Looking at a surface in 3-space: Topology of singular views

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This primitive art was created approximately 17,300 years ago in the Lascaux Cave, France. The painter definitely knew, at least in a practical way, about the roundness of surfaces (i.e., curvature), the perspective and singularities of apparent contours (i.e., fold, cusp and double fold). The geometry of apparent contours is the main theme of this talk.

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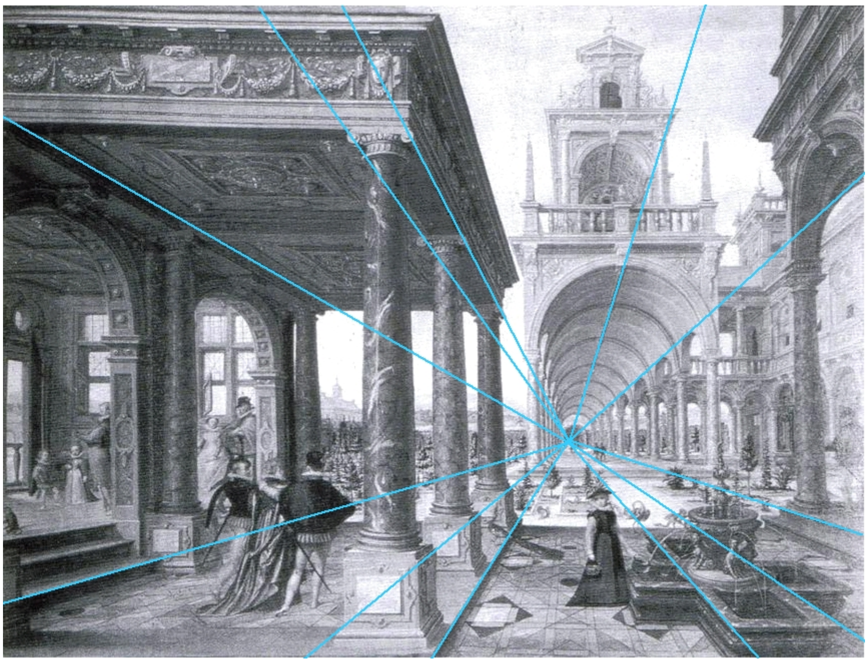
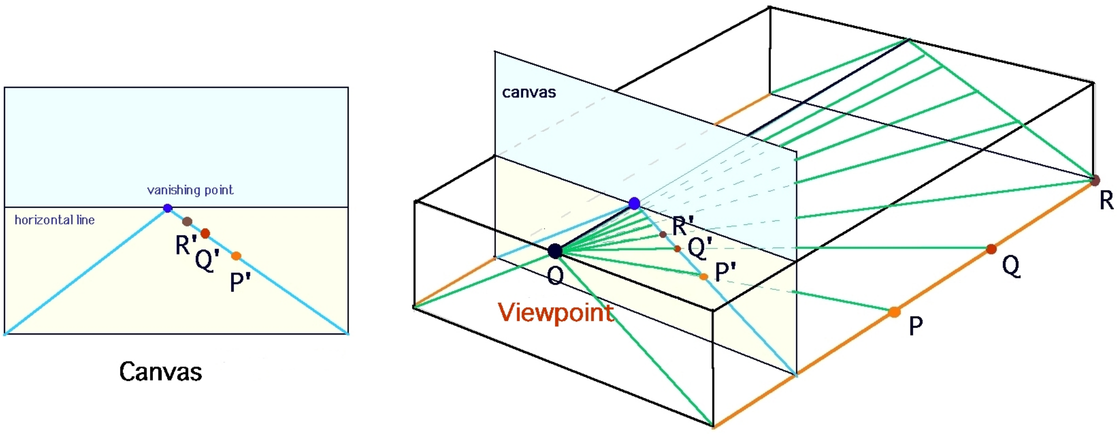
1.1 Perspective - Projection from a viewpoint

Perspective drawing in art is a technique based on facts in Projective Geometry.



Note that parallel lines are drawn on the canvas to have the same *vanishing point*. That is explained in the following picture:



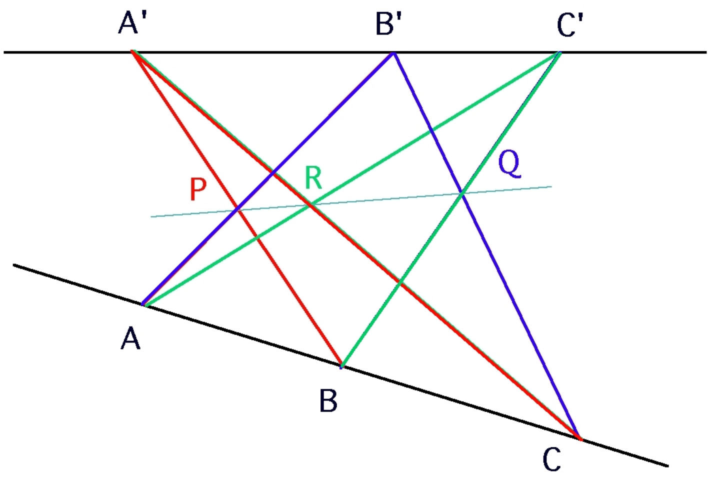
Natural law of perspective:
- A point of the canvas represents a line passing through the viewpoint (called a *viewline*).
- A line on the canvas represents a plane containing the viewpoint.
- Any line in 3-space has the *vanishing point* on the canvas.
- Any plane in 3-space has the *hozisontal line* on the canvas.
- Parallel lines in 3-space have the same vanishing point.
- Parallel planes in 3-space have the same hozisontal line.

A canvas is just a local chart of the "space of viewlines". Take the origin as the viewpoint. Our space of viewlines should be taken as

- **the projective plane** $\mathbb{P}^2$ = the space of all *viewlines* = $S^2/\{\pm 1\}$
- **the** 2**-sphere** S^2= the space of all *oriented viewlines*

1.2 Pappus' Theorem

Example 1.1 (Pappus' Theorem B.C. 300 ?) Let ℓ and ℓ' be two lines on the plane, and take three points on each line and draw six lines as below. Then the points P, Q, R are colinear.

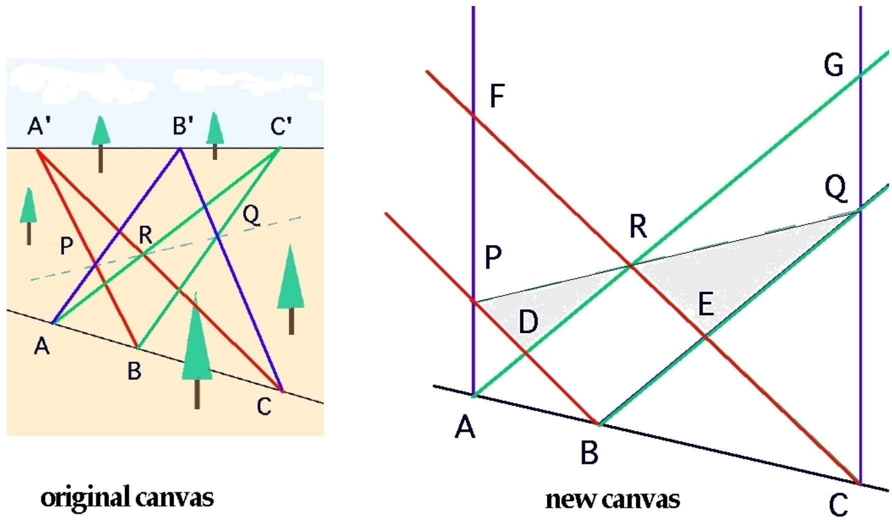


Here is a simple proof using projective geometry.

