1.7). Let $f : X \to Y$ be a morphism and let Γ_f be the graph of the morphism f. Then the divergent diagram $X \overset{p}{\leftarrow} \Gamma_f \overset{q}{\rightarrow} Y$, where p and q are the restrictions of the projections, is called the *graph divergent diagram of f*, and is denoted by γ_f.

Observation (1.8). We note that the graph divergent diagram γ_f of f is isomorphic to the divergent diagram $|f> := (id_X : X : f) : X \overset{id_X}{\leftarrow} X \overset{f}{\rightarrow} Y$.

Next we show that the usual category Var, of complex algebraic varieties, can be embedded into the divergent category Div.

Theorem (1.9). **(The graph functor Γ)** *Let $\Gamma : Var \to Div$ be the map defined as follows:*
(i) *for an object $X \in Obj(Var)$, define $\Gamma(X) := X$, and*
(ii) *for a morphism $f \in Hom_{Var}(X,Y)$, define $\Gamma(f)$ by*

$$\Gamma(f) := [\gamma_f] \in Hom_{Div}(X,Y),$$

where $[\gamma_f]$ is the isomorphism class of the graph divergent diagram γ_f.

Then the map $\Gamma : Var \to Div$ is a faithful covariant functor.

Proof. (1) To prove covariance we have to show that $\Gamma(f \circ g) = \Gamma(f) \circ \Gamma(g)$, given $g \in \mathrm{Hom}_{Var}(X, Y)$ and $f \in \mathrm{Hom}_{Var}(Y, Z)$. By definition, $\Gamma(f \circ g) = [\gamma_{f \circ g}]$ and $\Gamma(f) \circ \Gamma(g) = [\gamma_f] \circ [\gamma_g] = [\gamma_f \circ \gamma_g]$, hence we have to show that $\gamma_{f \circ g} \sim \gamma_f \circ \gamma_g$. We can show this directly, but using Observation (1.8) above, it suffices to show that $|f \circ g > \sim |f > \circ |g >$. By the definition of $|f > \circ |g >$ we have the following diagram:

$$\begin{tikzcd} & & X \times_Y Y \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & & \\ & X \arrow[dl, "id"'] \arrow[dr, "g"] & & Y \arrow[dl, "id"'] \arrow[dr, "f"] & \\ X & & Y & & Z \end{tikzcd}$$

We note that $X \times_Y Y$ is the graph Γ_g of the map $g : X \to Y$. The map $pr_1 : \Gamma_g \to X$ is an isomorphism and the following diagram is clearly commutative:

$$\begin{tikzcd} & \Gamma_g \arrow[dl, "pr_1"'] \arrow[dr, "f \circ pr_2"] \arrow[dd, "pr_1"] & \\ X & & Z \\ & X \arrow[ul, "id"] \arrow[ur, "f \circ g"'] & \end{tikzcd}$$

Thus we have the equivalence of $|f \circ g > \sim |f > \circ |g >$.

(2) To prove faithfullness we have to prove, given $f, g \in \mathrm{Hom}_{Var}(X, Y)$, that $\Gamma(f) = \Gamma(g)$ implies that $f = g$. By definition $\Gamma(f) = \Gamma(g)$ means that $[\gamma_f] = [\gamma_g]$,which means that $|f > \sim |g >$ via Observation (1.8). This means that there is an isomorphism $h : X \to X$ such that the following diagram commutes, i.e., that $id_X = id_X \circ h$ and $f = g \circ h$.

$$\begin{tikzcd} & X \arrow[dl, "id_X"'] \arrow[dr, "f"] \arrow[dd, "h"] & \\ X & & Y \\ & X \arrow[ul, "id_X"] \arrow[ur, "g"'] & \end{tikzcd}$$

This means that $h = id_X$ and hence $f = g$. $\square$

Similarly we can show the following:

Theorem (1.10). (The reverse graph functor Γ^*) *Let $\Gamma^* : Var \to Div$ be a map defined as follows:*

(i) *for each object $X \in Obj(Var)$, $\Gamma^*(X) = X$, and*

(ii) *for a morphism* $f \in Hom_{Var}(X,Y)$, $\Gamma^*(f)$ *is defined by*

$$\Gamma^*(f) := [\gamma_f^*] \in Hom_{Var}(Y,X)$$

where γ_f^* *is the reverse of the graph divergent diagram* γ_f.
Then the map $\Gamma^* : Var \to Div$ *is a faithful contravariant functor.*

Remark (1.11). Theorem (1.10) can be shown similarly as in the proof of Theorem (1.9), but we can easily see this because the map $^* : Div \to Div$, taking the reverse, is a faithful contravariant functor.

Remark (1.12). Any divergent diagram $a = (f : M : g) : X \leftarrow M \to Y$ can be expressed as the composite $|g> \circ <f|$ of the divergent diagrams $|g>$ and $<f|$.

