CHERN CLASSES AND THOM POLYNOMIALS

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1. Introduction

1.1. Following Jean-Paul Brasselet's lecture [2] on *Chern classes of singular varieties* in this ICTP summer school, I introduce the theory of *equivariant Chern-Schwartz-MacPherson classes* and show two different types of applications.

We work in the complex algebraic context for simplicity; for a variety X let $H_*(X)$ denote the Borel-Moore homology group of the underlying analytic space. (In an algebraic (quasi-projective) context over a field of characteristic 0, the homology means the Chow group $A_*(X)$ of algebraic cycles under rational equivalence.) Our main theorem is

Theorem 1.1. ([22]) *Let G be a complex algebraic group. For the category of complex algebraic G-varieties X and proper G-morphisms, there is a natural transformation from the equivariant constructible function functor to the equivariant homology functor*

$$C_*^G : \mathcal{F}^G(X) \to H_*^G(X)$$

such that if X is non-singular, then $C_*^G(1\!\!1_X) = c^G(TX) \frown [X]_G$ where $c^G(TX)$ is G-equivariant total Chern class of the tangent bundle of X. C_*^G is unique in a certain sense. In particular, for the trivial G-action, C_*^G coincides with ordinary C_*.

Remark that in general the quotient X/G does not make sense as a variety or a scheme, but *the quotient stack* $[X/G]$ exists (see 3.5). Thus the above theorem may be regarded as an extension of original MacPherson's transformation C_* to a wider category of spaces, *quotient stacks.*

This equivariant setting is based on Totaro-Edidin-Graham's "algebraic Borel construction" of classifying spaces ([35], [7]), so first I will talk about the basic idea of this construction (§3). Second, I will show some applications of C_*^G. For a compact G-variety X, the constant term (degree) and the top term of our Chern class $C_*^G(1\!\!1_X)$ coincide with the Euler characteristic and the *equivariant fundamental class* respectively:

$$C_*^G(1\!\!1_X) = \chi(X)[pt] + \cdots + [X]_G.$$

So, if we are given some formulae of Euler characteristics or fundamental classes, we may expect similar type formulae for total Chern classes. The

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following §4 and §5 are devoted to two short surveys on such "total class versions". We outline about those below.

1.2. "Thom polynomials" (Fundamental class ⇒ total class): [22], [24]
The *Tp theory* has been newly developed since the mid of 90's, see M. Kazarian, e.g., [13], [14], R. Rimányi [30], Fehér-Rimányi [9]. Given a pair of a nonsingular G-variety V and an invariant subvariety η, the *Thom polynomial* $Tp(\eta)$ of η in V is defined to be the G-equivariant Poincaré dual to $\iota_*[\eta]_G \in H_*^G(V)$, where ι is the inclusion. A particularly interesting case is that V is a G-affine space, then the Thom polynomial of η is expressed by an universal polynomial in G-characteristic classes:

$$Tp(\eta) := \mathrm{Dual}_G \iota_*[\eta]_G \in H^*_G(V) \simeq H^*_G(pt) \simeq H^*(BG).$$

The recent Tp theory provides a systematic study of such universal polynomials, including especially the method to compute Tp for any "singularity types" η.

Here I propose a "total class version of $Tp(\eta)$" as the "Segre class" for $C_*^G(1\!\!1_\eta)$, that gives an "integration of invariant functions" $s^{SM} : \mathcal{F}^G_{inv}(V) \to H^*(BG)$, (see 4.2). In fact the lowest degree homogeneous term of $s^{SM}(1\!\!1_\eta)$ is just $Tp(\eta)$,

Historically, Tp has appeared in a modern enumerative theory of singularities of complex analytic or real smooth maps. I will talk a bit about $s^{SM}(\mu)$ where μ is the Milnor number function. In fact the integration of such local invariants of maps has been a missing part in the Tp theory so far.

1.3. "Orbifold Chern classes" (Euler characterisitcs ⇒ total class): [23]
Let us consider a typical example, the symmetric product S^nX of a complex (possibly singular) variety X. There have been many studies on *generating funcitons* for several "orbifold Euler characteristics" of S^nX: Euler characteristics (Macdonald [17]), orbifold Euler characterisitics (e.g., Hirzebruch-Höfer [12]) and its generalization (Bryan-Fulman [6]). As "total class versions" of these formulae, I will give *generating functions of orbifold Chern homology classes* of S^nX. For instance, the generating function formula for orbifold Euler characterisitics χ^{orb} of S^nX is generalized to

$$\sum_{n=0}^{\infty} C_*^{orb}(S^nX)z^n = \prod_{k=1}^{\infty}(1-z^kD^k)^{-C_*(X)} \qquad (ob)$$

in the $\mathbb{Q}$-algebra $\sum_{n=0}^{\infty} z^n H_*(S^nX;\mathbb{Q})$ of the formal power series whose coefficients are total homology classes. Here D is a letter indicating *diagonal operators*. The result is stated more generally as *the "Dey-Wohlfahrt formula"* (an exponential formula) for equivariant Chern classes of X associated to S_n-representations of a group A. In particular, if X is a point, this recovers the exponential formula for the numbers $|\mathrm{Hom}(A,S_n)|$ (the classical Dey-Wohlfahrt formula). There is an equivariant version, i.e., the quotient via a wreath product $G \sim S_n$ (semidirect product), but we omit it here.

This direction tends to the further theory of Chern classes and their generating functions for more complicated graded spaces arising in several moduli problems.

1.4. I should mention to other characteristic classes or natural transformation: Brasselet-Schürmann-Yokura [3] have introduced the theory of motivic Chern classes and Hirtzeburch classes, which unifies the Chern-Schwartz-MacPherson class class and Baum-Fulton-MacPherson's (singular) Todd class. So, it would be a promising task to look for a similar type formulae in (singular) Todd classes as in (*ob*) above by passing through thier theory.

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2. Preliminary

This is a quick introduction to Chern classes in connection with [2].

