Vassiliev type invariants for generic mappings, revisited

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Abstract. We discuss Vassiliev-type invariants for isotopy classes of generic maps $M^m \to \mathbb{R}^n$ $(m, n \geq 2)$ using a kind of infinite dimensional Alexander duality. First we describe a general framework based on Mather's stratification of the space $C^\infty(M, \mathbb{R}^n)$ of smooth mappings, and then deal with *finite type invariants* for generic maps in an axiomatic way. Also we mention about a relation with characteristic classes of manifold-bundles with fiber M as a sort of degeneracy loci problem.

Introduction

In this short note, we revisit an 'old and new' theme on *the topology of the discriminant hypersurface* Γ *in the infinite dimensional space* $\mathcal{M}$ *of smooth mappings from an* m*-dimensional manifold* M *to Euclidean space*, initiated by R. Thom [**27, 28**],

$$\mathcal{M} := C^\infty(M, \mathbb{R}^n) \supset \Gamma := \{C^\infty \text{ unstable maps}\}.$$

In particular we are interested in

(A) $H^0(\mathcal{M} - \Gamma)$ as the space of all isotopy invariants of generic maps;
(B) the space $\mathcal{M}$ as a linear representation of the group $\operatorname{Diff} M$;
(C) the space of all invariants of smooth oriented closed d-manifolds belonging to a fixed cobordism class.

The topic (A) leads us to the theory of **Vassiliev-type invariants for generic maps** [**30, 3, 6, 7, 20, 31, 22, 5, 9, 13, 14**]. We focus on an elementary common feature of the theory in general dimensions m, n of the source and target manifolds greater than or equal to 2. That would be much different from the theory of knot invariants. For instance, according to the $\mathcal{A}$-classification theory of map-germs, *first order invariants* of generic maps $M^m \to \mathbb{R}^n$ $(m, n \geq 2)$ can be quite rich, while the first order invariant for knots $S^1 \to \mathbb{R}^3$ is trivial.

First we overview a general framework based on an invariant stratification of the mapping space $\mathcal{M}$ due to J. Mather [**18**] and a characteristic spectral sequence due to V. A. Vassiliev [**29**]. Second, in case of $m, n \geq 2$, *naïve finite type invariants for generic maps* are introduced in an axiomatic way similar to the case of knots (Definition 3.2). We then show that such naïve finite type invariants are basically reduced to polynomials of first order invariants (Theorem 3.7). In some particular dimensions (e.g. in case of generic immersions of a surface into $\mathbb{R}^4$), similar results

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have been reported in [**14, 13, 9**], but the common essential reason has not been quite clear so far. Our purpose is to figure it out in most general form.

The main point is as follows: (i) The *transverse k-th self-intersection locus* of Γ consists of irreducible components $\Gamma(\boldsymbol{a})$ labeled by k-tuples $\boldsymbol{a} = (\alpha_j)$ of local singularity types of generic 1-parameter families of maps, and each $\Gamma(\boldsymbol{a})$ turns to be *connected but not simply-connected* in general; (ii) the value of the jump $\nabla^k v$ of any invariant v of order k vanishes on *coherence relations for local invariants*, i.e., the relations determined by local data of adjacencies around singularity types of codimension 2. Contrarily, in case of knots [**30**], (i) the k-th self-intersection locus consists of many contractible components labeled by different 'k-chord diagrams', and (ii) the value of the jump $\nabla^k v$ vanishes on the so-called 1- *and* 4-*term relations*, which are in fact the coherence relations associated to adjacencies to cusps and triple points, respectively. That is to say, in our case, the complexity of combinatorics among local singularity types much increases, while the complexity of chord-diagrams (=configurations of unstable singular points) is much simplified, because such a configuration space on M is connected. The lost information must be hidden beyond the first cohomology groups $H^1(\Gamma(\boldsymbol{a}))$, non-transverse self-intersection loci, some combinatorial data on the global configuration of critical point sets, ... etc, which may suggest further studies on true finite type invariants for generic maps.

As for (B) we describe an equivariant version of some tools used in (A) in order to obtain a geometric presentation of **characteristic classes of manifold-bundles** with fiber M and group Diff(M), based on Kazarian [**15, 16, 17**]. We will briefly discuss a few examples and proposals.

Finally, we remark about a counterpart under **global contact equivalence** (C) which has already been suggested in Thom [**27**]. For instance, consider $\mathcal{M}_0 = C^\infty(S^{n+d}, S^n)_{base}$ ($n \gg 0$) and the discriminant $\mathcal{D}$ consisting of maps having critical value at a fixed point 0. Then $H^0(\mathcal{M}_0 - \mathcal{D})$ may be regarded as the space of invariants of null-cobordant smooth closed manifolds of dimension d. In case of $d = 1$ Saeki [**23**] has worked out some details on the classification of singular fibers. In case of $d = 3$, Otsuki's theory of finite type invariants [**21**] for $\mathbb{Z}$-homology 3-spheres is the case when restricted to an open set of $\mathcal{M}_0 - \mathcal{D}$ so that $f^{-1}(0)$ have the fixed Betti numbers. There still remain further researches in this area.

All spaces and mappings are of class C^∞ throughout this paper.

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1. Mapping space and Discriminant

1.1. $\mathcal{A}$-equivalence and invariant stratification. Let M be a compact m-dimensional manifold without boundary, and N an n-dimensional manifold without boundary. Denote the space of smooth maps, equipped with C^∞ topology, by $\mathcal{M} := C^\infty(M, N)$.

We say a map $\varphi : U \to \mathcal{M}$ from a finite dimensional manifold to $\mathcal{M}$ is *smooth* if the evaluation map $M \times U \to N$ is a C^∞ map. This gives a Frechet manifold structure on $\mathcal{M}$.

The $\mathcal{A}$-*equivalence group* or the *right-left group* is the direct product of diffeomorphism groups, $\mathcal{A}_{M,N} := \mathrm{Diff}(M) \times \mathrm{Diff}(N)$, which acts on $\mathcal{M}$ by $(\varphi, \tau).f := \tau \circ f \circ \varphi^{-1}$. Put $\mathcal{A}^0_{M,N}$ to be the connected component containing (id_M, id_N).

DEFINITION 1.1. C^∞ maps $f, g : M \to N$ are $\mathcal{A}$-*equivalent* if $\mathcal{A}_{M,N}.f = \mathcal{A}_{M,N}.g$, i.e., there exists $(\varphi, \tau) \in \mathcal{A}_{M,N}$ so that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \varphi \downarrow \simeq & & \simeq \downarrow \tau \\ M & \xrightarrow{g} & N \end{array}$$

Further, f, g are C^∞-*isotopic* if $\mathcal{A}^0_{M,N}.f = \mathcal{A}^0_{M,N}.g$.

DEFINITION 1.2. We say f is a C^∞-*structurally stable map* if the orbit $\mathcal{A}_{M,N}.f$ is an open set in $\mathcal{M}$. If the pair of dimension (m, n) is in so-called *Mather's nice range*, C^∞-structurally stable maps form a residual open subset in $\mathcal{M}$, that is known as the C^∞-*Structural Stability Theorem* proved by J. Mather. In this paper we often call such f by a *generic map* for short.

We are interested in $\mathcal{A}_{M,N}$-orbits (or families of orbits) of finite codimension in $\mathcal{M}$, which are invariant Frechet submanifold of $\mathcal{M}$. By definition, f is C^∞-stable if and only if codim $\mathcal{A}_{M,N}.f = 0$. Put

$$\Gamma_\infty := \{\ f \in \mathcal{M} \mid \mathrm{codim}\ \mathcal{A}_{M,N}.f = \infty\ \}, \quad \mathcal{U}_0 := \mathcal{M} - \Gamma_\infty.$$

An $\mathcal{A}_{M,N}$-invariant subset K of $\mathcal{M}$ is called to be *pseudo-algebraic* if for any $f \in K \cap \mathcal{U}_0$ and for any smooth map $\varphi : U \to \mathcal{M}$ with $\varphi(u_0) = f$ which is transverse to the $\mathcal{A}_{M,N}$-orbit of f at u_0, the preimage $\varphi^{-1}(K) \subset U$ is a semi-algebraic subset in some local coordinate centered at u_0. A pseudo-algebraic set K is said to be of codimension s if the semi-algebraic subset $\varphi^{-1}(K)$ is of codimension s around u_0.