2. The Radon functor

We shall here define a covariant functor from the divergent category Div to the category Ab of abelian groups. We will denote the covariant functor of constructible functions with push-forward by $\mathcal{F}_*$, to emphasize the covariance. Given a morphism $f : X \to Y$ the push-forward is denoted $f_* : \mathcal{F}(X) \to \mathcal{F}(Y)$. With the pull-back $f^* : \mathcal{F}(Y) \to \mathcal{F}(X)$ the constructible functions become a contravariant functor, which will be denoted by $\mathcal{F}^*$, to emphasize the contravariance. Motivated by the Radon transformations of constructible sheaves [3], and constructible functions [7,8,18], we introduce the Radon functor $\mathcal{F}^{Rad} : Div \to Ab$. The Radon functor $\mathcal{F}^{Rad}$ specializes, when restricted to the faithful subcategories $\Gamma(Var)$ and $\Gamma^*(Var)$, to the covariant functor $\mathcal{F}_* : Var \to Ab$ and the contravariant functor $\mathcal{F}^* : Var \to Ab$, respectively.

Definition (2.1). (**Radon transformations**) For a divergent diagram

$$\alpha : X \overset{p}{\leftarrow} M \overset{q}{\to} Y$$

we define the homomorphism $\mathcal{F}^{Rad}(\alpha) : \mathcal{F}(X) \to \mathcal{F}(Y)$ to be the composite $q_* \circ p^*$, where
(i) $p^* : \mathcal{F}(X) \to \mathcal{F}(M)$ is the pull-back, i.e., $p^*(\lambda) := \lambda \circ p$,
(ii) $q_* : \mathcal{F}(M) \to \mathcal{F}(X)$ is the push-forward, which is defined by

$$q_*(\lambda)(y) := \sum_W a_W \chi(q^{-1}(y) \cap W),$$

where $\lambda = \sum_W a_W 1_W$ and each W is a subvariety of M.
We call this homomorphism $\mathcal{F}^{Rad}(\alpha)$ *the Radon transformation associated to* α.
The reverse Radon transformation associated to α, is defined to be the Radon transformation $\mathcal{F}^{Rad}(\alpha^*)$, where $\alpha^* : Y \overset{q}{\leftarrow} M \overset{p}{\to} X$.

It is easy to see the following:

Lemma (2.2). *Let* $\Sigma : X \overset{p}{\leftarrow} M \overset{p}{\to} X$ *be a symmetric divergent diagram. Let* W *be a subvariety of* X. *Then we have*

(i)

$$\mathcal{F}^{Rad}(\Sigma)(1_W)(y) = \begin{cases} \chi(p^{-1}(y)) & \text{if } y \in W \\ 0 & \text{if } y \notin W\ , \end{cases}$$

in particular

(ii) *if p is surjective and the Euler-Poincaré characteristic $\chi(p^{-1}(y))$ of each fiber $p^{-1}(y)$ is equal to 1, then $\mathcal{F}^{Rad}(\Sigma) = 1_{\mathcal{F}(X)}$ the identity homomorphism on the abelian group of constructible functions on X.*

Theorem (2.3). *The map $\mathcal{F}^{Rad} : Div \to Ab$ defined below, is a covariant functor.*

(i) For each object $X \in Obj(Div)$ set $\mathcal{F}^{Rad}(X) := \mathcal{F}(X)$, the abelian group of constructible functions on X.

(ii) For a morphism $[\alpha] \in Hom_{Div}(X,Y)$, set $\mathcal{F}^{Rad}([\alpha]) := \mathcal{F}^{Rad}(\alpha)$, the Radon transformation.

We call the functor $\mathcal{F}^{Rad} : Div \to Ab$ the Radon functor.

Proof. We have to show: (1) that $\mathcal{F}^{Rad}([\alpha])$ is well-defined, and (2) the functoriality of the map $\mathcal{F}^{Rad} : Div \to Ab$.

(1) Let $\alpha : X \overset{p}{\leftarrow} M \overset{q}{\to} Y$ be isomorphic to $\alpha' : X \overset{p'}{\leftarrow} M' \overset{q'}{\to} Y$, i.e., there is an isomorphism $h : M \to M'$ such that $p = p' \circ h$ and $q = q' \circ h$. Then we have

$$\begin{aligned} \mathcal{F}^{Rad}(\alpha) &= q_* p^* \\ &= (q' \circ h)_*(p' \circ h)^* \\ &= (q')_* h_* h^* (p')^* \\ &= (q')_*(p')^* \qquad \text{(since } h_* h^* = 1_{\mathcal{F}(M')} \text{ by Lemma (2.2) (ii))} \\ &= \mathcal{F}^{Rad}(\alpha') \end{aligned}$$

Thus the definition of $\mathcal{F}^{Rad}([\alpha]) := \mathcal{F}^{Rad}(\alpha)$ is independent of the choice of the representative from the isomorphism class $[\alpha]$; i.e., $\mathcal{F}^{Rad}([\alpha])$ is well-defined.