Imagine that you are drawing a picture of the ground plane on your canvas. Suppose that ℓ' is now the horizontal line on the canvas, so A', B', C' are vanishing points, which correspond to points at infinity. Suppose now that A, B, C, P, Q, R are *points on the ground plane*, and each pair of above lines colored by red, blue and green represents *a pair of two parallel lines in* 3*-space lying on the ground.*

Take the ground plane itself as a new canvas. Then, our points and lines are placed like as in the right hand side; lines BP and CR, AG and BQ, AP and CQ are parallel on the new canvas (=grand plane). In the Euclidean geometry, it is easy to show that $\triangle DRP$ is similar to $\triangle EQR$, thus lines PR and RQ coincide. The line PRQ in 3-space must be depicted as a *line in original canvas*, so the claim is verified. This completes the proof.

The key idea in this proof is **to change canvases**, i.e., **change the local chart** of the space of viewlines. Indeed, we simply take a **projective transform**, which sends the horizontal line to infinity and preserves (co)linearity.



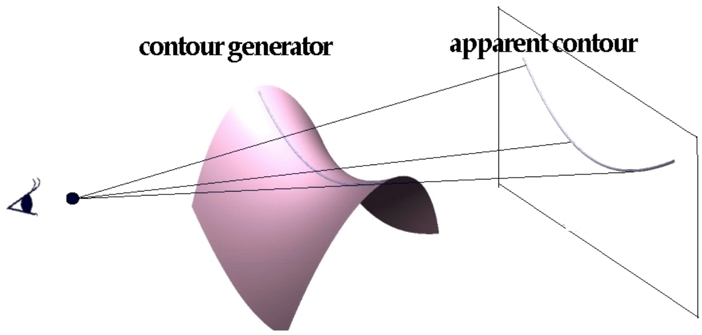
Remark 1.2 Pappus's theorem deals with two lines on the (projective) plane. It is generalized to a theorem about any conic, that is *Pascal's theorem*. In later centuries, those theorems were quite much developed in the context of *projective algebraic geometry*: Of particular importance is *Bezout's theorem*, which says that the sum of multiplicities at the intersection points of given two projecitve plane curves of degree m and n, respectively, is exactly equal to mn. In this talk we will speak about *enumerative geometry* later, whose prototype goes back to Bezout's theorem.

2 Looking at a surface in 3-space

Let M be a surface in $\mathbb{R}^3$ and take a view point $p \in \mathbb{R}^3 - M$. Instead of $\mathbb{R}^3$, we may consider S^3 or $\mathbb{P}^3$ (also $\mathbb{C}^3, \mathbb{CP}^3$). We can define a natural projection centered at p

$$\varphi_p : M \to \mathbb{P}^2$$

by sending each $x \in M$ to the viewline $\overline{px}$. Also let
contour generator := the set of critical points of φ_p (curve on M)
apparent contour := the set of critical values of φ_p (curve on $\mathbb{P}^2$)



In Sigularity Theory, we classify map-germs under a natural equivalence relation: $f, g : \mathbb{R}^m, 0 \to \mathbb{R}^n, 0$ are $\mathcal{A}$-equivalent if there are diffeomorphism-germs σ, τ of source and target, respectively, so that the following diagram commutes:

$$\begin{tikzcd} \mathbb{R}^m, 0 \arrow[r, "f"] \arrow[d, "\sigma"', "\simeq"] & \mathbb{R}^n, 0 \arrow[d, "\simeq"', "\tau"] \\ \mathbb{R}^m, 0 \arrow[r, "g"'] & \mathbb{R}^n, 0 \end{tikzcd}$$

Theorem 2.1 (Arnold ('79), Bruce ('84), Platonova ('86))
For a generic surface M, it holds that for any viewpoint p and any point $x \in M$, the germ

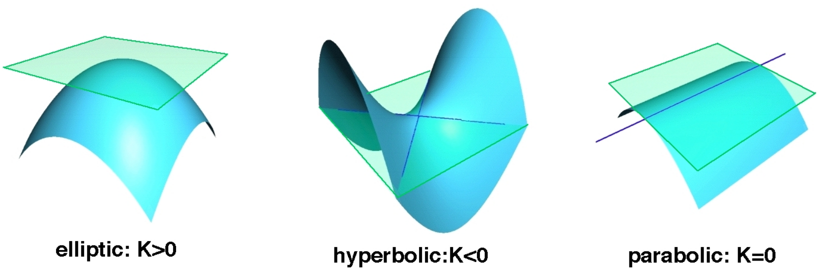
$$\varphi_p : M, x \to \mathbb{P}^2, \varphi(x)$$

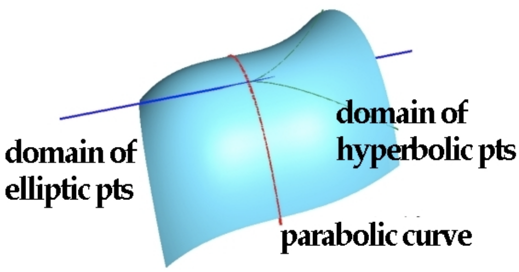
is equivalent to the germ $(x, y) \mapsto (f(x, y), y)$ where f is one of the following list:

type	*codim.*	$f(x,y)$	*type*	*codim.*	$f(x,y)$
$0(regular)$	0	x	$7(seagull)$	2	$x^4 + x^2y + xy^2$
$1(fold)$	0	x^2	$8, 9(butterfly)$	2	$x^5 \pm x^3y + xy$
$2(cusp)$	0	$x^3 + xy$	$10, 11$	3	$x^3 \pm xy^4$
$3, 4(lips/beaks)$	1	$x^3 \pm xy^2$	12	3	$x^4 + x^2y + xy^3$
$5(goose)$	2	$x^3 + xy^3$	13	3	$x^5 + xy$
$6(swallowtail)$	1	$x^4 + xy$			

2.1 Curvature, Asymptotic line

According to the Gaussian curvature of the surface, points on the surface are classified into three types. The parabolic curve separates M into two open domains; the domain of elliptic points and the domain of hyperbolic points.