2.1. **Chern classes of vector bundles.** As seen in [2], the top Chern class $c_n(TM)$ for a compact complex manifold M ($n = \dim M$) is defined as the obstruction class for non-zero vector fields (the Poincaré-Hopf theorem). Also the i-th Chern class $c_i(TM)$ is given by the obstruction class for the existence of $(n-i+1)$-frames over M: roughly saying, let s be a generic collection of $n-i+1$ vector fields over M, then the singular set $\eta(s)$, at which s is linearly dependent, represents the obstruction class (ι is the inclusion),

$$\iota_*[\eta(s)] = c_i(TM) \frown [M]$$

(cf. Example 4.1 (Thom-Porteous formula); later we will define *Thom polynomials* as this kind of obstruction classes). The total Chern class $c(TM)$ means the formal sum $1 + c_1(TM) + \cdots + c_n(TM) \in H^*(M)$.

Chern classes are actually defined for (topological) complex vector bundles, not only tangent bundles TM. Then Chern classes are characterized by the function c assigning to a complex vector bundle $E \to M$ a total class $c(E) = \sum c_i(E)$ where $c_i(E) \in H^{2i}(M)$ so that it satisfies the following axiom (for the detail, in topology, see [19]; in algebraic geometry, see [10]):

- $c_0(E) = 1$ and $c_i(E) = 0$ for $i >$ rank E;
- $c(E) = c(E')c(E'')$ for any exact sequence $0 \to E' \to E \to E'' \to 0$;
- $c(f^*E) = f^*c(E)$ where f^*E is the pullback bundle of $E \to N$ via a base change $f : M \to N$;
- $c_1(\bar{\gamma}^1) \frown [\mathbb{C}P^1] = 1$, for the canonical line bundle $\bar{\gamma}^1 (= O(1))$ over the projective space $\mathbb{C}P^1$.

An important fact in topology is that any rank n vector bundle can be obtained from a universal vector bundle ξ_n over *the classifying space* $BGL(n)$: for any $E \to M$, there is *a classifying map* (unique up to homotopy) $\rho : M \to$

$BGL(n)$ so that $E = \rho^*\xi_n$. Here $BGL(n)$ and ξ_n are given by the inductive limit of the grassmannian of n-dimensional subspaces in $(n+k)$-dimensional affine spaces ($k \to \infty$) and the limit of tautological vector bundles over the grassmanianns, respectively. Since $H^*(BGL(n)) = \mathbb{Z}[c_1, \cdots, c_n]$ (degree of $c_i (= c_i(\xi_n))$ is $2i$), any Chern classes are obtained from those generators: $c_i(E) = \rho^* c_i$.

In algebraic geometry, Totaro [35] introduced an algebraic construction of the classifying space BG for algebraic group G, which we will use later. An algebraic counterpart of classifying maps is given also in [35].

2.2. **Chern class for singular varieties.** For a singular algebraic variety X, the tangent bundle does not exist, so we need some "substitutes" for frames or tangent bundles in order to define reasonable "Chern classes" for X, see [2]. The most particular feature is that those "Chern classes" are no longer cohomology classes of X, but are homology classes, because of the lack of Poincaré duality. In 1965 M. Schwartz [32] defined a certain obstruction class for "radial vector fields (frames)" on X, which is today called the Chern-Schwartz-MacPherson class $C_*(X) \in H_*(X)$: The degree $C_0(X)$ is equal to $\chi(X)$ and the top component $C_n(X)$ is equal to the fundamental class $[X]$.

The axiomatic description is due to R. MacPherson: he showed in [18] (as a solution of Deligne-Grothendieck conjecture) that there exists a unique natural transformation $C_* : \mathcal{F}(X) \to H_*(X)$, where $\mathcal{F}(X)$ is the group of constructible functions over X, so that

- (natrural transform) C_* is a homomorphism of additive groups, and $f_* \circ C_* = C_* \circ f_*$ for proper morphisms $f : X \to Y$;
- if X is non-singular, then $C_*(1\!\!1_X) = c(TM) \frown [X]$.

In [4], it is shown that Schwartz's class and MacPherson's one coincide as $C_*(X) = C_*(1\!\!1_X)$.

In algebraic context, G. Kennedy [15] reformulated MacPherson's transformation, that is, $C_* : \mathcal{F}(X) \to A_*(X)$, through the Lagrange cycle approach.

In the following sections, based on the algebraic Borel construction, we will combine two related but different stories as mentioned above.

3. Equivariant Chern class theory

A *G-action* on a variety X is a morphism $G \times X \to X$, $(g, x) \to g.x$, with properties $h.(g.x) = (hg).x$ ($h, g \in G$) and $e.x = x$ (e is the identity element of G). We call X a G-variety for short. A morphism $f : X \to Y$ between G-varieties is called *G-equivariant* if $f(g.x) = g.f(x)$ for any $g \in G$, $x \in X$. (Precisely, those properties (identities) mean the commutativity of corresponding diagrams of morphims.)

3.1. **Totaro's construction of BG.** Let G be a complex linear algebraic group of dimension g. Take an l-dimensional linear representation V of G

and a G-invariant Zariski closed subset S in V so that G acts on $U := V - S$ freely. Let $I(G)$ denote the collection of such U (that is, all pairs (V, S)). We say $U < U'$ (where $U = V - S$, $U' = V' - S'$) if there is a representation V_1 satisfying $V \oplus V_1 = V'$, $U \oplus V_1 \subset U'$ and $\operatorname{codim}_V S < \operatorname{codim}_{V'} S'$. Then $(I(G), <)$ is a directed set (in fact $U, U' < U \oplus U'$). All quotients $U \to U/G$ form an inductive system over $I(G)$ (via inclusion maps). The inductive limit is the algebraic counterpart of the universal principal bundle $EG \to BG$ ([35], [7]).

Let X be a G-variety. For any $U \in I(G)$, the diagonal action of G on $X \times U$, which is always a free action, gives a principal bundle $X \times U \to X \times_G U = (X \times U)/G$, and thus the equivariant projection $X \times U \to U$ serves the fibre bundle $X \times_G U \to U/G$ with fibre X. We denote $X_U := X \times_G U$ for short. Roughly saying, the universal fibre bundle $X \times_G EG \to BG$ is approximated by those mixed quotients $X_U \to U/G$ ([7]).