(2) Let $\alpha : X \overset{p}{\leftarrow} M \overset{q}{\to} Y$ and $\beta : Y \overset{r}{\leftarrow} N \overset{s}{\to} Z$ be two divergent diagrams. We want to show that $\mathcal{F}^{Rad}([\beta] \circ [\alpha]) = \mathcal{F}^{Rad}([\beta]) \circ \mathcal{F}^{Rad}([\alpha])$. It follows from the definition of composition in Div that $\mathcal{F}^{Rad}([\beta] \circ [\alpha]) = \mathcal{F}^{Rad}([\beta \circ \alpha])$. By (1), it suffices to show that $\mathcal{F}^{Rad}(\beta \circ \alpha) = \mathcal{F}^{Rad}(\beta) \circ \mathcal{F}^{Rad}(\alpha)$. Recalling the definition of the composite $\beta \circ \alpha$ (using the same symbols as in Definition (1.4)) and the definition of $\mathcal{F}^{Rad}(\beta \circ \alpha)$, we have that

$$\begin{aligned} \mathcal{F}^{Rad}(\beta \circ \alpha) &= (s \circ pr_2)_*(p \circ pr_1)^* \\ &= s_*(pr_2)_*(pr_1)^* p^*. \end{aligned}$$

Here we recall the following lemma:

Lemma (2.4). **([8, Proposition 3.5])** *If the following square is a fiber square; i.e., W is the fiber product of $f : Y \to X$ and $p : Z \to X$ and q and g are the projections,*

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "q"'] & Z \arrow[d, "p"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

then it holds that $p^* f_* = g_* q^*$.

Proof. Given a subvariety V of Y, it suffices to prove that $p^* f_*(1_V) = g_* q^*(1_V)$. Fix $z \in Z$ and set $x = p(z)$. Then by definition we have that

$$(p^* f_*(1_V))(z) = \chi(f^{-1}(x) \cap V) \quad \text{and} \quad (g_* q^*(1_V))(z) = \chi(g^{-1}(z) \cap q^{-1}(V)).$$

Furthermore, since W is the fiber product $Y \times_X Z$, we have that $g^{-1}(z) = f^{-1}(x) \times z$. We see that $g^{-1}(z) \cap q^{-1}(V) = (f^{-1}(x) \cap V) \times z$. Thus, we have that $(p^* f_*(\lambda))(z) = (g_* q^*(\lambda))(z)$; i.e., $p^* f_*(1_V) = g_* q^*(1_V)$ since z is an arbitrary point of Z . □

We apply this lemma to the following fiber square,

$$\begin{tikzcd}
 & M \times_Y N \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\
M \arrow[dr, "q"'] & & N \arrow[dl, "r"] \\
 & Y &
\end{tikzcd}$$

we have that $(pr_2)_*(pr_1)^* = r^* q_*$. Thus we get that

$$\begin{aligned}
\mathcal{F}^{Rad}(\beta \circ \alpha) &= (s \circ pr_2)_*(p \circ pr_1)^* \\
&= s_* {pr_2}_* {pr_1}^* p^* \\
&= s_* r^* q_* p^* \\
&= \mathcal{F}^{Rad}(\beta) \circ \mathcal{F}^{Rad}(\alpha).
\end{aligned} \qquad \square$$

Corollary (2.5). *The Radon functor* $\mathcal{F}^{Rad}$ *is related to the push-forward* $\mathcal{F}_*$ *and the pull-back* $\mathcal{F}^*$ *via the graph functor* Γ*:*
(i) $\mathcal{F}^{Rad} \circ \Gamma = \mathcal{F}_* : Var \to Ab$.
(ii) $\mathcal{F}^{Rad} \circ (\Gamma^*) = \mathcal{F}^* : Var \to Ab$.

Next we give some specific examples of divergent diagrams.

Example (2.6).

(1) A divergent diagram $\Pi : X \overset{pr_1}{\leftarrow} X \times Y \overset{pr_2}{\to} Y$ is called *a product divergent diagram.* For W a subvariety of X we have that $\mathcal{F}^{Rad}(\Pi)(1_W) = \chi(W) 1_Y$. Thus, $\mathcal{F}^{Rad}(\Pi)(\lambda) = \chi(\lambda) 1_Y$, where $\chi(\lambda) = \sum_W a_W \chi(W)$ if $\lambda = \sum_W a_W 1_W$.

(2) *A Grassmannian divergent diagram* is a diagram $\Gamma_k : P^N \overset{p}{\leftarrow} I_k \overset{q}{\to} Gr_k(P^N)$, where $Gr_k(P^N)$ is the Grassmannian of k-dimensional planes of the projective space P^N,

$$I_k := \{(x, l) \in P^N \times Gr_k(P^N) \mid x \in l\}$$

is the k-th incidence variety and p and q are the restrictions of the projections. A specific description of the topological Radon transformation $\mathcal{F}^{Rad}(G)$ of the Grassmannian divergent diagrams is one of the main results of Ernström [7,8]. We will get back to this topic in §4.