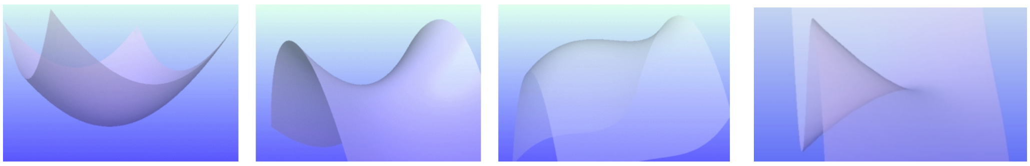
The *asymptotic line* at $p \in M$ is the line having contact with M of order ≥ 3 at p, i.e., the line along which the second fundamental form of M at p vanishes [*1]

[*1] The 'contact order' may depend on the context: Here we regard the line transverse to M has first order contact (revised in Feb. 2014).

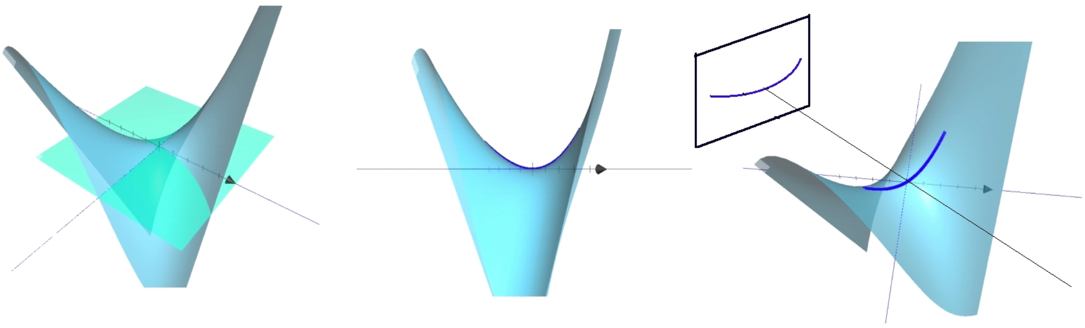
Every hyperbolic point has exactly two asymptotic lines, and those two lines degenerate into one line at a parabolic point, and there is no asymptotic line at any elliptic point.

2.2 Stable projection: Codimension 0

type	normal form
$0(regular)$	$(x, y) \mapsto (x, y)$
$1(fold)$	$(x, y) \mapsto (x^2, y)$
$2(cusp)$	$(x, y) \mapsto (x^3 + xy, y)$

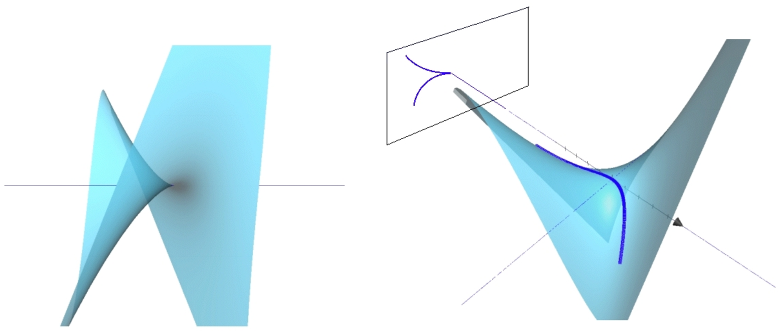


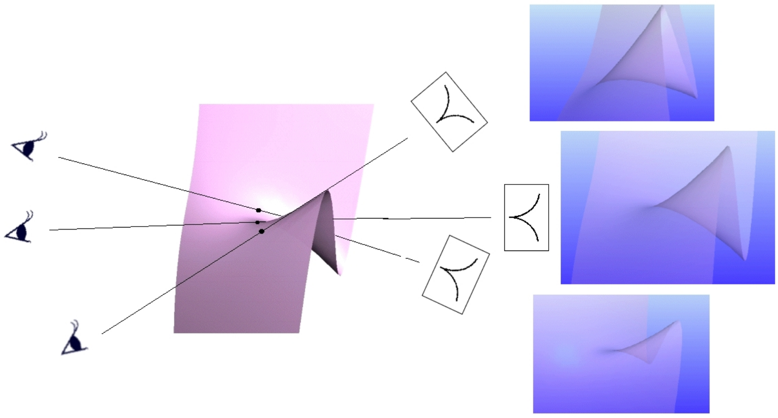
$\boxed{\text{1: Fold}}$: Suppose that our viewline is tangent to M but **not asymptotic**, i.e., 2nd order contact. Then, the projection produces a fold singularity.



Apparent contours at elliptic, hyperbolic and parabolic points, unless the viewline is asymptotic, correspond to fold singularities. Note that the above condition on viewlines is an open condition, that implies that the fold singularity is stable (codimension 0).

$\boxed{\text{2: Cusp}}$: **Let p be a hyperbolic point.** Suppose that our viewline is **asymptotic** in the simplest way, i.e., 3rd order contact with M. Then, the projection produces a cusp singularity.

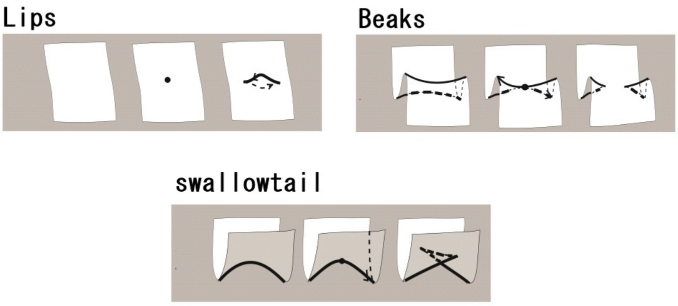




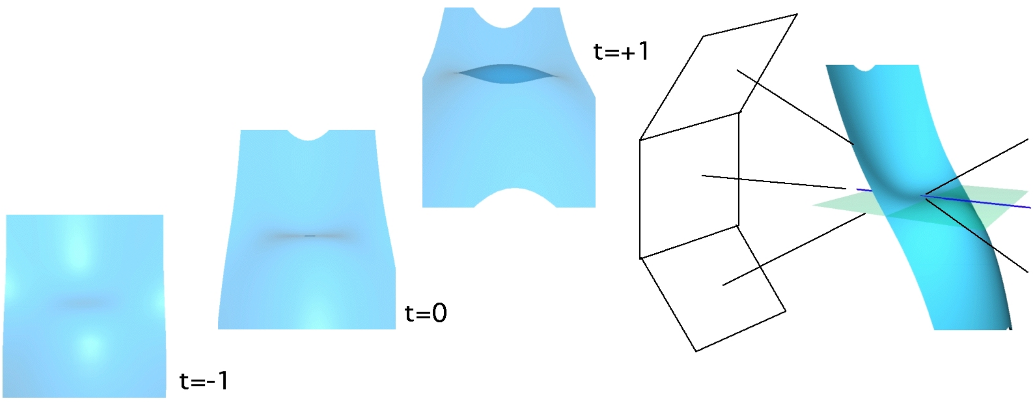
If you move the viewpoint, you always catch some other point close to p, at which the viewline is asymptotic. This means that the cusp singularity is *stable*.