THEOREM 1.3. (Mather [**18**])
Assume that (m, n) belongs to the nice range. Then Γ_∞ has infinite codimension, and there exists a filtration

$$\mathcal{M} \supset \Gamma := \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_s \supset \cdots \supset \Gamma_\infty,$$

by $\mathcal{A}_{M,N}$-invariant closed pseudo-algebraic subsets Γ_s of codimension s such that it admits a topologically locally trivial fibration $\pi_s : \Gamma_s - \Gamma_{s+1} \to Y_s$ so that each fiber is an $\mathcal{A}_{M,N}$-orbit and Y_s is a finite dimensional manifold.

REMARK 1.4. 1) In Theorem 1.3, $\Gamma_s - \Gamma_{s+1}$ is a disjoint union of single $\mathcal{A}_{M,N}$-orbits of codimension s or moduli strata of $\mathcal{A}_{M,N}$-orbits whose base (parameter) spaces are components of Y_s, see §1.2 below.
2) By the definition, the complement $\mathcal{M} - \Gamma$ consists of all stable maps. An isotopy invariant of stable maps is a locally constant function over $\mathcal{M} - \Gamma$, thus we regard the 0-th singular cohomology group $H^0(\mathcal{M} - \Gamma)$ as the space of all isotopy invariants of stable maps.
3) The ranks of $H^0(\mathcal{M} - \Gamma)$ and of $H^0(\Gamma_s - \Gamma_{s+1})$ are at most countable.

1.2. Multi-singularity types. We outline the construction of the $\mathcal{A}_{M,N}$-invariant filtration in Theorem 1.3.

A *multi-germ* is a map-germ $\varphi : M, S \to N, p$ at finite points S mapped to a single point $p = \varphi(S)$ (including a *mono*-germ as the case when S consists of a single point). We denote by $\alpha, \beta, \cdots$, $\mathcal{A}$-equivalent classes (or $\mathcal{A}$-moduli families) of multi-germs, and also denote by $\boldsymbol{a} = (\alpha_1, \cdots, \alpha_l)$ an l-tuple of those $\mathcal{A}$-classes of multi-germs. The *codimension* $|\boldsymbol{a}|$ is given by the sum of $\mathcal{A}_e$-codimension of α_j. Note that α is a stable (multi-)singularity if and only if its $\mathcal{A}_e$-codimension $= 0$.

We say $f : M \to N$ *has multi-singularities of type* $\boldsymbol{a}$ (at S) if there are mutually disjoint finite subsets $S_1, \cdots, S_l$ in M ($S = \coprod S_j$) and l distinct points $p_1, \cdots, p_l \in N$ so that $f(S_j) = p_j$ and the germ $f : M, S_j \to N, p_j$ is of type α_j for each j.

For a multi-singularity type α of codimension ≥ 1, we put

$$\Gamma(\alpha) := \mathrm{Cl}\,\{\ f \in \Gamma \mid f \text{ has a multi-singularity of type } \alpha\ \}$$

(Cl stands for the closure), and for a l-tuple $\boldsymbol{a} = (\alpha_1, \cdots, \alpha_l)$ with $|\alpha_j| \geq 1$

$$\Gamma(\boldsymbol{a}) := \mathrm{Cl}\,\{\ f \in \Gamma \mid f \text{ has multi-singularities of type } \boldsymbol{a}\ \} = \bigcap_{j=1}^{l} \Gamma(\alpha_j).$$

The locus Γ_s in Theorem 1.3 is given by

$$\Gamma_s := \bigcup \Gamma(\boldsymbol{a})$$

taken over all multi-singularity types $\boldsymbol{a}$ of codimension $\geq s$. In particular we obtain a stratification

$$\Gamma_s - \Gamma_{s+1} = \bigsqcup_{|\boldsymbol{a}|=s} (\Gamma(\boldsymbol{a}) - \Gamma_{s+1}).$$

Note that

- $\Gamma(\boldsymbol{a})$ is a closed pseudo-algebraic subset of codimension s;
- $f \in \Gamma(\boldsymbol{a}) - \Gamma_{s+1}$ if and only if f has multi-singularities of type $\boldsymbol{a}$ at some S, any point in $f^{-1}(f(S)) - S$ is not critical, and f is infinitesimally C^∞-stable at any finite points in $M - S$;
- $\Gamma(\boldsymbol{a}) - \Gamma_{s+1}$ is a Frechet submanifold of codimension s in $\mathcal{M}$ having possibly countably many connected components: Each component is a single $\mathcal{A}^0_{M,N}$-orbit (if $\boldsymbol{a}$ consists of simple singularity types), or a moduli family of orbits (if $\boldsymbol{a}$ contains a moduli singularity type);
- $\Gamma(\boldsymbol{a}) - \Gamma_{s+1}$ is the transverse intersection of submanifolds $\Gamma(\alpha_j) - \Gamma_{|\alpha_j|+1}$ in $\mathcal{M}$.

The filtration in Theorem 1.3 is obtained by the above definitions and properties. A technical detail in the proof belongs to the geometry of real algebraic group action [**18**].

Definition 1.5. We say a multi-singularity type α is *coorientable* if the normal bundle of $\Gamma(\alpha) - \Gamma_{|\alpha|+1}$ is orientable. Also $\boldsymbol{a} = (\alpha_1, \cdots, \alpha_l)$ is coorientable if each α_j is so.

Definition 1.6. We define the *s-th transverse self-intersection locus of the discriminant hypersurface* Γ to be a union of $\Gamma(\boldsymbol{a})$ of all s-tuples $\boldsymbol{a} = (\alpha_1, \cdots, \alpha_s)$ with $|\alpha_j| = 1$ ($\forall j$); The rest is the *non-transverse* self-intersection locus of Γ, that is a union of $\Gamma(\boldsymbol{a})$ for $\boldsymbol{a}$ where some α_j has codimension greater than one.

Example 1.7. Any $f \in \Gamma_2 - \Gamma_3$ has either
- only a couple of multi-singularities of codim $= 1$ with distinct two critical values (i.e., f belongs to the transverse double point locus of Γ); or
- only one multi-singularity of codim $= 2$ with a single critical value (i.e., f belongs to the non-transverse intersection locus of Γ).

For instance, if $m = 1$ and $n = 3$, a map in the former case is an immersion with two double points (two crossing changes), and a map in the latter case is an embedding except for one ordinary cusp point.