(3) Let X be a variety and let $\sigma : E \to F$ be a homomorphism of vector bundles E and F of ranks e and f over X. Let k be any integer such that $0 \le k \le \min(e, f)$. Then the degeneracy locus $D_k(\sigma)$ of the bundle homomorphism σ is defined by (see Fulton's book [11]):

$$D_k(\sigma) := \{x \in X | \text{rank}\,(\sigma(x)) \le k\}.$$

Let $\text{Hom}(E, F)$ be the vector space of all the homomorphisms from E to F. Since for any non-zero scalar $c \in \mathbb{C}$, the complex numbers, $D_k(\sigma) = D_k(c\sigma)$, we consider the projective space $P(\text{Hom}(E, F))$. Then we consider the following incidence variety:

$$I_k := \{(x, \sigma) \in X \times P(\text{Hom}(E, F)) \mid \text{rank}\,(\sigma(x)) \le k\}.$$

A diagram of the following type is called a *degeneracy divergent diagram*:

$$\Delta_k(E, F) : X \overset{p}{\leftarrow} I_k \overset{q}{\to} P(Hom(E, F)).$$

It would be interesting to find a specific description of the topological Radon transformation $\mathcal{F}^{Rad}(\Delta_k(E, F))$. By the definition we see that

$$(\mathcal{F}^{Rad}(\Delta_k(E, F))(1_W))(\sigma) = \chi(W \cap D_r(\sigma)).$$

Thus the problem is equivalent to finding the Euler characteristic of the degeneracy loci of a homomorphism of vector bundles. Parusiński and Pragacz [15] have found a formula for the Euler characteristic degeneracy loci $D_r(\varphi)$ for generic φ.

Remark (2.7). For a divergent diagram $\alpha : X \overset{p}{\leftarrow} M \overset{q}{\to} Y$ the composite $\alpha^* \circ \alpha$ of α and its reverse α^* is a symmetric divergent diagram from X to itself. In the case of the above Grassmannian divergent diagram

$$\Gamma_k : P^N \overset{p}{\leftarrow} I_k \overset{q}{\to} Gr_k(P^N),$$

Ernström [8, Proposition 3.6] gave an explicit description of $\mathcal{F}^{Rad}(\Gamma_k^* \circ \Gamma_k)$: For a constructible function $\lambda \in \mathcal{F}(P^N)$,

$$\mathcal{F}^{Rad}(\Gamma_k^* \circ \Gamma_k)(\lambda) = (\binom{N}{k} - \binom{N-1}{k-1})\lambda + \binom{N-1}{k-1}\chi(\lambda)1_{P^N}.$$

In particular when $k = N - 1$ we have

$$\mathcal{F}^{Rad}(\Gamma_{N-1}^* \circ \Gamma_{N-1})(\lambda) = \lambda + (N-1)\chi(\lambda)1_{P^N}.$$

This was previously proved by Viro [18]. The above formulas are closely related to the problem of finding inversion formulas for Radon transforms (see Schapira [17]).

Remark (2.8). There is a dual notion of divergent diagram, which we denote convergent diagram. For a convergent diagram,

$$\begin{tikzcd} X \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & M & \end{tikzcd}$$

one considers the homomorphism $q^* \circ p_* : \mathcal{F}(X) \to \mathcal{F}(Y)$, which is also called a topological Radon transformation. The Radon transformation $q^* \circ p_*$ is equal to the Radon transformation associated to the divergent diagram $X \overset{pr_1}{\leftarrow} X \times_M Y \overset{pr_2}{\rightarrow} Y$ because of the following fiber square

$$\begin{tikzcd} & X \times_M Y \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ X \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & M & \end{tikzcd}$$

and Lemma (2.4) : $q^* p_* = (pr_2)_* (pr_1)^* : \mathcal{F}(X) \to \mathcal{F}(Y)$. Thus, as far as we are concerned with Radon transformations of constructible functions, it suffices to consider divergent diagrams.

Remark (2.9). In a natural way we can make the category *Div* additive. Let $\alpha : X \overset{p}{\leftarrow} M \overset{q}{\rightarrow} Y$ and $\beta : X \overset{p'}{\leftarrow} M' \overset{q'}{\rightarrow} Y$ be two divergent diagrams. Then the summation operation $\alpha + \beta$ is defined by

$$\alpha + \beta : X \overset{p+p'}{\longleftarrow} M \coprod M' \overset{q+q'}{\longrightarrow} Y$$

where $M \coprod M'$ is the direct sum and $p + p'$ and $q + q'$ are the direct sums of morphisms. We define the summation $[\alpha] + [\beta]$ to be $[\alpha + \beta]$. Then $Hom_{Div}(X, Y)$ becomes an additive monoid group with zero element $0 : X \leftarrow \emptyset \rightarrow Y$, where $\emptyset$ is the empty set. Now, by group completion, we can get an abelian group, denoted $Hom_{Div}(X, Y)^+$, the construction of which is standard and so omitted. We denote the inverse of the class $[\alpha]$ of the divergent diagram $\alpha : X \overset{p}{\leftarrow} M \overset{q}{\rightarrow} Y$, by $-[\alpha] := [-\alpha]$. Here

$$-\alpha : X \overset{-p}{\leftarrow} -M \overset{-q}{\rightarrow} Y,$$

is a virtual divergent diagram, where $-M$ is the virtual variety and $-p$ and $-q$ are virtual morphisms.