3.2. **G-equivariant (co)homology.** Let X be a G-variety of equidimensional n. We recall the definition of the equivariant (co)homology ([35], [7]).

The i-th equivariant cohomology of X is given as the projective limit

$$H^i_G(X) = \varprojlim H^i(X_U).$$

We denote $H^*_G(X) = \varprojlim H^*(X_U)$. This becomes a contravariant functor (the pullback of a G-morphism f is denoted by f^*_G).

Let ξ be a G-equivariant vector bundle $E \to X$ (i.e., E, X are G-varieties and the projection is G-equivariant so that the action of g sends each fibre E_x to $E_{g.x}$ linearly). Then we have a vector bundle $E_U \to X_U$ over the mixed quotient for each U, denoted by ξ_U, and define the *G-equivariant Chern class* $c^G(\xi) \in H^*_G(X)$ to be the projective limit of Chern classes $c(\xi_U)$. When $X = \{pt\}$, an equivariant vector bundle is a representation $V(\to \{pt\})$; its equivariant Chern class is denoted by $c^G(V) \in H^*_G(V) \simeq H^*(BG)$.

Next, let us define the homology as follows: a key point is *shifting dimensions via pullback*. For any pair $U < U'$, we have a diagram of natural projection and inclusion

$$X_U \overset{p}{\leftarrow} X_{U \oplus V_1} \overset{\iota}{\to} X_{U'} \qquad (*)$$

Put $g = \dim G$, $l = \dim U$, $l' = \dim U'$ and $s = \operatorname{codim}_V S$ ($U = V - S$). Note that ι is an open embedding, so the pullback ι^* is defined and is in fact isomorphic if $2(n - s) < i \leq 2n$ because of the (co)dimensional reason. We prefer to denote its inverse by $\iota_* := (i^*)^{-1}$, and then the diagram $(*)$ induces

$$H_{i+2(l-g)}(X_U) \underset{\simeq}{\overset{p^*}{\to}} H_{i+2(l'-g)}(X_{U \oplus V_1}) \underset{\simeq}{\overset{\iota_*}{\to}} H_{i+2(l'-g)}(X_{U'}) \qquad (*_H)$$

We then define *the i-th equivariant homology group* to be

$$H^G_i(X) = H_{i+2(\dim U - g)}(X_U)$$

for U with $\operatorname{codim} S$ high enough. This group is trivial for $i > 2n$ and nontrivial for any negative i in general. The direct sum is denoted by $H_*^G(X) = \oplus H_i^G(X)$. For a proper G-morphism $f : X \to Y$, the push-forwad f_*^G is defined by taking limit of $(f_U)_* : X_U \to Y_U$; thus H_*^G becomes a covariant functor.

For any U, the cycle $[X_U]$ tends to a unique element of $H_{2n}^G(X)$, denoted by $[X]_G$, called *the G-equivariant fundamental class* of X. It induces a homomorphism

$$\frown [X]_G : H_G^{2n-i}(X) \to H_i^G(X), \quad a \mapsto r_U(a) \frown [X_U]$$

where r_U denotes the restriction. If X is nonsingular, this is isomorphic, called *the G-equivariant Poincaré dual*.

Example 3.1. $G = GL(1)$, $X = \{pt\}$ and $U_m = \mathbb{C}^{m+1} - \{0\}$ in $I(G)$:

$$\begin{array}{ccccccc} & & (U_m \times \mathbb{C})/GL(1) & = \mathbb{P}^{m+1} - \{pt\} & & & \\ & \cdots \overset{p}{\swarrow} & & & \overset{\iota}{\searrow} \cdots & & \\ \cdots \subset & \mathbb{P}^m & & \subset & & \mathbb{P}^{m+1} & \subset \cdots \subset \mathbb{P}^\infty \end{array}$$

Then $H_{-2n}^{GL(1)}(pt) \simeq H_{GL(1)}^{2n}(pt) = \mathbb{Z}$ for $n \geq 0$, and trivial otherwise.

3.3. **Equivariant constructible functions.** Let $\mathcal{F}(X)$ denote the Abelian group consisting of all constructible functions over X. The subgroup of G-invariant constructible functions is denoted by

$$\mathcal{F}_{inv}^G(X) := \{\ \alpha \in \mathcal{F}(X) \mid \alpha(g(x)) = \alpha(x),\ x \in X, g \in G\ \}.$$

For any $U < U'$ with the projection $p : V' = V \oplus V_1 \to V$, we have $p^* : \mathcal{F}_{inv}^G(X \times V) \to \mathcal{F}_{inv}^G(X \times V')$ $(\alpha \mapsto \alpha \circ (id \times p))$. We define *the group of G-equivariant constructible functions associated to X* to be the inductive limit

$$\mathcal{F}^G(X) := \varinjlim \mathcal{F}_{inv}^G(X \times V).$$

For a proper G-morphism f, *pushforward* f_*^G are defined in an obvious way, so $\mathcal{F}^G$ becomes a covariant functor. There is a canonical inclusion, denoted by ϕ_0,

$$\mathcal{F}_{inv}^G(X) \subset \mathcal{F}^G(X), \quad \alpha_0 \mapsto \lim(\alpha_0 \times 1\!\!1_V).$$

Note that $\mathcal{F}_{inv}^G(pt) = \mathbb{Z}$, but $\mathcal{F}^G(pt)$ consists of functions over representations.

Now let us think of the group $\mathcal{F}(X_U)$ of (ordinary) constructible functions over mixed spaces. The previous diagram $(*)$ induces

$$\mathcal{F}(X_U) \xrightarrow{p^*} \mathcal{F}(X_{U \oplus V_1}) \xrightarrow{\iota_*} \mathcal{F}(X_{U'}) \qquad (*_F)$$

Note that $\mathcal{F}(X_U)$ is canonically identified with $\mathcal{F}_{inv}^G(X \times U)$. Then the above composed map $\iota_* p^*$ commutes with $p^* : \mathcal{F}_{inv}^G(X \times V) \to \mathcal{F}_{inv}^G(X \times V')$ via restrictions caused by $U \subset V$ and $U' \subset V'$.