2. Vassiliev complex

2.1. Characteristic spectral sequence. Let us think of the situation as in Theorem 1.3. Put

$$\mathcal{U}_s := \mathcal{M} - \Gamma_{s+1},$$

then we have invariant open subsets of $\mathcal{M}$:

$$\mathcal{M} - \Gamma = \mathcal{U}_0 \subset \mathcal{U}_1 \subset \mathcal{U}_2 \subset \cdots \subset \mathcal{U}_s \subset \cdots \subset \mathcal{U}_\infty \subset \mathcal{M}.$$

The cohomology spectral sequence associated to this filtration is called the *Vassiliev-Kazarian spectral sequence for* $\mathcal{M}$ [**15, 29, 30**]. The E_1-term is

$$E_1^{s,t} := H^{s+t}(\mathcal{U}_s, \mathcal{U}_{s-1}; R) \overset{\simeq}{\longleftarrow} H^t(\Gamma_s - \Gamma_{s+1}; R)$$

with a coefficient ring R. The arrow indicates the *Alexander duality for functional spaces* in the sense of Eells [**8**], that is, the Thom isomorphism for *coorientable* components of the s-codimensional Frechet manifold $\Gamma_s - \Gamma_{s+1}$ (for a non-coorientable connected component the Thom class within integer coefficients vanishes, but it works within $\mathbb{Z}_2$-coefficients). Thus $E_1^{s,0}$ is the R-module generated by coorientable connected components in $\Gamma_s - \Gamma_{s+1}$, especially,

$$E_1^{0,0} = H^0(\mathcal{U}_0) = H^0(\mathcal{M} - \Gamma).$$

The first cochain complex is

$$0 \to E_1^{0,t} \to E_1^{1,t} \to E_1^{2,t} \to \cdots$$

where the operator $d_1 : E_1^{s,t} \to E_1^{s+1,t}$ is the connection homomorphism ∂ of cohomology exact sequence for the triple $(\mathcal{U}_{s+1}, \mathcal{U}_s, \mathcal{U}_{s-1})$. As usual, we put for $r \geq 1$,

$$E_{r+1}^{s,t} := \frac{\ker[d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}]}{\mathrm{Im}\,[d_r : E_r^{s-r,t+r-1} \to E_r^{s,t}]}$$

with d_{r+1}, and we have a natural homomorphism $E_\infty^{s,t} \to H^*(\mathcal{M})$.

Instead, we may take

$$0 \to E_1^{1,t} \to E_1^{2,t} \to E_1^{3,t} \to \cdots$$

and $E_r^{s,t}$ $(s \geq 1)$, that approximates $H^*(\mathcal{U}, \mathcal{U}_0) = H^*(\mathcal{M}, \mathcal{M} - \Gamma)$, the cohomology with support on Γ.

Remark 2.1. The main technical invention in Vassiliev's original approach [**30**] is a *simplicial resolution* $\Gamma' \to \Gamma$ of the discriminant hypersurface, constructed in $\mathcal{M} \times \mathbb{R}^\infty$ in a purely combinatorial way. A natural filtration of Γ' produces a spectral sequence converging to a subspace in $H^0(\mathcal{M} - \Gamma)$. That spectral sequence is an entirely different one from the characteristic spectral sequence described above, so the readers should not confuse them. In this paper we wouldn't develop such a simplicial resolution technique, although it might be possible theoretically. Instead we use a more easier cochain subcomplex of our $E_1^{s,t}$ given by data of local singularity types and some additional semi-global data of configurations of singular points/values, [**29, 19, 23**]. A substitute to the 'E_1-term' A_* for the simplical resolution Γ' will be given in §3.3 by (the dual to) some cochain complex of singularities, cf. [**30, 2**].

2.2. Local Vassiliev complex of multi-singularities. Let $C^0(\mathcal{A}) = 0$ and for $s \geq 1$ let

$$C^s(\mathcal{A}) = \bigoplus R \cdot \boldsymbol{a}$$

the R-module generated by *coorientable* $\mathcal{A}$-classes $\boldsymbol{a}$ of multi-singularities of codimension s (for a technical detail, see [**29, 19**]). This is regarded as a submodule

$$C^s(\mathcal{A}) \subset E_1^{s,0} = H^0(\Gamma_s - \Gamma_{s+1}; R)$$

by identifying $\boldsymbol{a}$ with the constant function on $\Gamma(\boldsymbol{a}) - \Gamma_{s+1}$ (taking 1, otherwise 0). The coboundary operator $\partial : C^s(\mathcal{A}) \to C^{s+1}(\mathcal{A})$ is induced from d_1.

Definition 2.2. The cochain complex $(C^*(\mathcal{A}), \partial)$ is called the *local Vassiliev complex for $\mathcal{A}$-classification of multi-singularities.*

The operator $\partial : C^s(\mathcal{A}) \to C^{s+1}(\mathcal{A})$ can explicitly be written down as follows. Let $\boldsymbol{a} \in C^s(\mathcal{A})$ and $\boldsymbol{b} \in C^{s+1}(\mathcal{A})$ (we fix coorientations of $\boldsymbol{a}$, $\boldsymbol{b}$ in advance). Take a versal deformation of $\boldsymbol{b}$. On the parameter space, the bifurcation diagram $\Psi(\boldsymbol{a})$ of type $\boldsymbol{a}$ is defined: It is either empty or 1-dimensional semi-algebraic curves approaching the origin (That corresponds to that the stratum $\Gamma(\boldsymbol{a})$ is pseudo-algebraic and of codimension s). Count the *incidence coefficient* $[\boldsymbol{a}; \boldsymbol{b}]$, defined by the algebraic intersection number of $\Psi(\boldsymbol{a})$ with a small oriented sphere centered at the origin. Then $\partial \boldsymbol{a} = \sum [\boldsymbol{a}; \boldsymbol{b}]\, \boldsymbol{b}$, the sum taken over all generators $\boldsymbol{b}$.

We use the $\mathcal{A}^0$-classification when the source and target manifolds M, N are oriented. Its Vassiliev complex is denoted by $C(\mathcal{A}^0)$.

Remark 2.3. Notice that the local Vassiliev complex is determined only by the local classification of singularities. In fact, although there are possibly many connected components in each $\Gamma(\boldsymbol{a}) - \Gamma_{s+1}$, they are regarded as just 'one stratum' in the complex $C^*(\mathcal{A})$. Instead, a more finer subcomplex can be considered by adding some 'non-local' data to $C^*(\mathcal{A})$, which we provisionally call an *enriched Vassiliev complex*,

$$C^s_{en}(\mathcal{A}) \subset E_1^{s,0}.$$

The additional data in consideration would be, for instance, types of configurations of subsets $S_1, \cdots, S_k$ on M at each of which a local bifurcation occurs (in case of knots in 3-space, these data are called *chord-diagrams* or *weight systems*), the placement of the singular set in M and of the singular value sets in N, the topological types of singular fibers (if $m \geq n$), and so on. $E_1^{s,0}$ itself can be regarded as the finest refinement among all the enriched complexes.

2.3. Local invariants for generic maps. There is a natural homomorphism

$$H^s(C(\mathcal{A})) \to E_2^{s,0} \to E_\infty^{s,0} \to H^s(\mathcal{M}, \mathcal{M} - \Gamma) \to H^s(\mathcal{M}).$$

In particular, if $H^{k+1}(\mathcal{M}) = 0$, we have

$$\rho : H^{k+1}(C(\mathcal{A})) \to H^k(\mathcal{M} - \Gamma) \quad \text{modulo } H^k(\mathcal{M}).$$

This map is explained as follows. For simplicity, assume that $\mathcal{M}$ is contractible (e.g., M is connected and $N = \mathbb{R}^n$). Let $c = \sum \lambda_{\boldsymbol{a}} \cdot \boldsymbol{a}$ be a cocycle of $C^{k+1}(\mathcal{A})$, and D^{k+1} and $S^k = \partial D^{k+1}$ denote the standard oriented disk and the sphere. Given any smooth map $f : S^k \to \mathcal{M} - \Gamma$, we can take a smooth map $\gamma : D^{k+1} \to \mathcal{M}$ which is an extension of f, i.e., $\gamma|_{\partial D^{k+1}} = f$, and is transverse to Γ_{k+1}. Then the value of $\rho([c])$ on the cycle $[f]$ is given by

$$\rho([c])([f]) = \sum \lambda_{\boldsymbol{a}} \cdot \sharp\, [\Gamma(\boldsymbol{a}) \cap \gamma].$$

Here $\sharp\, [\Gamma(\boldsymbol{a}) \cap \gamma]$ stands for the intersection number, that is, the number of points $b \in D^{k+1}$ taking into account the sign ± 1 so that the C^∞-map $\gamma_b : M \to N$ admits multi-singularities of type $\boldsymbol{a}$ at some finite points in M. The sign means the coincidence between a given orientation of D^{k+1} and the coorientation of $\boldsymbol{a}$.