Let us call the following enlarged category *the additive divergent category* and denote it by *ADiv*:

$$\mathrm{Obj}(ADiv) = \mathrm{Obj}(Div) = \mathrm{Obj}(Var)$$
$$Hom_{ADiv}(X, Y) = Hom_{Div}(X, Y)^+ .$$

Then we can naturally extend the topological Radon functor $\mathcal{F}^{Rad} : Div \to Ab$ to the additive divergent category *ADiv* as follows: define $\mathcal{F}^{Rad} : ADiv \to Ab$ by

$$\mathcal{F}^{Rad}(X) = \mathcal{F}(X), \text{ and}$$
$$\mathcal{F}(m[\alpha] + n[\beta]) = m\mathcal{F}^{Rad}([\alpha]) + n\mathcal{F}^{Rad}([\beta]).$$

If we assume that $Hom_{Div}(X, Y)$ is a set for any objects X, Y, then

$$Hom_{ADiv}(X, Y) = \mathbb{Z}[Hom_{Div}(X, Y)],$$

i.e., the free abelian group generated by $Hom_{Div}(X,Y)$. In particular, $Hom_{ADiv}(X,X)$ becomes a ring, which is the so-called *Hecke ring* over X.

In fact, we can always assume that $Hom_{Div}(X,Y)$ is a set because, as the referee pointed out, the Radon transform $\mathcal{F}^{Rad}([\alpha])$, associated to a divergent diagram $\alpha : X \overset{p}{\leftarrow} M \overset{q}{\rightarrow} Y$, is equal to the Radon transform of a correspondence between X and Y.

Lemma (2.10). *Given a commutative diagram*

$$\begin{tikzcd} & M \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, "f"] & \\ X & & Y \\ & N \arrow[ul, "r"] \arrow[ur, "s"'] & \end{tikzcd}$$

If $f_*1_M = \sum a_V 1_V$ *then we have the following identity of homomorphism of constructible functions:*

$$q_*p^* = \sum a_V (s|_V)_*(r|_V)^*.$$

Proof. First, note that for any point p in N, we have that

$$\chi(f^{-1}(p)) = \sum_{V \text{ containing } p} a_V.$$

Thus, for any subvariety T of N, we get

$$f_*f^*(1_T) = \sum a_V 1_{T\cap V}.$$

Therefore, for any subvariety W of X, we deduce

$$\begin{aligned} q_*p^*1_W &= (sf)_*(rf)^*(1_W) = s_*f_*f^*r^*(1_W) = s_*f_*f^*(1_{r^{-1}(W)}) \\ &= s_* \sum a_V 1_{V\cap r^{-1}(W)} \\ &= \sum a_V (s|_V)_*(r|_V)^*(1_W). \quad \square \end{aligned}$$

Next, for a divergent diagram $X \overset{p}{\leftarrow} M \overset{q}{\rightarrow} Y$, we define a map

$$P : \mathcal{F}(M) \to Hom_{ADiv}(X,Y)$$

$$1_W \mapsto [X \overset{p|_W}{\leftarrow} W \overset{q|_W}{\rightarrow} Y]$$

and extending linearly. Then for a commutative diagram as in Lemma (2.10) we use P and the push-forward of constructible functions to define a push-forward $f_* : Hom_{ADiv}(X,Y) \to Hom_{ADiv}(X,Y)$ by

$$f_*([X \leftarrow M \rightarrow Y]) = P(f_*(1_M)),$$

and extending linearly. a

We apply the lemma with $N = X \times Y$ and $f = p \times q$.

Proposition (2.11). *Let* $\alpha : X \overset{p}{\leftarrow} M \overset{q}{\rightarrow} Y$ *be a divergent diagram. Then*

$$\mathcal{F}^{rad}(\alpha) = \mathcal{F}^{Rad}((p\times q)_*\alpha).$$

3. Verdier-type Riemann-Roch and homological Verdier-Radon transformations

The Chern-Schwartz-MacPherson transformation C_* is a transformation from the functor of constructible functions F to homology groups H_* with coefficients in $\mathbb{Z}$. Using the Radon transformations of constructible functions and the Chern-Schwartz-MacPherson transformation we will construct a homological Radon transformation.

Definition (3.1). A divergent diagram $\alpha : X \overset{p}{\leftarrow} M \overset{q}{\rightarrow} Y$ is said to be *smooth* if X, M and Y are smooth and α is said to be *Euler* if furthermore the morphism $p : M \to X$ is Euler; i.e., the constructible function 1_M on M satisfies the local Euler condition (see [12, page 77]). The latter means that 1_M is an element of the bivariant group of constructible functions $F(M \to X)$.

For details about bivariant groups of constructible functions, see [1] and [12].

Definition (3.2). Given a smooth divergent diagram $\alpha : X \overset{p}{\leftarrow} M \overset{q}{\rightarrow} Y$, the composite

$$H^{Rad}(\alpha) := q_* \circ p^! : H_*(X;\mathbb{Z}) \to H_*(Y;\mathbb{Z})$$

is called *the homological Radon transformation.* Here $q_* : H_*(M;\mathbb{Z}) \to H_*(Y;\mathbb{Z})$ is the usual push-forward and $p^! : H_*(X;\mathbb{Z}) \to H_*(M;\mathbb{Z})$ is the usual Gysin homomorphism defined by $p^! = \mathcal{D}_M \cdot p^* \cdot \mathcal{D}_X^{-1}$, where $\mathcal{D}_Z : H^*(Z;\mathbb{Z}) \to H_*(Z;\mathbb{Z})$ is the Poincaré duality isomorphism for a smooth variety Z.