3.4. **Equivariant natrual transformation.** The space $X \times_G EG$ is the *inductive limit* (via inclusions) of mixed quotients X_U, while the definitions of H_*^G and $\mathcal{F}^G$ involve a *contravariant* operation p^*. Roughly saying, our idea to define $C_*^G(1\!\!1_X)$ is to take the inductive limit of $C_*(X_U)$ and multiply it by the inverse Chern class factor "$c(T_{BG})^{-1}$". This factor relates to p^* via *the Verdier-Riemann-Roch theorem* (VRR theorem).

Recall that the VRR refers to C_* with the contravariant operation induced by $f : X \to Y$ ([11]):

Theorem 3.2. ([37], cf. [31]) *For a smooth morphism $f : X \to Y$, let $c(f)$ be the Chern class of the relative tangent bundle. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{F}(Y) & \xrightarrow{C_*} & H_*(Y) \\ f^* \downarrow & & \downarrow\ c(f) \frown f^* \\ \mathcal{F}(X) & \xrightarrow{C_*} & H_*(X) \end{array}$$

Now we put

$$C_{U,*} := c^G(V)^{-1} \frown C_* : \mathcal{F}(X_U) \to H_{tr}(X_U).$$

where H_{tr} means the direct sum of H^i over $2(n-s+l-g) < i \le 2(n+l-g)$ ($l = \dim U$, $s = \operatorname{codim}_V S$). Note that $c^G(V)$ is the Chern class of the vector bundle $X \times_G TU \to X_U$.

• We apply the VRR theorem for $p : X_{U \oplus V_1} \to X_U$, then it turns out that $C_{U\oplus V_1,*} \circ p^* = p^* \circ C_{U,*}$.

• For an open embedding $\iota : X_{U\oplus V_1} \to X_{U'}$, it holds that $\iota_* \circ C_{U\oplus V_1,*} = C_{U',*} \circ \iota_*$ in this renge of dimension.

Consequently, "Radon transforms" $(*)_F$ and $(*)_H$ commute as follows (cf., [8]):

$$\begin{array}{ccc} \mathcal{F}(X_U) & \xrightarrow{C_{U,*}} & H_{tr}(X_U) \\ \iota_* p^* \downarrow & & \downarrow \iota_* p^* \\ \mathcal{F}(X_{U'}) & \xrightarrow{C_{U',*}} & H_{tr}(X_{U'}) \end{array}$$

Since $p^* : \mathcal{F}^G_{inv}(X \times V) \to \mathcal{F}^G_{inv}(X \times V')$ commutes with the left $\iota_* p^*$ (as noted in the end of 3.3) , we can take the limit homomorphism

$$C_*^G := \varinjlim C_{U,*} : \mathcal{F}^G(X) \to H_*^G(X).$$

This is our equivariant Chern-MacPherson transformaition in Theorem 1.1.

3.5. **Quotient stacks.** By definition, a *quotient stack* $\mathcal{X} = [X/G]$ is a category itself, whose objects are principal G-bundles $p : P \to B$ together with G-equivariant morphism $\varphi : P \to X$ and its arrows are morphisms between principal bundles which make their equivariant morphisms to X commute. In [7] (Proposition 16) , Edidin-Graham introduced *the integral Chow groups of a quotient stack* $\mathcal{X} = [X/G]$ by $A_*(\mathcal{X}) := A^G_{*-g}(X)$ where $g = \dim G$; in fact it is independent from the choice of the presentation X and G. In the exactly same way, $\mathcal{F}(\mathcal{X}) := \mathcal{F}^G(X)$ is well-defined.

As for "arrows", $f : \mathcal{X} \to \mathcal{Y}$ under consideration is assumed to have presentations $\mathcal{X} = [X/G]$ and $\mathcal{Y} = [Y/H]$ so that f is induced from a morphism $\bar{f} : X \to Y$ and a group-scheme homomorphism $h_f : G \to H$ with $\bar{f}(g.x) = h_f(g).\bar{f}(x)$ (i.e, Y has a G-action via h_f and $\bar{f}$ is G-equivariant). Then, Theorem 1.1 can be translated as follows:
For the (above) category of quotient stacks $\mathcal{X}$, there is a natural transformation $C_* : \mathcal{F}(\mathcal{X}) \to A_*(\mathcal{X})$ so that for any nonsingular varieties $\mathcal{X} = X$ (with $G = \{e\}$), it holds that $C_*(1\!\!1_X) = c(TX) \frown [X]$.

Note that an object of $[X/G]$ means "a family of G-orbits in X which is parametrized by B", that admits another interpretation: $[X/G]$ can be also regarded as the universal space for *sections* of associated bundles with fibre X and structure group G. In fact, let $P \to B$ be a principal G-bundle and set $E = P \times_G X$, then an equivariant morphism $\varphi : P \to X$ corresponds to a section $s : B \to E$ $(s(B) = (\mathrm{graph}\,\varphi)/G)$.

4. Thom polynomials

4.1. **Universality.** Let V be a G-affine space and η an invariant subvariety. Let $i : \eta \to V$ be the inclusion map. The *Thom polynomial* $Tp(\eta) \in H^*(BG)(= H^*_G(V))$ is defined to be the G-Poincaré dual to $i^G_*[\eta]_G \in H^G_*(V)$ ([13], [9], [29]). By definition, $Tp(\eta)$ satisfies the following universality:
(u): For any bundle $E \to M$ with fibre V and the structure group G over a manifold M of dimension m, let $E_\eta \to M$ be the associated fibre bundle with fibre η. For a "generic" section $s : M \to E$ (e.g., s is transverse to the subvariety E_η in E) , we define *the singular set of type* η

$$\eta(s) := s^{-1}(E_\eta),$$

which has the expected codimension $l = \mathrm{codim}\,\eta$. Let $i : \eta(s) \to M$ be the inclusion. Then, the fundamental class of the singular set is expressed in M by

$$i_*[\eta(s)] = Tp(\eta)(c(E)) \frown [M] \in H_{2(m-l)}(M)$$

after substituting $c_i(E)$ to universal classes c_i.

As a remark, we may drop the condition in (u), "genericity" of the section and/or "smoothness" of the base space, if we correct the formula by replacing $i_*[\eta(s)]$ by a certain localized class (a residue class) of expected dimension.