Definition 2.4. An element $\rho(c) \in H^k(\mathcal{M} - \Gamma)$ (determined up to $H^k(\mathcal{M})$) is called a *local invariant* for k-cycles in $\mathcal{M} - \Gamma$. In particular, in case of $k = 0$, a *local invariant for generic maps* is a locally constant function $v : \mathcal{M} - \Gamma \to R$ determined by some element of $H^1(C(\mathcal{A})) = \ker \partial_1$ up to constants (Goryunov [**6**]). If we take some enriched Vassiliev complex instead, then we say $v = \rho(c)$ is *semi-local* or *enriched-local*.

Example 2.5. For generic plane curves, there are Arnold's basic invariants: $J^\pm$ are local, while St is *enriched-local* in our sense, because its definition involves the cyclic order of the preimage of a triple point. There are several higher dimensional analogies to J^+, J^-, St: local invariants for $M^2 \to \mathbb{R}^3$ are related to inverse self-tangencies and quadric points [**6**]; for generic immersions $M^3 \to \mathbb{R}^5$ the Smale invariant is related [**9**], and so on. A list of some works is noted below (this list is not complete!).

(m,n)	object/singularities
$(1,1)$	functions on circle bundles/global max.-min. [**15**]
$(1,2)$	generic immersed curves, wavefronts [**2, 1**]
$(1,3)$	knots and links [**30**]
$(2,1)$	Morse function/singular fibers [**23**]
$(2,2)$	generic maps [**20, 12**]
$(2,3)$	generic maps [**6**]
$(2,4)$	generic immersions [**14, 13**], ribbon knots [**11**]
$(3,2)$	generic maps [**31**]
$(3,3)$	geneirc maps [**22, 7**]
$(3,4)$	generic immersions [**5**]
$(3,5)$	generic immersions [**9**]
$(4,3)$	generic maps/singular fibers [**23, 32**]

Example 2.6. For instance, look at the case of $m = n = 2$, studied in [**20**]. In that case, stable singularity types (codimension 0) are of fold, cusps and double folds, and there are 10 multi-singularities of codimension one and 20 multi-singularities of codimension two (Here couples of codimension one singularities are omitted, since they do not effect coherence relations for local invariants):

$$0 \longrightarrow C^1(\mathcal{A}_{2,2}) \simeq \mathbb{Z}^{10} \xrightarrow{\partial} C^2(\mathcal{A}_{2,2}) \simeq \mathbb{Z}^{20}$$

It turns out that $H^1(C(\mathcal{A});\mathbb{Z}) = \ker\partial$ has rank 3 (the generators are denoted by ΔI_i $(i = 1,2,3)$ in [**20**]) and local invariants (over $R \ni \frac{1}{4}$) are generated modulo constants by

$$\begin{array}{ll} \Delta c = 2\Delta I_1 : & \sharp \text{ of cusps;} \\ \Delta d = \Delta I_2 + \Delta I_3 : & \sharp \text{ of double folds;} \\ \Delta\nu = \Delta I_1 - \Delta I_2 + \Delta I_3 : & \text{projective Bennequin invariant.} \end{array}$$

The last one is an interesting invariant for apparent contours obtained as a variant of the Bennequin number for Legendre curves (e.g., [**4**] for a computer program to compute $\Delta\nu$).

When we take the $\mathcal{A}^0$-equivalence instead of $\mathcal{A}$, Δc breaks into two independent invariants $\Delta c_\pm$ for the numbers of positive and negative cusps.

Further, note that type B (beak-to-beak) of codimension one can be separated into two types according to how components of contour curves are mutually connected. This yields an *enriched-local* invariant [**12**]:

$$\Delta I_4 = \Delta l - \Delta b_1 + \Delta b_2 : \ \sharp \text{ of components of critical set } C(f).$$

3. Finite type invariants for generic maps

3.1. A naïve approach. A path transverse to the discriminant Γ is a smooth homotopy which causes a generic local bifurcation, say of type α. We denote such a path by

$$\Xi^{(\alpha)} : [-1,1] \longrightarrow \mathcal{M},\ \epsilon_1 \mapsto \Xi^{(\alpha)}_{\epsilon_1},$$

so that it is transverse to $\Gamma(\alpha) - \Gamma_2$ at $\epsilon = 0$. For *cooriented* α, we always assume that $\Xi^{(\alpha)}$ is *compatible with the coorientation*, i.e., the path is directed to the given normal orientation.

Let $\boldsymbol{a} = (\alpha_1, \cdots, \alpha_s)$, all α_j being of codimension one. A normal slice to $\Gamma(\boldsymbol{a}) - \Gamma_{s+1}$ is a smooth family of maps which causes local bifurcations of type α_j $(j = 1, \cdots, s)$ around s distinct points in N independently. We denote it by

$$\Xi^{\boldsymbol{a}} : [-1,1]^s \longrightarrow \mathcal{M},\ (\epsilon_1, \cdots, \epsilon_s) \mapsto \Xi^{\boldsymbol{a}}_{\epsilon_1\cdots\epsilon_s}$$

so that $\Xi^{\boldsymbol{a}}_{\epsilon_1\cdots\epsilon_s} \in \Gamma(\alpha_i)$ if and only if $\epsilon_i = 0$. If $\boldsymbol{a}$ is cooriented, $\Xi^{\boldsymbol{a}}$ is assumed to be compatible with the coorientation for each α_j.

Example 3.1. In case of maps $S^1 \to \mathbb{R}^3$, there is only one singularity type of codimension one, $\delta :=$ *crossing change*. Then Ξ^{δ^s} for the s-tuple of δ means a family which causes crossing changes at s distinct points in $\mathbb{R}^3$ independently; In particular $\Xi^{\delta^s}_0 : S^1 \to \mathbb{R}^3$ is an immersion with s double points. In case of maps $M^2 \to \mathbb{R}^2$, there arise 10 different types of singularities of codimension one (Example 2.5).

To define 'finite type invariants' in general case, we take in mind an analogy to *partial derivatives* of functions in several variables.

Definition 3.2. For a locally constant function $v : \mathcal{M} - \Gamma \to R$, we define the *$s$-th partial derivative* $\nabla^{\boldsymbol{a}} v := \nabla^{\alpha_1} \cdots \nabla^{\alpha_s} v$ *with respect to* $\boldsymbol{a} = (\alpha_1, \cdots, \alpha_s)$, $|\alpha_j| = 1$, to be a locally constant function over $\Gamma(\boldsymbol{a}) - \Gamma_{s+1}$ given by

$$\nabla^{\boldsymbol{a}} v\,(\Xi^{\boldsymbol{a}}_{0\cdots0}) := \sum \epsilon_1 \cdots \epsilon_s \cdot v(\Xi^{\boldsymbol{a}}_{\epsilon_1\cdots\epsilon_s}),$$

the sum taken over 2^s combinations of $\epsilon_i = \pm 1$. We say that v is an invariant *of order at most* r if the following condition (T) is satisfied:

$$\nabla^{\boldsymbol{a}} v = 0 \qquad (\forall \boldsymbol{a} \text{ of codimension } s \geq r+1). \qquad (T)$$

In particular, in case that $m \geq 2$, such an invariant is called a *naïve finite type invariant.*

REMARK 3.3. The partial derivative

$$\nabla^{\alpha} v(\Xi_0) = v(\Xi_+) - v(\Xi_-)$$

defines an operator $\nabla^{\alpha} : H^0(\Gamma(\boldsymbol{a}) - \Gamma_{s+1}) \to H^0(\Gamma(\boldsymbol{a} \cup \alpha) - \Gamma_{s+2})$.