2.3 Singularities of Codimension 1

type	miniversal unfolding
$3, 4(lips/beaks)$	$(x, y) \mapsto (x^3 \pm xy^2 + ax, y)$
$6(swallowtail)$	$(x, y) \mapsto (x^4 + xy + ax^2, y)$

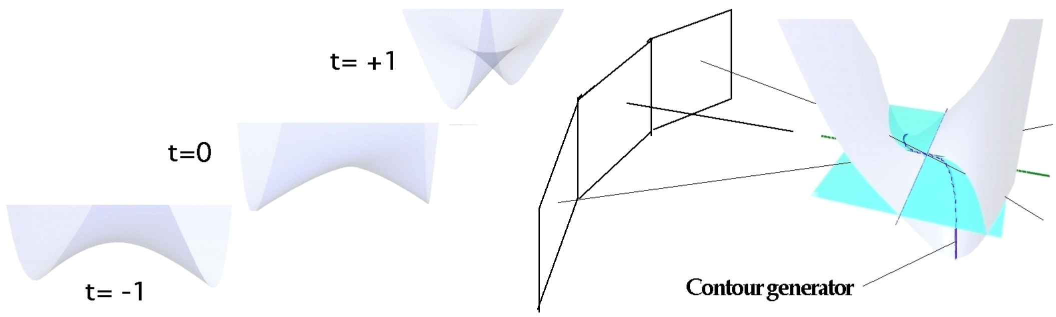


$\boxed{3, 4\textbf{: Lips/Beaks（くちびる，くちばし）}}$ **Let $p \in M$ be a parabolic point.** Then there is only one asymptotic line. Suppose that our viewline is **asymptotic**: It has generically **3rd order contact** with M. Then, we have singularities of type Lips or Beaks:



The set of all the asymptotic lines at parabolic points forms a *ruled surface* in 3-space. When our viewpoint comes across the ruled surface, we observe the bifurcation of lips or beaks. The ruled surface is called the *bifurcation set of lisp/beaks type*.

6: **Swallowtail （ツバメの尾）** **Let $p \in M$ be a hyperbolic point.** Suppose that our view-line is **asymptotic**. As seen above, if the contact is of order 3, we observe the cusp singularity. Now let it be **4th order contact** with M. Then, it creates a singularity of type swallowtail.



The locus of points of M corresponding to the swallowtail singularity is a curve in the hyperbolic domain of M. Take all the asymptotic lines at points of the curve, we obtain a *ruled surface* in 3-space, being different from the one for lips/beaks. This is called the *bifurcation set of swallowtail type*. When our viewpoint comes across this ruled surface, we observe the bifurcation of swallowtail.

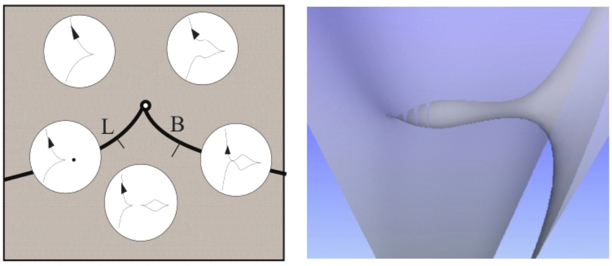
2.4 Singularities of Codimension 2

type	miniversal unfolding
$5(goose)$	$(x, y) \mapsto (x^3 + xy^3 + axy + bx, y)$
$7(seagull)$	$(x, y) \mapsto (x^4 + x^2y + xy^2 + axy + bx, y)$
$8, 9(butterfly)$	$(x, y) \mapsto (x^5 \pm x^3y + xy + ax^3 + bx^2, y)$

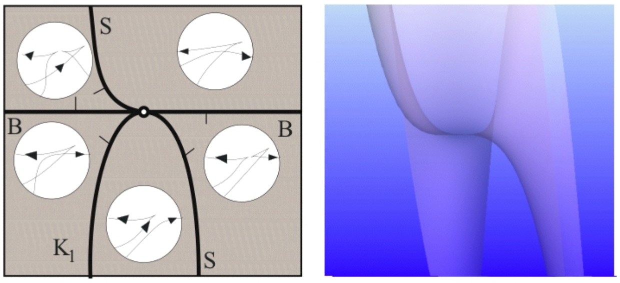
The bifurcation set of lips/beaks type may have singularities of itself. It is a ruled surface as seen above. Generically speaking singularities of ruled surfaces are of type cuspidal edge and swallowtail. Each point of the cuspidal edge of our bifurcation set of lips/beaks has a line which is asymptotic at some parabolic point (a turning point of the asymptotic lines). That line corresponds to the bifurcation of codimension two, named *goose*. Each point of the swallowtail of the ruled surface corresponds to a type of the bifurcation of codimension three (no. 10, 11 in the list of Platonova).

It also happens that the asymptotic line at a parabolic point is tangent to the parabolic line. This corresponds to the bifurcation of *gull*.

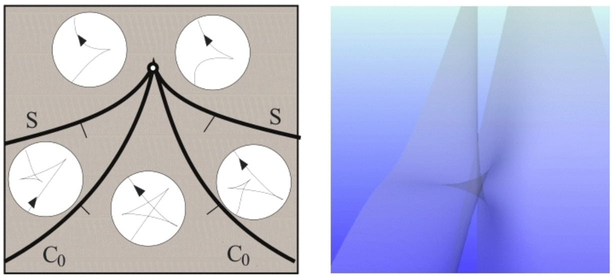
The bifurcation set of swallowtail type also has singularities of itself. That is the bifurcation of codimension two, named *butterfly*.



5: Goose（がちょう）



7: Seagull（かもめ）



8：Butterfly（蝶々）

2.5 Summary (recoginization up to codim. 2)

- **Elliptic pt** $K > 0$ $\Longrightarrow$ **Fold** $\boxed{1}$ **(stable)**
- **Parabolic pt** $K = 0$

$$\Longrightarrow \begin{cases} \textbf{viewline is not asymp.} \to \textbf{Fold } \boxed{1} \textbf{ (stable)} \\ \textbf{viewline is asymp.} \to \begin{cases} \textbf{3rd order} \to \begin{cases} \textbf{Lips/Beaks } \boxed{3,4} \textbf{ (codim } 1\textbf{)} \\ \textbf{Goose } \boxed{5} \textbf{ (codim } 2\textbf{)} \end{cases} \\ \textbf{4th order} \to \textbf{Gull } \boxed{7} \textbf{ (codim } 2\textbf{)} \end{cases} \end{cases}$$

- **Hyperbolic pt** $K < 0$

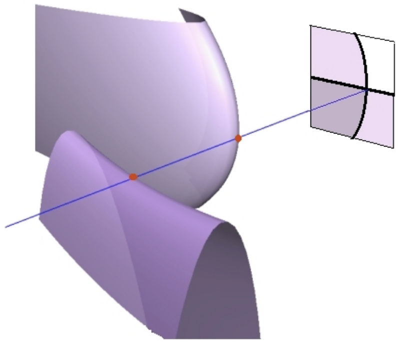
$$\Longrightarrow \begin{cases} \textbf{viewline is not asymp.} \to \textbf{Fold } \boxed{1} \textbf{ (stable)} \\ \textbf{viewline is asymp.} \to \begin{cases} \textbf{3rd order} \to \textbf{Cusp } \boxed{2} \textbf{ (stable)} \\ \textbf{4th order} \to \textbf{Swallowtail } \boxed{6} \textbf{ (codim } 1\textbf{)} \\ \textbf{5th order} \to \textbf{Butterfly } \boxed{8,9} \textbf{ (codim } 2\textbf{)} \end{cases} \end{cases}$$