Example 4.1. (Thom-Porteous formula) Let $V := Hom(\mathbb{C}^m, \mathbb{C}^n)$ on which the group $G = GL(m,\mathbb{C}) \times GL(n,\mathbb{C})$ acts as linear coordinate changes. Take $\eta := D^k$, the closure of the orbit of linear maps with $\dim\ker = k$. Then the Thom polyomial is given by a certain Schur polynomial: $Tp(D^k)(c) = \Delta^{(k)}_{k+l}(c)(:= \det[c_{k+l-i+j}])$, [29]. Namely, for a suitably generic vector bundle map $f : E \to F$ (i.e., a section $f : M \to \mathrm{Hom}(E,F)$), where E and F are vector bundles over a manifold M of rank m and n, respectively, then the degeneratcy loci $D^k(f)(= \{x \in M, \dim\ker f_x \geq k\})$ is expressed by

$$\iota_*[D^k(f)] = \det[c_{k+l-i+j}(F-E)]_{1\leq i,j\leq k} \frown [M].$$

where $c(F-E) = (1+c_1(F)+\cdots)(1+c_1(E)+\cdots)^{-1}$. To correct the formula without the genericity/smoothness condition, for instance, in algebraic context Fulton [10] defines *the degeneracy class* by the pushforward of the localized top Chern class of a bundle over the Granssmaiann bundle $Gr(k;E)$. As for a localization in complex analytic geometry, see e.g. Suwa [34].

4.2. **Segre-Schwartz-MacPherson class.** For a subvariety Z in a non-singular variety M, the *Segre-SM class* of Z in M is defined (cf. Aluffi [1], [22])

$$s^{SM}(Z,M) := c(TM|_Z)^{-1} \frown C_*(Z) \ \in \ H_*(Z)$$

which is an analogy to the relation between Fulton's canonical class $C^F(Z)$ and the Segre covariance class $s(Z,M)$: in fact $C^F(Z)$ is defined to be $c(TM|_Z) \frown s(Z,M)$ ([10]).

Now, for an invariant subvariety η in a non-singular G-variety V, we define *the equivariant Segre-Schwartz-MacPherson class* by

$$s_G^{SM}(\eta,V) := c^G(TV|_\eta)^{-1} \frown C_*^G(1\!\!1_\eta) \ \in \ H_*^G(\eta).$$

In particular, suppose V is a G-affine space, then we have an additive homomorphism

$$s^{SM} : \mathcal{F}_{inv}^G(V) \to H^*(BG), \ s^{SM}(1\!\!1_\eta) := \mathrm{Dual}_G\, \iota_* s_G^{SM}(\eta,V).$$

Theorem 4.2. *Let V and η be as above. Then*
(1) $s^{SM}(\eta)(:= s^{SM}(1\!\!1_\eta)) = Tp(\eta) +$ higher degree terms;
(2) For G-morphisms $\iota : V' \to V$ being transverse to η, it holds that

$$s^{SM}(\iota^* 1\!\!1_\eta) = \iota^* s^{SM}(1\!\!1_\eta).$$

(3) (universality) Let $E \to M$ and a generic section s be as in (u) of 4.1, then we have

$$i_* s^{SM}(\eta(s),M) = s^{SM}(\eta)(c(E)) \frown [M] \in H_*(M).$$

In particular, the Euler characteristic $\chi(\eta(s))$ admits a universal expression, that is, the degree $\int_M c(TM)s^{SM}(\eta)(c(E))$.

This is an immediate consequence from the definition. A prototype of s^{SM} appeared implicitly in Parsiński-Pragacz's work (Theorem 2.1 in [27]) for degeneracy loci of vector bundle maps (cf. Example 4.1).

As a remark on (3), the base space M can be possibly singular: under suitable "genericity" of s (cf. [27]), the formula is replaced by

$$i_* C_*(\eta(s)) = s^{SM}(\eta)(c(E)) \frown C_*(M) \ \in \ H_*(M).$$

4.3. **Singularities of maps.** Let $k = \mathbb{C}$ or $\mathbb{R}$. In the Tp theory in Singularity Theory of Differentiable Mappings, V is the space $\mathcal{E}(m,n)$ of map-germs $k^m, 0 \to k^n, 0$ and G is the $\mathcal{A}$-equivalence group $\mathcal{A}_{m,n}$ or the $\mathcal{K}$-equivalence group $\mathcal{K}_{m,n}$. More precisely, we think of their jet spaces which are finite dimensional. In case that G is the $\mathcal{K}$-equivalence group, we can take a stabilization of $\mathcal{K}$-orbits via embeddings $\to \mathcal{E}(m,n) \to \mathcal{E}(m+1,n+1) \to$

defined by trivial unfoldings. It turns out that for any $\mathcal{K}$-invariant subvariety η, $Tp(\eta)$ is a universal polynomial in the Chern classes of the virtual normal bundle $c_i(f) = c_i(f^*TN - TM)$ ([21], [13], [9]). We also denote $\bar{c}_i(f) = c_i(TM - f^*TN)$.

In the above setting, the Segre class s^{SM} can be also defined: $s^{SM}(\eta)$ is a formal sum of polynomials in $c_i = c_i(f)$ $(i \geq 0)$, whose leading term is $Tp(\eta)$. Let us see an example in case that $l := m - n \geq 0$. For the $\mathcal{K}$-invariant "Milnor number constructible function" $\mu : \mathcal{E}(m,n) \to \mathbb{Z}$ (off the germs whose Milnor number is not defined), it can be shown that

Theorem 4.3. ([23]) $s^{SM}(\mu) = (-1)^{l+1} \sum_{i,j\geq 0} c_i\bar{c}_{l+j+1}$.

This is a unversal expression arising from the following simple equality (e.g., Yomdin [39], Nakai [20]):

$$f_*(1\!\!1_M + (-1)^{m-n+1}\mu(f)) = \chi(F)1\!\!1_N$$

for a finite type morphism $f : M \to N$ betwen complex manifolds (i.e., the Milnor number $\mu(f)(x) < \infty$ for each $x \in M$) with generic fibre F. Further, this relates to a "relative version" of *the Milnor class* (cf., [33], [25], [26]; [28], [5]). That suggests a connection between the recent Tp theory and the geometry of polar varieties (e.g., Lê-Teissier [16]).