Denote by V_r the R-module generated by invariants of order $\leq r$:

$$V_0 \subset V_1 \subset \cdots \subset V_\infty := \bigcup_{r=0}^{\infty} V_r \subset H^0(\mathcal{M} - \Gamma; R).$$

Obviously, an invariant of order zero is a function which is constant over each connected component of $\mathcal{M}$, therefore $V_0 = H^0(\mathcal{M}; R)$.

LEMMA 3.4. *V_∞ is a graded R-algebra: If $v_1 \in V_r$ and $v_2 \in V_\ell$, then $v_1 v_2 \in V_{r+\ell}$, where $v_1 v_2(f) := v_1(f) v_2(f)$ for $f \in \mathcal{M} - \Gamma$.*

PROOF. Similarly as in the case of knots, a Leibniz type formula holds:

$$\begin{aligned} \nabla^{\alpha}(v_1 v_2)(\Xi_0) &= v_1(\Xi_+) v_2(\Xi_+) - v_1(\Xi_-) v_2(\Xi_-) \\ &= v_1(\Xi_+)(v_2(\Xi_+) - v_2(\Xi_-)) + (v_1(\Xi_+) - v_1(\Xi_-)) v_2(\Xi_-) \\ &= v_1(\Xi_+) \cdot \nabla^{\alpha} v_2(\Xi_0) + \nabla^{\alpha} v_1(\Xi_0) \cdot v_2(\Xi_-). \end{aligned}$$

By the induction, we observe that any value of $\nabla^{\boldsymbol{a}} v_1 v_2$ is written by a linear combination $\nabla^{\boldsymbol{a}'} v_1 \nabla^{\boldsymbol{a}''} v_2$ with $\boldsymbol{a} = \boldsymbol{a}' \cup \boldsymbol{a}''$. □

3.2. (Non-)transverse self-intersection. Let $\boldsymbol{a} = (\alpha_j)$, $|\alpha_j| = 1$. In general the manifold $\Gamma(\boldsymbol{a}) - \Gamma_{s+1}$ has many connected components separated by the discriminant hypersurface $\Gamma(\boldsymbol{a}) \cap \Gamma_{s+1}$. Any transverse self-intersection locus $\Gamma(\boldsymbol{a} \cup \alpha)$, where $|\alpha| = 1$, is a top stratum of $\Gamma(\boldsymbol{a}) \cap \Gamma_{s+1}$, but it can happen that some non-transverse self-intersection locus also becomes a top stratum. Provisionally we make an additional strong condition:

DEFINITION 3.5. We say that an invariant $v : \mathcal{M} - \Gamma \to R$ satisfies the *non-transverse loci condition* (NT) if for any $\boldsymbol{a} = (\alpha_1, \cdots, \alpha_s)$ with $|\alpha_j| = 1$ the value of $\nabla^{\boldsymbol{a}} v$ does not change when passing through any *non-transverse* self-intersection loci of codimension one in $\Gamma(\boldsymbol{a})$. We denote by V_r' ($r \geq 1$) the subspace of V_r consisting of invariants satisfying (NT) (for $r = 0$, $V_0' = V_0$).

LEMMA 3.6. *Any polynomial of local invariants is a naïve finite type invariant with (NT).*

PROOF. By Lemma 3.4, it is enough to see the case $r = 1$. If $v_c : \mathcal{M} - \Gamma \to R$ is a local invariant corresponding to a Vassiliev 1-cocycle $c = \sum_\alpha c_\alpha \cdot \alpha$, then $\nabla^{\alpha} v_c$ is a *constant function* with value c_α over $\Gamma(\alpha) - \Gamma_2$. Obviously v_c is of first order and satisfies (NT), since any jump of $\nabla^{\alpha} v_c$ on Γ_2 is zero. □

The converse is also true: We will show that *our operators* ∇^{α} *are linearly dependent* in the relation corresponding to the coherence condition $\partial = 0$ of local invariants: Let $R = \mathbb{Q}$ for simplicity.

THEOREM 3.7. *Let $m, n \geq 2$ and the pair (m, n) be in Mather's nice range. Let M be connected, oriented and of dimension m, and $N = \mathbb{R}^n$, oriented. Then,*

- *If $n > m+1$, naïve finite type invariants are polynomials of local invariants modulo constants:*
$$V_r/V_{r-1} = \mathrm{Sym}^r(V_1/V_0), \qquad V_1/V_0 = H^1(C(\mathcal{A}^0)).$$
- *If $n \leq m+1$, naïve finite type invariants with non-transverse loci condition are polynomials of local invariants modulo constants.*

REMARK 3.8. A similar theorem has already been known by different arguments for generic immersions in cases of $n = 2m(\geq 4)$ in [**14, 13**], and for $n = 2m-1(\geq 5)$ in [**9**]: For instance, naïve finite type invariants of generic immersions $M^2 \to \mathbb{R}^4$ are generated by the number of double points and the normal Euler number [**14**]. Notice that when taking n much less than $2m - 1$, the number of unavoidable singularities rapidly increases and the situation would be much more complicated.

3.3. Irreducibility. An elementary fact is that *the locus $\Gamma(\boldsymbol{a})$ is irreducible* in the sense that any $f_0, f_1 \in \Gamma(\boldsymbol{a}) - \Gamma_{s+1}$ are joined by generic paths within the locus $\Gamma(\boldsymbol{a})$:

LEMMA 3.9. *Under the same assumption as in Theorem 3.7, the following properties hold:*

(1) *Let $v \in V_r'$, then $\nabla^{\boldsymbol{a}} v$ is a constant for each r-tuple $\boldsymbol{a}$.*
(2) *If $n > m+1$, $V_r' = V_r$.*
(3) *$V_1'/V_0' = H^1(C(\mathcal{A}^0))$.*

PROOF. (1) Let $v \in V_r'$. By definition the r-th partial derivative $\nabla^{\boldsymbol{a}} v$ is a locally constant function $\Gamma(\boldsymbol{a}) - \Gamma_{r+1} \to \mathbb{Q}$. We show that $\nabla^{\boldsymbol{a}} v$ is constant.

Let $f_0 \in \Gamma(\boldsymbol{a}) - \Gamma_{r+1}$, which has multi-singularities of type $\boldsymbol{a}$ at $S \subset M$ and $T := f_0(S) \subset \mathbb{R}^n$ and no more complicated singularities. Put
$$\Gamma(\boldsymbol{a}; f_0) := \{\, g \in \mathcal{M} \mid g = f_0 \text{ on some neighborhood of } S \,\},$$
which is an affine subspace of $\mathcal{M}$ contained in $\Gamma(\boldsymbol{a})$. For any $g \in \Gamma(\boldsymbol{a}; f_0) - \Gamma_{r+1}$, we can take a generic path γ_t from f_0 to g in $\Gamma(\boldsymbol{a}; f_0)$. Since v satisfies conditions (NT) and (T), any jump of $\nabla^{\boldsymbol{a}} v$ along γ_t is zero, hence $\nabla^{\boldsymbol{a}} v(f_0) = \nabla^{\boldsymbol{a}} v(g)$. Now take another $f_1 \in \Gamma(\boldsymbol{a}) - \Gamma_{r+1}$ with multi-singularities of type $\boldsymbol{a}$ at some S_1 and $T_1 = f_1(S_1)$. Since M^m and $\mathbb{R}^n$ are connected and $m, n \geq 2$, there exists an 1-parameter family $(\sigma_t, \tau_t) \in \mathcal{A}^0_{M,N}$ so that

- $\sigma_0 = id_M$ and $\tau_0 = id_{\mathbb{R}^n}$ (trivial),
- $\sigma_1(S_1) = S$, $\tau_1(T_1) = T$,
- $g_1 := \tau_1 \circ f_1 \circ \sigma_1^{-1}$ coincides with f on a neighborhood of S.