2.6 Multi-singularities

Multi-singularity, or *multi-germ*, is a germ of a map at finite points $S = \{x_1, x_2, \cdots, x_s\}$ mapped onto a single point y

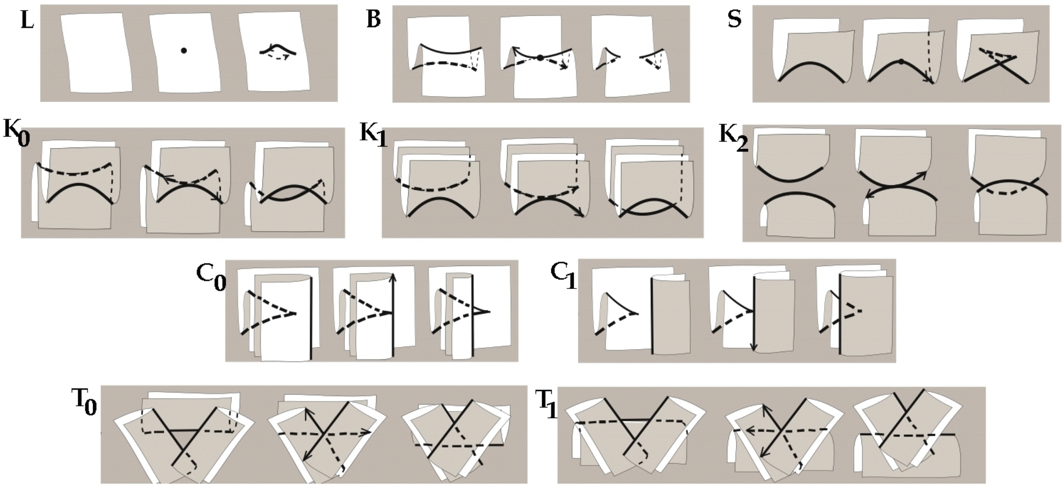
$$\pi : M, S \to \mathbb{P}^2, y$$

For instance, the double fold is a stable bi-germ (i.e., codim. 0):



- **Multi-singularities of codimension 1**

  There are 10 different types (3 mono, 5 bi, 2 tri-germs)

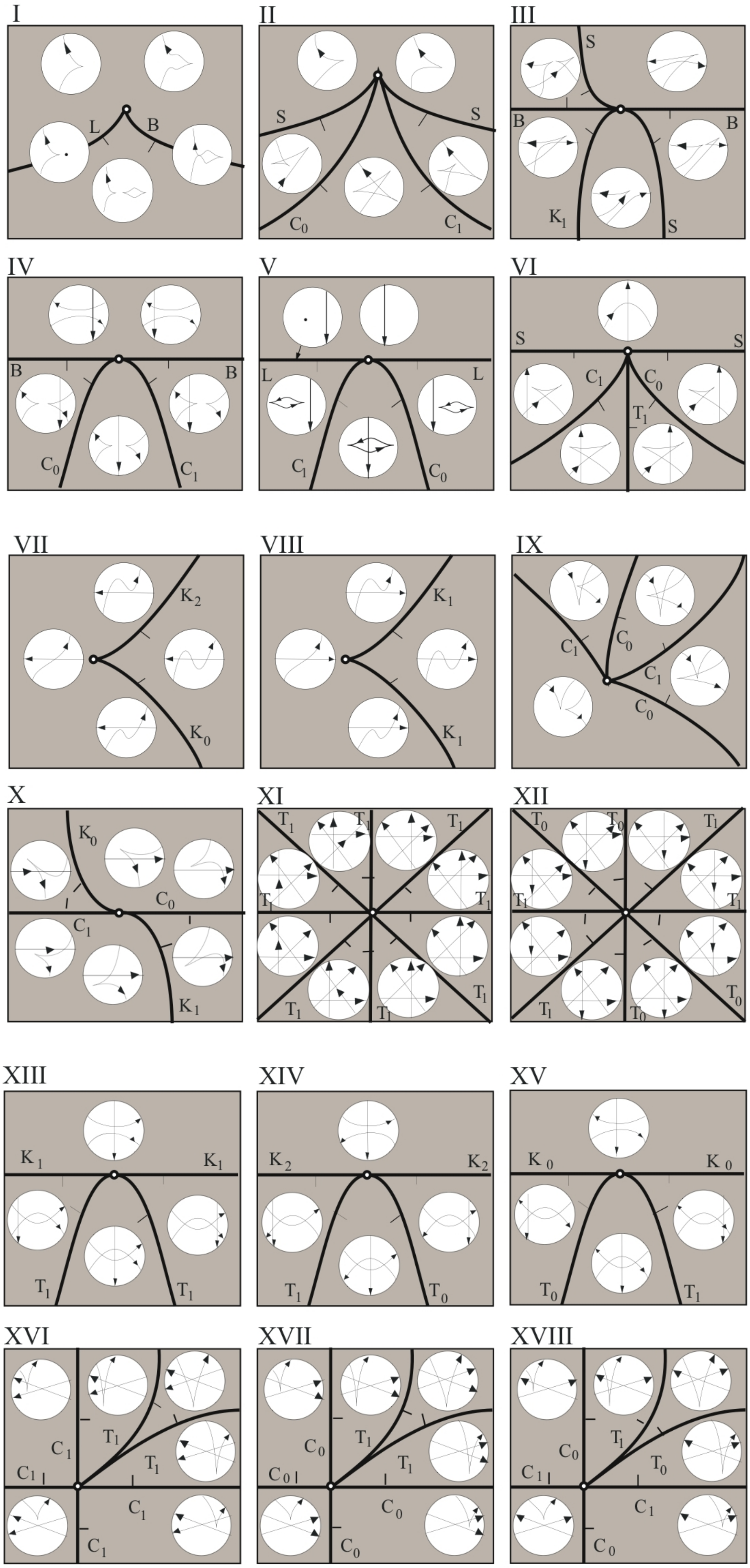


- **Multi-singularities of codimension 2**

  There are 18 different types, depicted in the next page.

- **Multi-singularities of codimension 3**

  There are 4 types of mono-singularity of codim. 3, (no. 10–13) as seen in the list of Platonova. But the list of multi-germs and bifurcation diagrams has not yet been completed.