In real analytic case ($k = \mathbb{R}$), C_* should be replaced by the Stiefel-Whitney homology class W_*: Sullivan's definition of $W_i(X)$ of a real analytic variety X is the sum of all i-simplices in the barycentric subdivision of a subanalytic triangulation of X. Let η be a $\mathcal{K}$-invariant subvariety in V and $f : M \to N$ a suitably generic map between real analytic manifolds. Then the fundamental $\mathbb{Z}_2$-cycle of $\eta(f)$ is expressed by a universal polynomial (with $\mathbb{Z}_2$-coefficient) $Tp(\eta)_2$ in $w_i(f) = w_i(f^*TN - TM)$, as well its "Segre-version" is: $i_*(w(TM)^{-1} \frown W_*(\eta(f))) = s^{SM}(\eta)_2(w(f)) \frown [M]_2$.

Finally, we remark that *multiple point formulae* are not Tp in the above sense (i.e., those are not the case of action of a group). In fact, multiple point formulae are included into *the Tp theory for multi-singularities* $k^m, S \to k^n, 0$ (S: finite, $m < n$), that was recently established in topology by Maxim Kazarian [14] using classical cobordism theory. The "classifying space $B\mathcal{K}_{multi}$ for multi-singularities" (where $\mathcal{K}_{multi}$ is a groupoid, not a group) admits a stratification by classifying spaces $B\eta$ for indivisual singularity types η at least up to homotopy (cf. Thom-Pontragin-Szücs construction for singular maps). Note that the algebraic counterpart has been missing.

5. Orbifold Chern classes

5.1. **Canonical constructible functions.** At first, recall that for a quotient variety (an orbifold) $\mathcal{X} = X/G$ of a possibly singular variety X with an action of a finite group G, there are two kinds of Euler characteristics (ordinary one and physist's one):

$$\chi(X/G) = \frac{1}{|G|}\sum_{g\in G}\chi(X^g), \quad \chi^{orb}(X;G) = \frac{1}{|G|}\sum_{gh=hg}\chi(X^{h,g}).$$

Here X^g is the set of fixed points of g and $X^{h,g} := X^h \cap X^g$, and the second sum runs over all pairs $(h,g) \in G\times G$ such that $gh = hg$ (see e.g., Hirzebruch-Höfer [12]).

Now let G be an algebraic group and X a G-variety. Let A be a group, and assume that $\mathrm{Hom}(A,G)$ is a scheme, on which G acts by $(g.\rho)(a) := \rho(a)g$. We put

$$Z := \{\, (x,\rho) \in X \times \mathrm{Hom}(A,G) \mid \rho(a).x = x \; (\forall a \in A) \,\}$$

(with the diagonal action of G). The projection to the first factor is denoted by $\pi : Z \to X$, which is a G-morphism. We define *the integer canonical constructible functions over X associated to (all G-representations of) a group A* by

$$\alpha^{(A)}_{X/G} := \pi^G_* 1\!\!1_{Z_{red}} \in \mathcal{F}^G_{inv}(X).$$

In particular, if G is a finite group, then we define *the rational cannonical function* to be the average $1\!\!1^{(A)}_{X/G} = \frac{1}{|G|}\alpha^{(A)}_{X/G}$ in $\mathcal{F}^G_{inv}(X)\otimes\mathbb{Q}$. If $A = \mathbb{Z}^m$, we denote the canonical funciton simply by $1\!\!1^{(m)}_{X/G}$.

We call $C^G_*(\alpha^{(A)}_{X/G}) \in H^G_*(X)$ (resp. $C^G_*(1\!\!1^{(A)}_{X/G})$) the integer (resp. rational) *canonical quotient Chern class* associated to A.

In the case that G is a finite group and X/G is a variety, it immediately follows that

$$\int_X 1\!\!1^{(1)}_{X/G} = \chi(X/G), \quad \int_X 1\!\!1^{(2)}_{X/G} = \chi^{orb}(X;G).$$

Furthermore, in this case, it holds that $H^G_*(X;\mathbb{Q}) \simeq H_*(X/G;\mathbb{Q})$ (cf., [7]). By this identification, in rational coefficients the canonical Chern class $C^G_*(1\!\!1^{(1)}_{X/G})$ is identified with the ordinary Chern-SM class $C_*(X/G)$ of the quotient variety (In (*ob*) in Introduction, we used the notation C^{orb} to mean the corresponding class to $C^G_*(1\!\!1^{(2)}_{X/G})$).

5.2. **Symmetric products.** From now on, we concentrate the case that $G = S_n$, the n-th symmetric group acting the n-th Cartesian product X^n of a variety X as $\sigma(x_1,\cdots,x_n) := (x_{\sigma^{-1}(1)},\cdots,x_{\sigma^{-1}(n)})$. $\mathcal{F}$ and H_* are assumed to have rational coefficients and omit the notation $\otimes\mathbb{Q}$.

For $\alpha \in \mathcal{F}^{S_m}_{inv}(X^m)$ and $\beta \in \mathcal{F}^{S_n}_{inv}(X^n)$ (or corresponding homologies), we define the product $\odot$ by

$$\alpha\odot\beta := \frac{1}{|S_{m+n}|}\sum_{\sigma\in S_{m+n}}\sigma_*(\alpha\times\beta),$$

where σ_* is the pushforward induced by $\sigma : X^{m+n} \to X^{m+n}$. This produces commutative and associative graded $\mathbb{Q}$-algebras of formal power series

$$\mathcal{F}_{X,sym}[[z]] := \sum_{n=0}^{\infty} z^n \mathcal{F}_{inv}^{S_n}(X^n), \quad H_{X,sym}[[z]] := \sum_{n=0}^{\infty} z^n H_{2*}^{S_n}(X^n).$$

We denote $\alpha \odot \cdots \odot \alpha$ (c times) by α^c or $\alpha^{\odot c}$. For a proper morphism $f : X \to Y$, the n-th cartesian product $f^n : X^n \to Y^n$ induces a homomorphism of algebras (as well homology case) $f_*^{sym} : \mathcal{F}_{X,sym}[[z]] \to \mathcal{F}_{Y,sym}[[z]]$ given by $f_*^{sym}(\sum \alpha_n z^n) := \sum f_*^n \alpha_n z^n$.