Since f_1 and g_1 are isotopic and $g_1 \in \Gamma(\boldsymbol{a}; f_0) - \Gamma_{r+1}$, we have
$$\nabla^{\boldsymbol{a}} v(f_1) = \nabla^{\boldsymbol{a}} v(g_1) = \nabla^{\boldsymbol{a}} v(f_0).$$
Thus $\nabla^{\boldsymbol{a}} v$ is constant on $\Gamma(\boldsymbol{a}) - \Gamma_{r+1}$.
(2) Let $n > m+1$. Then, in the proof of (1) above, we can choose the path γ_t so that the image $\gamma_t(M - S)$ does not pass through any point of $T = f_0(S)$, i.e., γ_t does not create any new multi-singularity mapped to a point of T. Namely, the path γ_t meets only transverse self-intersection locus of Γ, thus $V_r = V_r'$.

(3) Let $v \in V_1'$. It follows from (1) that $\nabla^\alpha v$ is constant for each α of codimension one. Put

$$c_v := \sum_\alpha (\nabla^\alpha v) \cdot \alpha \in C^1(\mathcal{A}^0).$$

Since v is a locally constant function $\mathcal{M} - \Gamma \to \mathbb{Q}$, the coherence relation around each $\Gamma(\beta)$ of codimension 2 is automatically satisfied:

$$\sum_\alpha [\alpha; \beta] \cdot \nabla^\alpha v = 0, \qquad (*)$$

i.e., $\partial c_v = 0$, that means that v is a local invariant modulo constant. Lemma 3.6 shows local implies order one, thus the claim follows. □

Proof. (Theorem 3.7): By Lemma 3.9 (1), one can define a functional $\nabla^{\boldsymbol{a}} : V_r' \to \mathbb{Q}$ for each r-tuple $\boldsymbol{a}$. Note that $v \in V_{r-1}'$ if and only if $\nabla^{\boldsymbol{a}} v = 0$ for all r-tuples $\boldsymbol{a}$. Given a $(r-1)$-tuple $\boldsymbol{a}'$ and a singularity type β of codimension 2, the relation $(*)$ in the proof of (3) generates a linear relation

$$\sum_\alpha [\alpha; \beta] \cdot \nabla^{\boldsymbol{a}' \cup \alpha} = 0 \qquad (*_r)$$

among functionals $\nabla^{\boldsymbol{a}} = \nabla^{\boldsymbol{a}' \cup \alpha} = \nabla^\alpha \cdot \nabla^{\boldsymbol{a}'}$ (That means a coherence relation for locally constant functions $\nabla^{\boldsymbol{a}'} v$ on $\Gamma(\boldsymbol{a}') - \Gamma_r$ around $\Gamma(\boldsymbol{a}' \cup \beta)$, where $v \in V_r'$).

Consider the vector space spanned by the functionals indexed by $\boldsymbol{a}$ ($|\boldsymbol{a}| = r$), and let A_r denote the quotient modulo the relations $(*_r)$ for all $\boldsymbol{a}'$ and β:

$$A_r := \frac{\bigoplus_{|\boldsymbol{a}|=r} \mathbb{Q} \cdot \nabla^{\boldsymbol{a}}}{\text{relations } (*)_r}.$$

Then

- A natural pairing $A_r \times V_r' \to \mathbb{Q}$ induces an injective linear map
$$V_r'/V_{r-1}' \to \mathrm{Hom}(A_r, \mathbb{Q})$$
- A_r is naturally isomorphic to the $\mathbb{Q}$-vector space of homogeneous differential operators of order r on the space V_1'/V_0'.

So we have an injective linear map $V_r'/V_{r-1}' \to \mathrm{Sym}^r(V_1'/V_0')$. It follows from Lemma 3.6 that this map is surjective, thus it is an isomorphism. This completes the proof. □

Remark 3.10. Theorem 3.7 says that the naïve definition of finite type invariants for generic maps is somehow irrelevant. For improving the definition, there come up several difficulties which depend on the pair of dimensions m, n. In case of $n \le m+1$ $(m, n \ge 2)$, for instance, $(m, n) = (2, 2)$ or $(2, 3)$, the next task would be to define 'second order' invariants for stable maps from M^2 to $\mathbb{R}^2$ (or $\mathbb{R}^3$) involving non-transverse self-intersection loci. To define 'non-local' invariants, we must add some data on the global configuration of critical curves or double point curves of maps: Those data are used to identify connected components of $\Gamma(\boldsymbol{a}) - \Gamma_{s+1}$. In case of $n > m+1$ $(m \ge 2)$, for instance, $(m, n) = (2, 4)$, the difficulty is different from the above, and we need some completely new idea for 'finite type'.

Remark 3.11. Case of knots in 3-space ($n = 1$, $m = 3$).
(1) In case of knots, we should replace r-tuples $\boldsymbol{a}$ in the above argument by equivalent classes of r-chord diagrams on S^1, and A_r is generated by those classes of r-chord diagrams divided by all 1- and 4-term relations, which are a sort of coherence relations like $(*_r)$ (see (2) below). As well-known, it holds that

$$V_r/V_{r-1} = \mathrm{Hom}(A_r, \mathbb{Q}),$$

proved by M. Kontsevich. A purely combinatorial construction of knot invariants using *Actuality Table* was given by Vassiliev [**30**], that might be applicable to our setting.
(2) We have seen that if $n > m + 1$ and $m \geq 2$, (NT) is automatically satisfied. For maps $S^1 \to \mathbb{R}^3$, it's not true. *The triple point stratum* $\Gamma(\tau)$, consisting of immersions with a triple point τ, belongs to the *non-transverse* self-intersection locus of Γ in our sense of Definition 1.6. Further, any path on the strata $\Gamma(\delta^2)$ of maps with two double points may come across $\Gamma(\tau)$ generically, so we can not ignore this triple point stratum. In fact, the value of the jumps of invariants $\nabla^{\delta^2} v$ at $\Gamma(\tau)$ yields a coherence relation, that is the 4-term relation, see [**2, 30**].

4. Characteristic classes for fiber bundles

4.1. Classifying space. Let M be a compact, connected oriented manifold. We regard the affine space $\mathcal{M} = C^\infty(M, \mathbb{R}^n)$ *as a representation of the diffeomorphism group* $G = \mathrm{Diff}\, M$.

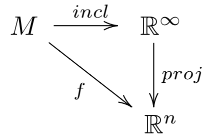
First, we shall recall a well-known construction (e.g., [**8**]) of the classifying space of the topological group $G = \mathrm{Diff}\, M$ of orientation preserving diffeomorphisms. If n is quite high, $C^\infty(M, \mathbb{R}^n) - \Gamma = \mathrm{Emb}(M, \mathbb{R}^n)$, the space of all embeddings of M in $\mathbb{R}^n$. Sending $n \to \infty$, we may identify the classifying space of G with the topological quotient

$$BG = B\,\mathrm{Diff}\, M = \mathrm{Emb}(M, \mathbb{R}^\infty)/\mathrm{Diff}\, M.$$

Put $EG = \mathrm{Emb}(M, \mathbb{R}^\infty)$, then EG is highly connected, thus the canonical map $EG \to BG$ gives the universal principal bundle for the group G. Let $BM := (EG \times M)/G$, the associated manifold-bundle with fiber M, then any manifold-bundle $E \to B$ (B paracompact), with fiber M and structure group G, can be obtained up to isomorphisms from the universal bundle $BM \to BG$ via a classifying map $\rho : B \to BG$. Any element of $H^*(BG)$ is called a universal G-characteristic class: G-characteristic classes of $E \to B$ are defined by their ρ^*-image in $H^*(B)$.