Given a smooth divergent diagram $\alpha : X \overset{p}{\leftarrow} M \overset{q}{\rightarrow} Y$, the following diagram is not always commutative.

$$\begin{array}{ccc}
\mathcal{F}(X) & \xrightarrow{C_*} & H_*(X;\mathbb{Z}) \\
{\scriptstyle \mathcal{F}^{Rad}(\alpha)}\downarrow & & \downarrow{\scriptstyle H^{Rad}(\alpha)} \\
\mathcal{F}(Y) & \xrightarrow[C_*]{} & H_*(Y;\mathbb{Z})
\end{array}$$

However, the drawback of the above (non-commutative) diagram can be remedied by introducing the following *Verdier-Gysin homomorphism.*

Definition (3.3). Given a morphism $f : X \to Y$ of compact smooth manifolds X and Y, denote by $c(T_f)$ the total Chern class of the virtual relative tangent bundle $T_f := TX - f^*TY$. The homomorphism

$$c(T_f) \cap f^! : H_*(Y;Z) \to H_*(X;\mathbb{Z})$$

is called *the Verdier-Gysin homomorphism* and denoted by $f^{!!}$ with a double shriek.

Remark (3.5). It is straightforward to see that, just like the Gysin homomorphism, the above Verdier-Gysin homomorphism is functorial; i.e., for morphisms $f : X \to Y$ and $g : Y \to Z$, we have that

$$(gf)^{!!} = f^{!!}g^{!!}.$$

To check functoriality, we need to observe that for a cohomology class β of Y,

$$f^!(\beta \cap g^!) = f^*\beta \cap (gf)^!.$$

This functoriality and the Verdier formula of [12, §10.4, page 110-111], imply that C_* is a natural transformation from the contravariant functor F^* to the contravariant homology functor with the Verdier-Gysin homomorphism, on the category of smooth varieties with Euler morphisms .

Definition (3.6). For a smooth divergent diagram $\alpha : X \xleftarrow{p} M \xrightarrow{q} Y$, the composite

$$H^{V-Rad}(\alpha) := q_* \circ p^{!!} : H_*(X;\mathbb{Z}) \to H_*(Y;\mathbb{Z})$$

shall be called *the homological Verdier-Radon transformation.*

Theorem (3.7). *For an Euler divergent diagram $\alpha : X \xleftarrow{p} M \xrightarrow{q} Y$ the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{F}(X) \arrow[r, "C_*"] \arrow[d, "\mathcal{F}^{Rad}(\alpha)"'] & H_*(X;\mathbb{Z}) \arrow[d, "H^{V-Rad}(\alpha)"] \\
\mathcal{F}(Y) \arrow[r, "C_*"'] & H_*(Y;\mathbb{Z})
\end{tikzcd}$$

Proof. The result follows from the fact that the diagram below is commutative.

$$\begin{tikzcd}
\mathcal{F}(X) \arrow[r, "C_*"] \arrow[d, "p^*"'] & H_*(X;\mathbb{Z}) \arrow[d, "c(T_p)\cap p^!"] \\
\mathcal{F}(M) \arrow[r, "C_*"'] \arrow[d, "q_*"'] & H_*(M;\mathbb{Z}) \arrow[d, "q_*"] \\
\mathcal{F}(Y) \arrow[r, "C_*"'] & H_*(Y;\mathbb{Z})
\end{tikzcd}$$

The top square is commutative because of the Verdier formula of Fulton-MacPherson [12, §10.4, page 110-111]. The bottom square is commutative because of MacPherson's theorem which states that the Chern-Schwarz-MacPherson class C_* commutes with push-forwards [14]. □

The Verdier formula is a consequence of the fact that there is a bivariant version of Chern-Schwartz-MacPherson transformation C_*. Brasselet extended C_* to a Grothendieck transformation from the bivariant theory of constructible functions to the bivariant homology theory [1]. The proof in [12] on the other hand, occured before Brasselet's construction in [1].

Bivariant Chern classes. *There exists a Grothendieck transformation $\gamma : \mathbf{F} \to \mathbf{H}$ from the bivariant theory $\mathbf{F}$ of constructible functions to the bivariant homology theory $\mathbf{H}$ such that if X is smooth, then*

$$\gamma(1_X) = c(TX) \cap [X]$$

where $c(TX)$ is the usual Chern cohomology class of the tangent bundle TX and $[X]$ is the fundamental class of X.

A bivariant version of the Verdier formula can be stated as follows.

Proposition (3.8). *Let M and X be smooth compact complex algebraic varieties and let $f : M \to X$ be an Euler morphism. Then the following equality holds in $\mathbf{H}(M \to X)$:*

$$\gamma(1_f) = c(T_f) \cap [f]$$

where 1_f is the canonical orientation for the bivariant constructible function group $\mathbf{F}(M \to X)$, the class $[f]$ is the canonical orientation for the bivariant homology group $\mathbf{H}(M \to X)$ and $T_f := TM - f^*TX$ is the virtual relative tangent bundle of the morphism f.