Theorem 5.1. *The following C_*^{sym} gives a natural transformation (as a $\mathbb{Q}$-algebra homomorphism)*

$$C_*^{sym} : \mathcal{F}_{X,sym}[[z]] \to H_{X,sym}[[z]], \quad \sum_{n=0}^{\infty} \alpha_n z^n \mapsto \sum_{n=0}^{\infty} C_*^{S_n}(\alpha_n) z^n.$$

5.3. **Types of permutations.** A collection $\mathbf{c} = [c_1, \cdots, c_n]$ of non-negative integers satisfying $|\mathbf{c}| := \sum_{i=1}^n i c_i = n$ is called *a type of weight* n. We put $\sharp\mathbf{c} := n!/1^{c_1}c_1!2^{c_2}c_2! \cdots n^{c_n}c_n!$, which is equal to the number of elements of a conjugacy class in S_n (each element has c_i cycles of length i).

For a group A, we denote by $j_r(A)$ the number of the subgroups of A with index r. Here is the "Dey-type formula" on canonical constructible functions:

Lemma 5.2. *The canonical function $1\!\!1_{X^n/S_n}^{(A)}$ is equal to the sum*

$$\sum_{|\mathbf{c}|=n} \frac{\sharp[c_1, \cdots, c_n]}{n!} \cdot (j_1(A) \cdot 1\!\!1_{\Delta X})^{c_1} \odot \cdots \odot (j_n(A) \cdot 1\!\!1_{\Delta X^n})^{c_n}.$$

5.4. **Diagonal operators.** *The standard n-th diagonal operator D^n* ($n = 0, 1, \cdots$) is defined to be the pushforward homomorphisms: $D^0 = 1$, $D^1 = D = id_*$ ($id : X \to X$) and

$$D^n := (\Delta^n)_* : \mathcal{F}(X) \to \mathcal{F}_{inv}^{S_n}(X^n)$$

(as well homology case) where $\Delta^n : X \to X^n$ is the diagonal inclusion map, $\Delta^n(x) := (x, \cdots, x)$. We call $U := \sum_{n\geq 1} a_n z^n D^n$ ($a_n \in \mathbb{Q}$) *a standard diagonal operator*; in particular, we put $Log(1 + zD) := \sum_{n\geq 1} \frac{z^n}{n} D^n$.

The mixed n-th diagonal operator of type $\mathbf{c} = [c_1, \cdots, c_n]$ means the maps $\mathcal{D}^{\mathbf{c}} : \mathcal{F}(X) \to \mathcal{F}_{inv}^{S_n}(X^n)$ (as well homology case) given by

$$\mathcal{D}^{[c_1,\cdots,c_n]}(\alpha) := (D^1(\alpha))^{c_1} \odot (D^2(\alpha))^{c_2} \odot \cdots \odot (D^n(\alpha))^{c_n}.$$

We also define *a formal diagonal operator* as a formal series $T = \sum_{n=0}^{\infty} z^n T_n$ of linear combinations $T_n = \sum_{|\mathbf{c}|=n} v_{\mathbf{c}} \mathcal{D}^{\mathbf{c}}$. Every formal operator T operates on $\mathcal{F}(X)$ and $H_*(X)$. By using $\odot$ we define $exp(T) := \sum_{n=0}^{\infty} \frac{1}{n!} T^{\odot n}$ for T with zero constant term, and we make a convention of notation

$$(1 + U)^\alpha := exp(Log(1 + U)(\alpha)) \quad \text{for a standard } U.$$

5.5. **Exponential formula.** For a group A, let $\Omega_A(r)$ (resp. Ω_A) denote the set of all subgroups B in A of index $|A:B| = r$ (resp. subgroups of finite index) and $j_r(A) := |\Omega_A(r)|$. A direct computation shows

Proposition 5.3. *It holds that*

$$\sum_{n=0}^{\infty} 1\!\!1^{(A)}_{X^n/S_n} z^n = exp\left(\sum_{r=1}^{\infty} \frac{j_r(A)}{r} z^r D^r(1\!\!1_X)\right).$$

Apply C_*^{sym} to the both sides of this equality of constructible functions. Since it holds that $T \circ C_* = C_*^{sym} \circ T$ for any T, we obtain the following theorem:

Theorem 5.4. "**Dey-Wohlfahrt formula for Chern classes**": *Assume that $j_r(A) < \infty$ for any r. Then it holds that*

$$\sum_{n=0}^{\infty} C_*^{S_n}(1\!\!1^{(A)}_{X^n/S_n}) z^n = exp\left(\sum_{B\in\Omega_A} \frac{1}{|A:B|}(zD)^{|A:B|} C_*(1\!\!1_X)\right).$$

Remark 5.5. When $X = pt$, this theorem gives the classical Dey-Wohlfahrt formula on the enumeration of S_n-representations of a group, [36]

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(A, S_n)|}{n!} z^n = exp\left(\sum_{B\in\Omega_A} \frac{z^{|A:B|}}{|A:B|}\right).$$

5.6. **Examples.**

(1) $A = \mathbb{Z}$ ($\mathrm{Hom}(\mathbb{Z}, S_n) \equiv S_n$). Our Chern class formula (Theorem 5.4) is written as

$$\sum_{n=0}^{\infty} C_*(S^n X) z^n = (1 - zD)^{-C_*(X)}.$$

This generalizes well-known Macdonald's formula of Euler characteristics of $S^n X$ ([18]):

$$\sum_{n=0}^{\infty} \chi(S^n X) z^n = (1 - z)^{-\chi(X)}.$$

(2) $A = \mathbb{Z}^m$ ($\mathrm{Hom}(\mathbb{Z}^m, S_n)$ corresponds to the set of mutually commuting m-tuples of S_n);

$$\sum_{n\geq 0} C_*^{S_n}(1\!\!1^{(m)}_{X^n/S_n})\, z^n = \prod_{r=1}^{\infty} (1 - z^r D^r)^{-j_r(\mathbb{Z}^{m-1}) C_*(X)}.$$