4.2. Degeneracy loci and universal polynomials. We are concerned with the "*geometric realization problem*" of a given characteristic class for a manifold-bundle $E \to B$.

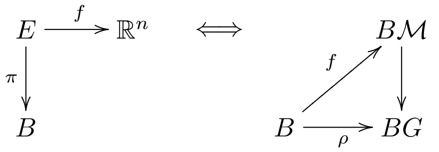
A Singularity Theory approach to this problem is as follows: Let us think of the composition of an embedding of M and a projection onto $\mathbb{R}^n$ for some n small enough,



Put $\mathcal{M} = C^\infty(M, \mathbb{R}^n)$ and $B\mathcal{M} \to BG$ to be the associated bundle with fiber $\mathcal{M}$ and group G. Now $\mathcal{M}$ is a contractible space, hence the Borel cohomology $H^*_G(\mathcal{M}) := H^*(B\mathcal{M})$ is isomorphic to $H^*_G(pt) = H^*(BG)$. The Vassiliev complex has much meanings in this equivariant setting: There is a natural homomorphism

$$Tp : H^s(C(\mathcal{A})) \to E_\infty^{s,0} \to H^s_G(\mathcal{M}) \simeq H^*(BG).$$

We denote by $Tp_c \in H^*(BG)$ the G-characteristic class associated to a Vassiliev cocycle $c = \sum \lambda_i \boldsymbol{a}_i \in C^s(\mathcal{A})$, $\partial c = 0$. This means the following: Suppose that we are given a bundle $\pi : E \to B$ with fiber M over a manifold B and a smooth map $f : E \to \mathbb{R}^n$ over the total space of the bundle. Denote by $\rho = \rho_\pi : B \to BG$ the classifying map associated to the bundle.



To any multi-singularity type $\boldsymbol{a}$ we associate the *bifurcation locus* $B_{\boldsymbol{a}}(f) \subset B$, which is a locally closed submanifold consisting of points $b \in B$ so that the map $f_b : E_b \simeq M \to \mathbb{R}^n$ obtained by the restriction to a fiber $E_b = \pi^{-1}(b)$ admits the multi-singularity of type $\boldsymbol{a}$ at some finite points in E_b. Given a Vassiliev cocycle $c := \sum \lambda_i \boldsymbol{a}_i$ and appropriately generic $f : E \to \mathbb{R}^n$, we can define the *bifurcation cycle* $B_c(f)$ to be the geometric cycle $\sum \lambda_i B_{\boldsymbol{a}_i}(f)$ in B: It is a geometric presentation of the G-characteristic class so that

$$\mathrm{Dual}\,[\overline{B_c(f)}] = \rho^* Tp_c.$$

The Thom polynomial theory for $\mathcal{A}$-equivalence of map-germs (e.g. [**19, 16, 17, 29**]) says that for a Vassiliev cocycle of mono-singularities $c = \sum \lambda_\alpha \alpha$, Tp_c is written by a universal polynomial in the relative Novikov-Landweber classes

$$\pi_* cl^I(T_\pi) = \pi_*(cl_1^{i_1}(T_\pi) \cdots cl_k^{i_k}(T_\pi))$$

where $T_\pi (= \ker d\pi)$ is the relative tangent bundle of $\pi : E \to B$ and cl means a certain characteristic class of vector bundles, e.g., Pontrjagin class, Euler class (with $\mathbb{Z}$-coefficients), Stiefel-Whitney class (with $\mathbb{Z}_2$-coefficients). For cocycles of $\mathcal{A}$-multi-singularities, the same conjecturally holds, cf. Kazarian [**16**].

Here n should be reasonably small: For if we take n to ∞, any cocycles of BG live in $\mathcal{M} - \Gamma$, thus there is no chance to obtain nontrivial geometric presentations by bifurcation loci.

We may think of the same story for some enriched Vassiliev complex instead of the local complex of singularities. Some natural questions are, e.g., to find

- the precise forms of Tp_c for given classes $[c] \in H^*(C(\mathcal{A}))$,
- nontrivial relations among those Tp_c's,
- elements in $H^*(C_{en}(\mathcal{A}))$ representing torsion elements of $H^*(BG; \mathbb{Z})$.

Example 4.1. For example, in case that M is oriented circle S^1,

$$H^*(BS^1) = H^*(BU(1)) = \mathbb{Z}[c_1]$$

where c_1 is the first Chern class of complex line bundles. In [**15**] it is shown that the class c_1 can be realized by some bifurcation locus of functions $E \to \mathbb{R}$ or maps $E \to \mathbb{R}^2$ over total space E of a S^1-bundle, and also universal polynomials Tp_Σ in c_1 for several types Σ are computed. When taking the target space $\mathbb{R}^n$ $(n \geq 3)$, it seems that c_1 can not be realized by any bifurcation points, i.e., c_1 lives in the space of embeddings.

Example 4.2. Recall that for an oriented C^∞-surface bundle $\pi : E \to B$ with fiber a closed oriented surface M, the r-th *Morita-Miller-Munford class* $e_r(E) \in H^{2r}(B;\mathbb{Z})$ is defined to be the pushforward $\pi_* e(T_\pi)^{r+1}$ where T_π is the relative tangent bundle over the total space E and $e(T_\pi) \in H^2(E;\mathbb{Z})$ is the Euler class. It is easy to see that for $r \geq 1$

$$Tp_{\Sigma^2}[f : E \to \mathbb{R}^{r+1}] = e_r(E),$$

that is, the class $e_r(E)$ is realized by the Σ^2-bifurcation locus of generic maps $E \to \mathbb{R}^{r+1}$, where Σ^2 means corank 2 singularities of germs $\mathbb{R}^2, 0 \to \mathbb{R}^{r+1}, 0$ (the differential has kernel dimension ≥ 2):

$$[\overline{B_{\Sigma^2}(f)}] = \pi_*[\overline{\Sigma^2(f)}] = \pi_* e(T_\pi^* \otimes f^*\epsilon^{r+1}) = \pi_* e(T_\pi^*)^{r+1} = e_r(E).$$

For instance, the first Morita-Miller-Mumford class $e_1(E)$ is geometrically observed by using generic maps $f : E \to \mathbb{R}^2$. If B is a closed surface, $B_{\Sigma^2}(f)$ consists of finite points in B, over which $f : E \to \mathbb{R}^2$ has the singularity of type

$$I_{22} + II_{22} : (x, y; a, b) \mapsto (x^2 \pm y^2 + x^3 + ay, xy + bx)$$

where x, y are local coordinates of fiber M and a, b are some local coordinates of the base space B. As another example, there is a work by Saeki-Yamamoto [**24**] which shows that $e_1(E)$ is realized by the codimension 2 bifurcation locus corresponding to a special topological type of singular fiber of generic functions $f : E \to \mathbb{R}$: The singular fiber contains three circle components each two of which meet at a nodal point.

5. Contact equivalence for mappings

5.1. Global contact equivalence. Let M be a compact manifold of dimension $m = n+d$ and N of dimension $n+k$. Let B be a k-dimensional compact closed (regular) submanifold of N and $p : M \times N \to M$ the projection to the first factor. If $f : M \to N$ is transverse to B, then $f^{-1}(B)$ is a d-dimensional submanifold of M; If f is tangent to B at some points, then the preimage $f^{-1}(B)$ has singularities of contact ($\mathcal{K}$-)type at those points. Let $p \in f^{-1}(B)$ be a singular point, then the contact singularity type at p is determined by the contact equivalence class of the germ

$$\pi \circ f : \mathbb{R}^{n+d}, 0 \to \mathbb{R}^n, 0$$

where we take local coordinates of M and N centered at p and $f(p)$ respectively, and $\pi : \mathbb{R}^{n+k}, 0 \to \mathbb{R}^n, 0$ denotes a submersion-germ defining $B = \pi^{-1}(0)$ locally.