Proof. (cf. [12, §6, Proposition 6A]) We consider the following diagram, where the elements at the arrows are bivariant homology classes of the corresponding morphism. Composing theses classes, the diagram is commutative, except possibly at the bottom triangle, where commutativity is to be proved.

$$\begin{tikzcd}
M \arrow[rrrr, "{[f]}"] \arrow[drrr, "{[M]}"] \arrow[ddd, "f^*(c(T_X)^{-1})"'] & & & & X \arrow[dl, "{[X]}"'] \arrow[ddd, "c(T_X)^{-1}"] \\
& & & \mathrm{pt} & \\
& & M \arrow[uull, "c(T_M)"] \arrow[ur, "\gamma(1_M)"'] \arrow[dll, "c(T_f)"'] \arrow[drr, "\gamma(1_f)"] & & \\
M \arrow[rrrr, "{[f]}"'] & & & & X \arrow[uul, "\gamma(1_X)"']
\end{tikzcd}$$

The proof goes as follows:

$$\begin{aligned}
c(T_f).[f].\gamma(1_X) &= c(TM).f^*c(TX)^{-1}.[f].\gamma(1_X) && (\text{since } c(T_f) = c(TM).f^*c(TX)^{-1}) \\
&= c(TM).[f].c(TX)^{-1}.\gamma(1_X) && (\text{since } f^*c(TX)^{-1}.[f] = [f].c(TX)^{-1}) \\
&= c(TM).[f].[X] && (\text{since } \gamma(1_X) = c(TX).[X]) \\
&= c(TM).[M] && (\text{since } [f].[X] = [M]\ ) \\
&= \gamma(1_M) && \\
&= \gamma(1_f).\gamma(1_X) && (\text{since } 1_M = 1_f.1_X\ )
\end{aligned}$$

Thus we get that $c(T_f).[f].\gamma(1_X) = \gamma(1_f).\gamma(1_X)$. Now, since the operation $(\ ).\gamma(1_X)$ is an isomorphism, we conclude that $c(T_f).[f] = \gamma(1_f)$. $\square$

Remark (3.9). We will here use Brasselet's Grothendieck transformation of bivariant theories to define compatible Radon transforms of constructible functions and homology groups for more general divergent diagrams. Suppose that $\alpha : X \xleftarrow{p} M \xrightarrow{q} Y$

is a divergent diagram, where p is not necessarily Euler. If there is a nonzero element b_p in the bivariant group of constructible functions $\mathbf{F}(M \xrightarrow{p} X)$ then there is a homomorphism

$$\bar{b}_p : \mathcal{F}(X) \to \mathcal{F}(M)$$

defined by composing f^* with multiplication by b_p. We define a Radon transform of constructible functions as the composition:

$$\mathcal{F}^{Rad}_{b_p}(\alpha) : \mathcal{F}(X) \xrightarrow{\bar{b}_p} \mathcal{F}(M) \xrightarrow{q_*} \mathcal{F}(Y)$$

This Radon transform is of course dependent on the element b_p. However, for a certain class of morphisms, named "Sans éclatement en codimension 0" in French by Henry, Merle and Sabbah [13], there is a unique constructible function b_p in $\mathbf{F}(M \xrightarrow{p} X)$ such that, over a Zariski open dense set of M, the constructible function b_p is equal to one. If p is Euler then $b_p = 1_p$ and $\mathcal{F}^{Rad}_{b_p} = \mathcal{F}^{Rad}$. The Grothendieck transformation γ applied to b_p is a bivariant class $\gamma(b_p)$ in $\mathbf{H}(M \xrightarrow{p} X)$. The class $\gamma(b_p)$ induces a homomorphism

$$\overline{\gamma(b_p)} : H_*(X;\mathbb{Z}) \to H_*(M;\mathbb{Z})$$

compatible with $\bar{b}_p$. Define a Radon transform of homology groups as the composition:

$$H^{Rad}_{b_p}(\alpha) : H_*(X) \xrightarrow{\overline{\gamma(b_p)}} H_*(M) \xrightarrow{q_*} H_*(Y)$$

The axioms of Grothendieck transformations [12, §2], imply the commutativity of the following diagram:

$$\begin{tikzcd}
\mathcal{F}(X) \arrow[r, "C_*"] \arrow[d, "\overline{b_p}"'] & H_*(X) \arrow[d, "\overline{\gamma(b_p)}"] \\
\mathcal{F}(M) \arrow[r, "C_*"] \arrow[d, "q_*"'] & H_*(M) \arrow[d, "q_*"] \\
\mathcal{F}(Y) \arrow[r, "C_*"] & H_*(Y)
\end{tikzcd}$$

Therefore the two Radon transformations $\mathcal{F}^{Rad}_{b_p}(\alpha)$ and $H^{Rad}_{b_p}(\alpha)$ are compatible via C_*, and if p is Euler then $H^{Rad}_{b_p}(\alpha)$ is equal to the homological Verdier-Radon transform $H^{V-Rad}(\alpha)$.