I
L B
II
S S
C0 C1
III
S B B K1 S
IV
B B C0 C1
V
L L C1 C0
VI
S S C1 C0 T1
VII
K2 K0
VIII
K1 K1
IX
C1 C0 C1 C0
X
K0 C0 C1 K1
XI
T1 T1 T1 T1 T1 T1 T1 T1
XII
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XIII
K1 K1 T1 T1
XIV
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XV
K0 K0 T0 T1
XVI
C1 T1 T1 C1 C1 C1
XVII
C0 T1 T1 C0 C0 C0
XVIII
C0 T1 T0 C1 C1 C0

2.7 Incidence variety

Let $M \subset \mathbb{P}^3$ = the space of 1-dim. vector subspaces in $\mathbb{R}^4$. The Grassmannian of lines is

$$\begin{aligned} Gr(2,4) &:= \text{the space of 2-dim. vector subspaces in } \mathbb{R}^4 \\ &= \text{ the space of projective lines in } \mathbb{P}^3 \end{aligned}$$

Incidence variety of points and lines

$$I(\mathbb{P}^3) := \{\ (p,\ell) \in \mathbb{P}^3 \times Gr(2,4) \mid p \in \ell\ \}$$

$$\begin{tikzcd} & I(\mathbb{P}^3) \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{P}^3 & & Gr(2,4) \end{tikzcd}$$

Let

$$I_M := p^{-1}(M) = \{\ (p,\ell) \in \mathbb{P}^3 \times Gr(2,4) \mid p \in M,\quad p \in \ell\ \} \quad \subset \quad I(\mathbb{P}^3)$$

$$\begin{tikzcd} & I_M \arrow[dl, "p"'] \arrow[dr, "f:=q|_{I_M}"] & \\ M & & Gr(2,4) \end{tikzcd}$$

The singular set of $f : I_M \to Gr(2,4)$:

$$\begin{array}{lll} \Sigma^1(f) & = \{\ (x,\ell) \mid x \in \ell \subset T_xM\ \}, & (\dim = 3) \\ \Sigma^{1,1}(f) & = \{\ (x,\ell),\ \ell \text{ is asymptotic at } x\ \}, & (\dim = 2) \\ \Sigma^{1,1,1}(f) & = \{\ (x,\ell),\ \ell \text{ is asymp. at } x \text{ of 4th order }\ \}, & (\dim = 1) \\ \Sigma^{1,1,1,1}(f) & = \{\ (x,\ell),\ \ell \text{ is asymp. at } x \text{ of 5th order }\ \}, & (\dim = 0) \end{array}$$

$\Sigma^{1,1}(f) \xrightarrow{p} M$: double cover of M ramified along the parbolic curve.
$p(\Sigma^{1,1,1,0}(f))$ is a curve on $M \leftrightarrow$ the swallowtail (no. 6).
$p(\Sigma^{1,1,1,1}(f))$ consists of isolated pts $\leftrightarrow$ the butterfly (no. 8, 9).

Remark 2.2 If we take $Q(\mathbb{P}^3) := \{\ (p,H) \in \mathbb{P}^3 \times \mathbb{P}^{*3},\ p \in H\ \}$,

$$\begin{tikzcd} & Q(\mathbb{P}^3) \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{P}^3 & & \mathbb{P}^{*3} \end{tikzcd}$$

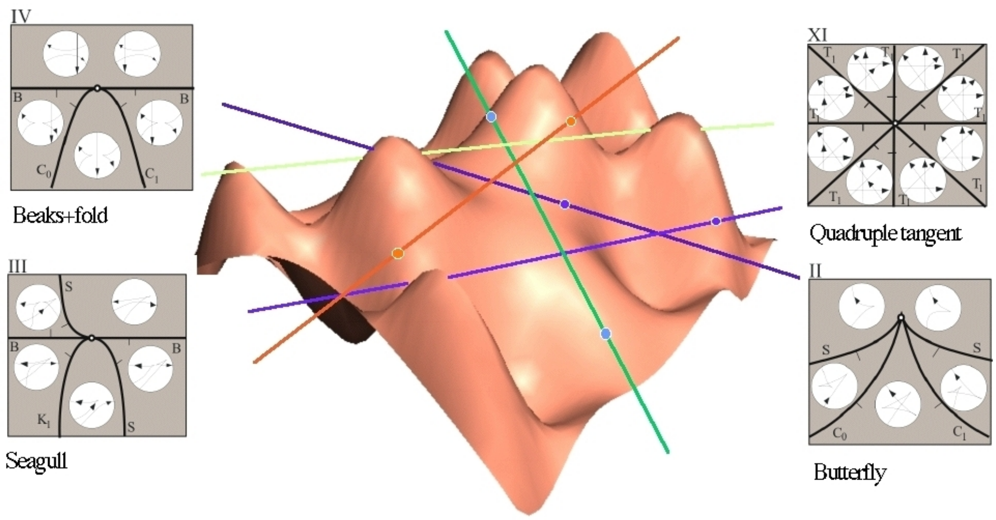
This diagram describes the projective duality and the Gauss map.

3 Topology of Singular Views (complex case)

Let M be a complex projective surface of degree d (≥ 4):

$$M := \{ \, p \in \mathbb{CP}^3 \mid f(p) = 0 \, \}, \quad (f \text{ is a homog. poly. of degree } d)$$

Given a singularity type of codimension two, e.g., butterfly, ... Then, the viewlines along which the projection is of that type are discrete.



The following questions are typical ones:

Q. How many viewlines of type butterfly bifurcation do exist?

Ans. $\sharp$Butterfly $= 5d(d-4)(7d-12)$

Q. How many viewlines tangent to the parabolic curve on M ?

Ans. $\sharp$Seagull $= 2d(d-2)(11d-24)$

Those kind questions can be solved by means of Thom polynomials.

3.1 Chern class of vector bundles

To a complex vector bundle ξ of rank n, the *Chern class of* ξ is defined in the integer cohomology ring of the base space:

$$1 + c_1(\xi) + \cdots + c_n(\xi) \in H^*(X;\mathbb{Z}), \quad c_k(\xi) \in H^{2k}(X).$$

Remark: For real vector bundle ξ of rank n, the Stiefel-Whitney class is defined the cohomology of the base space with $\mathbb{Z}_2$-coefficients: $w_k(\xi) \in H^k(X;\mathbb{Z}_2)$.

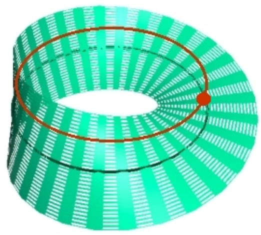
Chern classes measures the difference between the vector bundle ξ and from the trivial vector bundle ϵ_n, i.e., the direct product $\epsilon_n : \mathbb{C}^n \times X \to X$.