We introduce an equivalence relation on $\mathcal{M} = C^\infty(M, N)$ to measure how a map $f : M \to N$ is tangent to B: Define $\mathcal{K}_{M,N,B}$ (or $\mathcal{K}_B$ for short) to be the subgroup of Diff$(M \times N)$ consisting of diffeomorphisms H which preserves both the submanifold $M \times B$ and the fiber structure of $p : M \times N \to M$. C^∞-maps $f, g : M \to N$ are said to be *$\mathcal{K}_B$-equivalent* if there exists $H \in \mathcal{K}_B$ such that H sends the graph of f to the graph of g. Then, in particular, $f^{-1}(B)$ and $g^{-1}(B)$ are isomorphic via some diffeomorphism of the ambient manifold M.

Let $\mathcal{D}_\infty := \{f \in \mathcal{M}, \text{codim}\, \mathcal{K}_B.f = \infty\}$ and $\mathcal{U}_0 := \mathcal{M} - \mathcal{D}_\infty$. $f \in \mathcal{U}_0$ if and only if $f^{-1}(B)$ has finitely many singular points, each of which is of type $\mathcal{K}$-finite. In entirely the same way as in the case of Theorem 1.3, we have

THEOREM 5.1. *$\mathcal{D}_\infty$ has infinite codimension in $\mathcal{M}$, and there exists a filtration*

$$\mathcal{M} \supset \mathcal{D} := \mathcal{D}_1 \supset \mathcal{D}_2 \supset \cdots \supset \mathcal{D}_s \supset \cdots \supset \mathcal{D}_\infty,$$

by $\mathcal{K}_B$-invariant closed pseudo-algebraic subsets $\mathcal{D}_s$ of codimension s such that it admits a topologically locally trival fibration $\pi_s : \mathcal{D}_s - \mathcal{D}_{s+1} \to Y'_s$ so that each fiber is an $\mathcal{K}_B$-orbit and Y'_s is a finite dimensional manifold.

EXAMPLE 5.2. 0) $f \in \mathcal{M} - \mathcal{D}$ if and only if f is transverse to B.
1) $\mathcal{D}_1 - \mathcal{D}_2$ consists of maps f having one Morse singularity A_1 on $f^{-1}(B)$, whose contact type is $\mathcal{K}$-equivalent to

$$A_{1,k} : (x_1, \cdots, x_{d+1}, \boldsymbol{z}) \mapsto (-x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_{d+1}^2, \boldsymbol{z}),$$

for some $0 \le k \le [\frac{d+1}{2}]$. $A_{1,k}$ is *coorientable*, except for d odd, $k = [\frac{d+1}{2}]$. This is the surgery to attach a k-handle to a smooth fiber.
2) Also $\mathcal{D}_2 - \mathcal{D}_3$ consists of maps f having either of
- two Morse singularities, or
- one A_2-singularity (=cancelation of handle surgeries) :

$$A_{2,k} : (x_1, \cdots, x_{d+1}, y, \boldsymbol{z}) \mapsto (x_1^3 + yx_1 \pm x_2^2 \pm \cdots \pm x_{d+1}^2, y, \boldsymbol{z}).$$

5.2. Thom-Pontrjagin construction. The classical construction is as follows: Now let M be a compact oriented d-manifold, and embed it in Euclidean space $\mathbb{R}^{d+n}$ ($n \gg 0$). Let $B := BSO(n)$, the Grassmannian of oriented n-planes in $\mathbb{R}^{d+n}$, and let $\rho : M \to B$ the map sending $p \in M$ to the orthogonal plane to TM_p, i.e., the classifying map of the normal bundle of rank n. ρ is extended to a C^∞-map f from $S^{d+n} = \mathbb{R}^{d+n} \cup \infty$ to the Thom space $N_n := MSO(n)$ (a neighborhood of ∞ mapped to the base point of N_n), then $M = f^{-1}(B)$. Let us consider the space of such maps $S^{d+n} \to N_n$ (stationary around ∞), and denote it by

$$\mathcal{M} := C^\infty(S^{d+n}, N_n)_{base}, \quad B := BSO(n) \subset N_n \qquad (n \gg 0).$$

It follows from Theorem 5.1 that there is a $\mathcal{K}_B$-invariant stratification

$$\mathcal{M} \supset \mathcal{D} := \mathcal{D}_1 \supset \mathcal{D}_2 \supset \cdots .$$

If two points $f, g \in \mathcal{M} - \mathcal{D}$ are joined by a generic path γ in $\mathcal{M}$, then d-manifolds $f^{-1}(B)$ and $g^{-1}(B)$ are cobordant, that is, each connected component of $\mathcal{M}$ corresponds to a cobordism class: $\Omega^{ori}(d) = \pi_0(\mathcal{M})$ ($n \to \infty$). That was a basic theorem of R. Thom.

More generally, a sort of global version of *Martinet's versality theorem* holds, see Kazarian [**16, 17**] (also [**24**]): Let $e : S^{d+n} \times \mathcal{M} \to N_n$ be the evaluation map $e(p, f) := f(p)$, and $\mathcal{H} := e^{-1}(B)$ the preimage of B. Denote by $\pi_B : \mathcal{H} \to \mathcal{M}$ the restriction of the second factor projection. Then, we may regard π_B as the "universal C^∞ stable map", and $\mathcal{D}$ as the "discriminant set of π_B":

$$\begin{array}{ccccccc} Q^{d+p} & \longrightarrow & \mathcal{H} & \xrightarrow{incl} & S^{d+n} \times \mathcal{M} & \xrightarrow{e} & N_n \\ \Big\downarrow g & & \Big\downarrow \pi_B & & & & \\ P^p & \longrightarrow & \mathcal{M} & & & & \end{array}$$

That is, a suitably generic map $g : Q \to P$ corresponds to a smooth map $P \to \mathcal{M}$ which is transverse to each $\mathcal{D}_s$.

In case of $d = 1$, Saeki [**23**] has studied a cochain complex for topological types of singular 1-dimensional fiber, which is an enriched Vassiliev complex for $\mathcal{K}_B$-invariant filtration of $\mathcal{D}$.

5.3. Naïve invariants. Let us take a connected component $\mathcal{M}_1$ of $\mathcal{M}$, which corresponds to a cobordism class of oriented closed d-manifolds. Then $H^0(\mathcal{M}_1 - \mathcal{D})$ is regarded as the space of invariants of all such d-manifolds belonging to the fixed cobordism class.

EXAMPLE 5.3. The 'null-cobordant' component in $\mathcal{M}$ can be replaced by

$$\mathcal{M}_0 = C^\infty(S^{d+n}, S^n)_{base}, \qquad B = \{0\} \subset S^n, \quad n \gg 0.$$

Note $\Omega^{ori}(1) = \Omega^{ori}(2) = \Omega^{ori}(3) = 0$. So, in these cases, $\mathcal{M} = \mathcal{M}_0$.

Naïve finite type invariants can be defined in the same way as in §3.1. However, by entirely the same reason as seen in Theorem 3.7, the naïve finite type invariants are reduced to polynomials of local invariants, such as the Euler characteristics $\chi : \mathcal{M} - \mathcal{D} \to \mathbb{Z}$, $f \mapsto \chi(f^{-1}(B))$, cf. [**25**].

Instead, in order to keep the information of glueing maps of handle surgeries, we need more restrictions, i.e., not to be allowed to make other surgeries freely. A choice is to restrict us to a smaller mapping space: For instance, take an open subset $\mathcal{M}_{betti}$ of $\mathcal{M}_0$ so that $f^{-1}(B)$ has *fixed Betti numbers*. In case of $d = 3$, it is nothing but the theory of finite type invariants for homology 3-spheres (Ohtsuki [**21**]) and its generalization. We don't know any result at all in this direction for other dimensional cases.

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