4. A formula for the Grassmannian homological Radon transform of the Chern-Mather class.

As mentioned in the introduction, Ernström [7,8] studied the topological Radon transformation of the Grassmannian divergent diagram $\Gamma_k : P^N \xleftarrow{p} I_k \xrightarrow{q} Gr_k(P^N)$. Given a reduced subvariety X of P^N, he gave an explicit description of the image of the local Euler obstruction Eu_X, a constructible function on P^N, under the topological Radon transformation $\mathcal{F}^{Rad}(\Gamma_k)$. The k-dual variety $X^{<k>}$ is defined as the closure in $Gr_k(P^N)$ of the following set

$$\{L \in Gr_k(P^N) | \text{ there is a point } x \in X_{smooth} \cap L, \text{ and a hyperplane } H \supset TX_x,\ H \supset L\ \}.$$

Let $e^{<k>}$ be the generic value of $\mathcal{F}^{Rad}(Eu_X)$; i.e., $e^{<k>} = \chi(X \cap L, Eu_X)$ for a generic k-plane L. Set $n = \dim X$ and $n^{<k>} = \dim(X^{<k>})$. The main result is the following formula.

Theorem (4.1). ([8, Theorem (3.2)])

$$\mathcal{F}^{Rad}(Eu_X) = e^{<k>} 1_{Gr_k(P^N)} + (-1)^{n+k(N-k)-n^{<k>}} Eu_{X^{<k>}}$$

The special case when $k = N-1$ in the above formula is an affirmative solution to a conjecture due to Viro [18, 6D, p.132].

Definition (4.2). For a constructible function α on a variety X the integer $\chi(X, \alpha)$ shall be called

the topological Euler characteristic of the constructible function α:

$$\chi(X, \alpha) := \int_X C_*(\alpha), \qquad \text{the degree of the 0-th component of } C_*(\alpha).$$

Here $C_* : \mathcal{F}(X) \to H_*(X; \mathbb{Z})$ is the Chern-Schwartz-MacPherson homomorphism.

In particular, $\chi(X, Eu_X)$ is the degree of the 0-th component of the Chern-Mather class $C_M(X)$ of the variety X.

Lemma (4.3). ([7, Proposition 4.13]) *Let α be a constructible function on P^N. Then*

$$\chi(Gr_k(P^N), \mathcal{F}^{Rad}(\alpha)) = \binom{N}{k} \chi(P^N, \alpha).$$

The following formula (a generalized Plücker relation) follows from Theorem (4.1) and Lemma (4.3).

Theorem (4.4). ([7,Theorem (4.14)]) *Let X be a closed subvariety of P^N. Then we have*

$$\binom{N}{k} \chi(X, Eu_X) = e^{<k>} \binom{N+1}{k+1} + (-1)^{n+k(N-k)-n^{<k>}} \chi(X^{<k>}, Eu_{X^{<k>}}).$$

By the definition of the topological Euler characteristic of constructible functions, the above Plücker formula may be written as follows:

$$\binom{N}{k} \int C_M(X) = e^{<k>} \binom{N+1}{k+1} + (-1)^{n+k(N-k)-n^{<k>}} \int C_M(X^{<k>}),$$

or

$$\int C_M(X^{<k>}) = (-1)^{n+k(N-k)-n^{<k>}} \{ \binom{N}{k} /int C_M(X) - e^{<k>} \binom{N+1}{k+1} \},$$

where $\int C_M(Z)$ denotes the degree of the 0-th component of the Chern-Mather class $C_M(Z)$. This means that the degree of the 0-th component of the Chern-Mather class of the k-th dual variety $X^{<k>}$ can be described via the degree of the 0-th component of the Chern-Mather class of the source variety X. What we want to do is to describe the total Chern-Mather class of the k-th dual variety $X^{<k>}$ via the total Chern-Mather class of the source variety X. This naive wish was the very start of the present work. Now that we have Theorem (3.7), the solution for this problem follows immediately:

Theorem (4.5). *Let X be an n-dimensional reduced subvariety of P^N. Then we can describe the image of the total Chern-Mather class of the k-dual variety $X^{<k>}$ in the Grassmannian $Gr_k(P^N)$ as follows:*

$$\begin{aligned}&(i_{X^{<k>}})_* C_M(X^{<k>}) = \\ &\qquad (-1)^{n+k(N-k)-n^{<k>}}\{q_*(c(T_p)\cap p^!(i_X)_*C_M(X)) - e^{<k>}C_M(Gr_k(P^N))\},\end{aligned}$$

where $i_{X^{<k>}} : X^{<k>} \to Gr_k(P^N)$ and $i_X : X \to P^N$ are the inclusion maps.

Proof. It follows from Theorem (4.1) and Theorem (3.7). Note that $p : I_k \to P^N$ is Euler, because it is a locally trivial fibration with non-singular fibers. □

Remark (4.6). Theorem (4.4) is a special case of Theorem (4.5). This follows by considering the 0-th component of the formula in Theorem(4.5), and the fact that the top Chern class of T_p is equal to the Euler characteristic of the fibers of p; that is $\binom{N}{k}$.

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