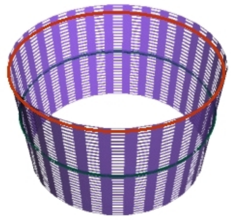
- If ξ adimits a trivial rank k subbundle $\epsilon_k \subset \xi$, then

$$c_j(\xi) = 0 \; (j \geq n-k+1)$$

- Let ξ_1 be the tautological line bdle of $\mathbb{CP}^1$ and ϵ_1 the trivial one. Then,

$$c_1(\xi_1) = -1, \quad c_1(\epsilon_1) = 0 \quad \in H^2(\mathbb{CP}^1;\mathbb{Z}) \simeq \mathbb{Z}$$





In fact, if one takes a section of ξ_1 transverse to the zero section, then the intersection number is expressed by $c_1(\xi_1)$. Note that this picture is a conceptual one for complex line bundles, but it is a precise picture for real line bundles. The total space of the tautological line bundle of $\mathbb{RP}^1 \simeq S^1$ is a Möbius strip, as depicted in the left, while the trivial line bundle is just an anulus, as depicted in the right. The intersection number of a generic section and the zero section is defined modulo 2: In the left (Möbius), the intersection number is odd, while in the right (trivial) it is even,

$$w_1(\xi_1) \equiv 1, \quad w_1(\epsilon_1) \equiv 0 \quad \in H^1(\mathbb{RP}^1;\mathbb{Z}_2) \simeq \mathbb{Z}_2.$$

3.2 Thom polynomials

Classification of map-germs $\Longrightarrow$ $\exists$ theory of Thom polynomials

$$\eta: \ \textbf{singularity type} \quad \longrightarrow \quad Tp(\eta) \in \mathbb{Z}[c_1, c_2, \cdots]$$

so that given a map $f : N^n \to P^p$ and a type η of codim $= n$*,*

$$Tp(\eta)(c(f)) = \sharp \ \eta\text{-type singular points of } f \quad \in H^{2n}(M) = \mathbb{Z}$$

Formal computation of $Tp(\Sigma^{1,1,1,1})$ etc applied to our map

$$f : I_M \to Gr(2,4)$$

gives the answer to the enumerative problem posed above.
E.g., In case of $m = n$,

$$Tp(\Sigma^{1,1,1,1}) = c_1^4 + 6c_1^2 c_2 + 2c_2^2 + 9c_1 c_3 + 6c_4$$

where $c_i = c_i(f^*TP - TN)$.

Remark 3.1 In real case, we may replace Chern class c_i by Whitney class w_i, then we can obtain $\mathbb{Z}_2$-enumeration, that is, the parity of the number of prescribed singularities. To obtain integer value enumerations, we have to consider the problem of *orientation*. In this case Tp is a polynomial in Pontryagin class and Euler class (with Whitney class for representing the mod. 2 torsion).

3.3 Tp for $\mathcal{A}$-classification

More directly, Tp for $\mathcal{A}$-classification of map-germs $\mathbb{C}^2, 0 \to \mathbb{C}^2, 0$ is defined as

$$Tp^{\mathcal{A}}(\eta) \in \mathbb{Z}[c_1, c_2, c_1', c_2']$$

where c_i, c_i' are Chern classes of the source and of the target mfd. The following list is new *2:

lips/beaks	$-2c_1^3 + 5c_1^2c_1' - 4c_1c_1'^2 - c_1c_2 + c_2c_1' + c_1'^3$
swallowtail	$-6c_1^3 + 11c_1^2c_1' - 6c_1c_1'^2 + 7c_1c_2 - 5c_1c_2' - 5c_1'c_2 + 3c_1'c_2' + c_1'^3$
goose	$2c_1^4 + 5c_1^2c_2 + 4c_2^2 - 7c_1^3c_1' - 10c_1c_2c_1' + 9c_1^2c_1'^2 + 5c_2c_1'^2$ $-5c_1c_1'^3 + c_1'^4 - 2c_1^2c_2' - 6c_2c_2' + 4c_1c_1'c_2' - 2c_1'^2c_2' + 2c_2'^2$
gull	$6c_1^4 - c_1^2c_2 - 4c_2^2 - 17c_1^3c_1' + 4c_1c_2c_1' + 17c_1^2c_1'^2 - 3c_2c_1'^2$ $-7c_1c_1'^3 + c_1'^4 + 2c_1^2c_2' + 6c_2c_2' - 4c_1c_1'c_2' + 2c_1'^2c_2' - 2c_2'^2$
butterfly	$24c_1^4 - 50c_1^3c_1' - 46c_1^2c_2 + 35c_1^2c_1'^2 + 25c_1^2c_2' + 55c_1c_2c_1' - 10c_1c_1'^3$ $-25c_1c_1'c_2' + 8c_2^2 - 15c_2c_1'^2 - 10c_2c_2' + c_1'^4 + 6c_1'^2c_2' + 2c_2'^2$
$I_{2,2}^{1,1}$	$c_2^2 - c_1c_2c'_1 + c_2c'^2_1 + c_1^2c'_2 - 2c_2c'_2 - c_1c'_1c'_2 + c'^2_2$

For $\mathcal{K}$-classification of map-germs, or stable map-germs, many actual computations have been known. On the other hand, there are $\mathcal{A}$-classification of map-germs: $\mathbb{C}^1, 0 \to \mathbb{C}^2, 0$, $\mathbb{C}^1, 0 \to \mathbb{C}^3, 0$, ... , $\mathbb{C}^2, 0 \to \mathbb{C}^2, 0$, $\mathbb{C}^2, 0 \to \mathbb{C}^3, 0$, ..., $\mathbb{C}^3, 0 \to \mathbb{C}^2, 0$, $\mathbb{C}^3, 0 \to \mathbb{C}^3, 0$, ... For each of them, one can consider Thom polynomials, but actual computations have not yet been done enough, although we know the method for computation.

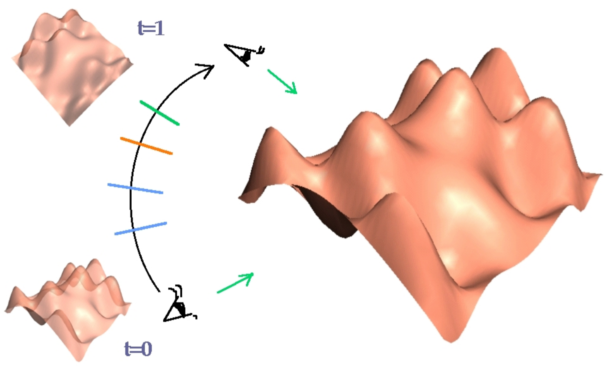
4 Topology of Singular Views (real case)

Let $M \subset \mathbb{R}^3$ be the graph of $z = f(x, y)$ ('the ground surface'), and let $\mathcal{U} := \{\ z > f(x, y)\ \} \subset \mathbb{R}^3$ as the space of viewpoints.

Imagine that *you are a bird flying over the ground*. Fly freely in the sky, get different views.

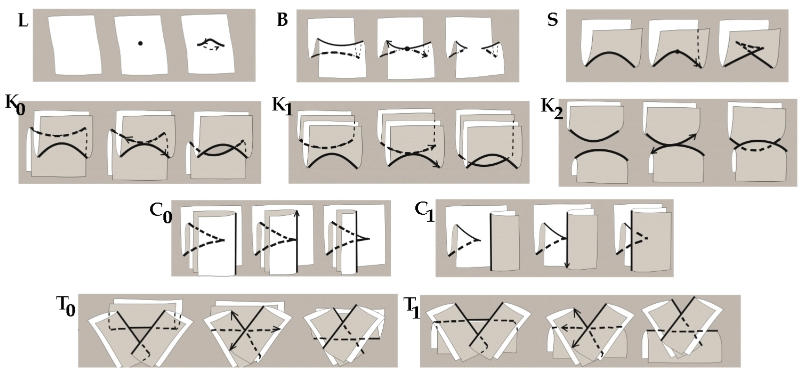
Let $p_0, p_1 \in \mathcal{U}$ be your initial viewpoint and the final viewpoint, respectively.

Moving along the path from p_0 to p_1, you meet bifurcations of codimension one several times. The view image is changed at each event when such a bifurcation occurs.



*2 revised in Feb. 2014

Remember the list of multi-singularities of codimension 1



These 10 types are *orientable*. Let it be directed from left to right.

Question: How many times does such a bifurcation occurs when moving along a path in $\mathcal{U}$, taking account of signs ?

Define a functional ΔL : { generic paths in $\mathcal{U}$ } $\to \mathbb{Z}$ by

$$\Delta L := \sum \pm 1 \quad \text{when the Lips bifurcation occurs}$$

$\Delta B, \Delta S, \Delta K_0, \Delta K_1, \Delta K_2, \cdots$ are also defined.

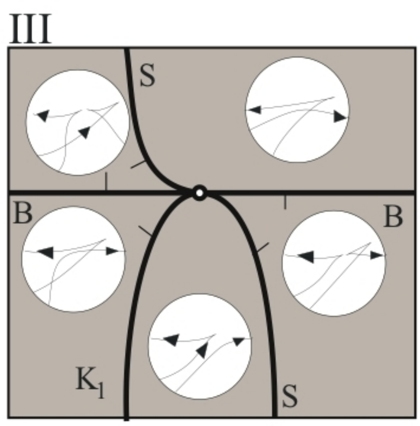
4.1 Vassiliev complex

We define an abstract cochain complex

$$0 \longrightarrow C^1 \xrightarrow{\partial} C^2 \xrightarrow{\partial} C^3 \longrightarrow 0$$

$C^1 = \mathbb{Z}^{10}$, freely generated by codim 1 multi-sing. $L, B, S, ...$
$C^2 = \mathbb{Z}^{18}$, freely generated by codim 2 multi-sing. $I, II, III, \cdots$
$C^3 = \mathbb{Z}^{??}$, freely generated by codim 3 multi-sing.
The coboundary operator $\partial : C^i \to C^{i+1}$ is defined:

$$\partial c = \sum_{\sigma} [c, \sigma]\sigma, \qquad (c = L, B, S, ...; \sigma = I, II, III, \cdots)$$



$$[S, III] = 2, \quad [B, III] = -2, \quad [K_1, III] = -1$$
$$[L, III] = [K_0, III] = \cdots = [T_1, III] = 0$$

Do the same computations for other 17 types, then you get $\partial : C^1 \to C^2$.

Theorem 4.1 $H^1(C) = \ker \partial_1$ has rank 3:

$$\Delta I_1 = \Delta L + \Delta B + \Delta S,$$

$$\Delta I_2 = \Delta S + 2\Delta K_1 + \Delta C_0 + \Delta C_1,$$

$$\Delta I_3 = 2\Delta K_0 + 2\Delta K_2 + \Delta C_0 + \Delta C_1$$

These values depend only on points p_0 and p_1 of the path,
do not depend on the choice of paths from p_0 to p_1.

Theorem 4.2

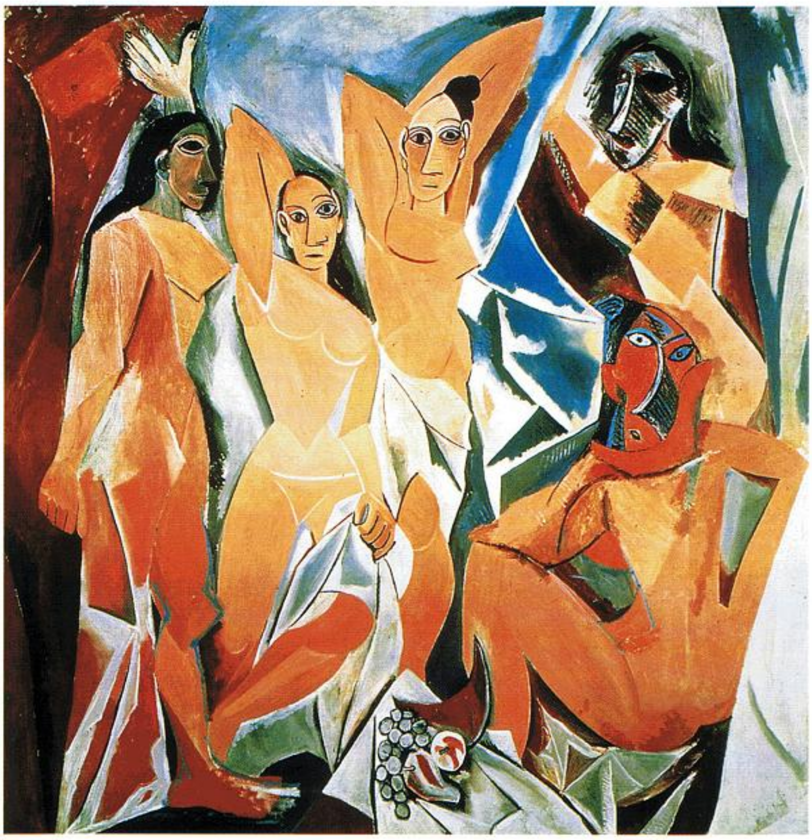
$$\begin{aligned} 2\Delta I_1 &= \text{increment of the number of cusps} \\ \Delta I_2 + \Delta I_3 &= \text{increment of the number of double folds} \\ \Delta I_1 - \Delta I_2 + \Delta I_3 &= \text{the projective Thurston-Bennequin number} \\ &\quad \text{(i.e., "self-linking number" of legendre lift)} \end{aligned}$$

Those are called *local first order invariants* of view maps $M \to \mathbb{P}^2$.
$\Longrightarrow$ *Theory of order one Vassiliv-type invariants* for $\mathcal{A}$-classification of map-germs (Vassiliev, Arnold, Goryunov, Ohmoto-Aicardi, ...)

(m, n)	object/singularities
$(1, 1)$	functions on circle bundles/global max.-min. (Kazarian)
$(1, 2)$	generic immersed curves, wavefronts (Arnold, Aicardi)
$(1, 3)$	knots and links (Vassiliev)
$(2, 1)$	Morse ft. (Saeki)
$(2, 2)$	generic maps (Ohmoto-Aicardi)
$(2, 3)$	generic maps (Goryunov)
$(2, 4)$	generic immersions (Kamada etc)
$(3, 2)$	generic maps (M. Yamamoto)
$(3, 3)$	geneirc maps (Oset, Goryunov)
$(3, 4)$	generic immersions (Catiana)
$(3, 5)$	generic immersions (Ekholm, Takase)

5 Summary

- Local geometry of projections and extrinsic differential geometry of a surface in 3-space is explained in detail.
- Enumerative problem of counting singularities can be solved by Thom polynomials
- The combinatorics among bifurcation diagrams for real singularities is encoded in the Vassiliev complex, which provides some interesting topological invariants of mappings, called local invariants.



Modern Art integrates multiple views.