The case $m = 2$ is the formula given in §1 (there, instead I wrote C_*^{orb} in LHS), which is the class version of the generating function of orbifold Euler charactersitics $\chi^{orb}(X^n; S_n)$ given by Hirzebruch-Höfer [12]. For general m, the degree part of the above formula gives the generating function of generalized orbifold Euler charactersitics in Bryan-Fulman [6]:

$$\sum_{n=0}^{\infty} \chi_m(X^n; S_n)\, z^n = \prod_{j_i \geq 1} (1 - z^{j_1 j_2 \cdots j_{m-1}})^{-j_1^{m-2} j_2^{m-3} \cdots j_{m-2}\, \chi(X)}.$$

(3) $A = \mathbb{Z}/d\mathbb{Z}$, the cyclic group of order d:

$$\sum_{n=0}^{\infty} C_*^{S_n}(1\!\!1_{X^n/S_n}^{(\mathbb{Z}/d\mathbb{Z})})\, z^n = exp\left(\sum_{r|d} \frac{1}{r}(zD)^r C_*(1\!\!1_X)\right).$$

(4) $A = \mathbb{Z}_p = \varprojlim \mathbb{Z}/p^k\mathbb{Z}$ (as an additive group, p is a prime number). "the Artin-Hesse exponential for the Chern class of X"

$$\sum_{n=0}^{\infty} C_*^{S_n}(1\!\!1_{X^n/S_n}^{(\mathbb{Z}_p)})\, z^n = exp\left(\sum_{k=0}^{\infty} \frac{1}{p^k}(zD)^{p^k} C_*(1\!\!1_X)\right).$$

References

[1] P. Aluffi, *Inclusion-Exclusion and Segre class, I and II*, arXiv:math:AG/0203122, 0203284.

[2] J. P. Brasselet, lecture note in this volume.

[3] J. P. Brasselet, J. Shürmann and S. Yokura, *Hirzebruch classes and motivic Chern classes for singular spaces*, preprint, math.AG/0503492.

[4] J. P. Brasselet and M. H. Schwartz, *Sur les classes de Chern d'une ensemble analytique complexe*, Astérisque vol 82–83 (1981), 93–148.

[5] J. P. Brasselet, D. Lehmann, J. Seade and T. Suwa, *Milnor classes of local complete intersections*, Trans. AMS. 354 (2002), no. 4, 1351–1371.

[6] J. Bryan and J. Fulman, *Orbifold Euler characteristics and the number of commuting m-tuples in the symmetric groups*, Ann. Combinatorics 2 (1998), 1–6.

[7] D. Edidin and W. Graham, *Equivariant intersection theory*, Invent. Math. 131 (1998), 595–634.

[8] L. Ernström, T. Ohmoto and S. Yokura, *Topological Radon transformations*, Jour. Pure and Applied Algebra, Vol. 120, No. 3, (1997), 235–254

[9] L. Fehér and R. Rimányi, *Calculation of Thom polynomials and other cohomological obstructions for group actions*, Real and Complex Singularities (Sao Carlos, 2002), Contemp. Math., 354, AMS, (2004), 69–93.

[10] W. Fulton, *Intersection Theory*, Springer-Verlag (1984)

[11] W. Fulton and R. MacPherson, *Categorical framework for the study of singular spaces*, Memoirs of the AMS, vol. 243 (1981).

[12] F. Hirzebruch and T. Höfer, *On the Euler number of an orbifold*, Math. Ann. 286 (1990), 255–260.

[13] M. E. Kazarian, *Characteristic Classes of Singularity Theory*, Arnold-Gelfand Seminars, Birkhäuser (1997), 325–340

[14] ———, *Multisingularities, cobordisms and enumerative geometry*, Russian Math. Survey 58:4 (2003). 665–724 (Uspekhi Mat. nauk 58:4 29–88).

[15] G. Kennedy, *MacPherson's Chern classes of singular algebraic varieties*, Comm. Algebra, 18(9) (1990), 2821–2839.

[16] D. -T. Lê and B. Teissier, *Variétés polaires locales et classes de Chern des variétés singulières*, Anal. of Math. 114 (1981), 457–491.

[17] I. G. Macdonald, *The Poincaré polynomial of a symmetric product*, Proc. Camb. Phil. Soc. 58 (1962), 563–568.

[18] R. MacPherson, *Chern classes of singular algebraic varieties*, Ann. of Math. , vol. 100, no.3 (1974), 421–432.

[19] J. Milnor and J. Stasheff, *Characteristic classes*, Ann. Math. Studies, vol. 76, Princeton Univ. Press, Princeton (1974).

[20] I. Nakai, *Elementary topology of stratified mappings*, Singularities (Proc. Sympos., Sapporo 1998), Adv. Stud. Pure Math. 29, Kinokuniya, Tokyo (2000), 221–243

[21] T. Ohmoto, *Vassiliev complex for contact classes of real smooth map-germs*, Rep. Fac. Sci. Kagoshima Univ., 1994, 1–12

[22] ———, *Equivariant Chern classes of singular algebraic varieties with group actions*, (math.AG/0407348) to appear in Math. Proc. Cambridge Phil. Soc..

[23] ———, *Generating functions of orbifold Chern classes I : Symmetric Products* preprint (2005)

[24] ———, *Thom polynomial expression for Segre type classes of singularity loci of maps* preprint (2005)

[25] T. Ohmoto, S.Yokura, *Product formula of Milnor class*, Bull. Polish Academy of Sciences, vol. 48, No. 4 (2000), 388–401

[26] T. Ohmoto, T. Suwa and S. Yokura *A Remark on Chern Classes of Complete Intersections*, Proccedings of Japan Academy, 73, Ser. A (1997), 93–95

[27] A. Parusiński and P. Pragacz, *Chern-Schwartz-MacPherson classes and the Euler characteristic of degeneracy loci and special divisors*, Jour. AMS, 8 (1995), no. 4, 793–817.

[28] ———, *Characteristic classes of hypersurfaces and characteristic cycles*, J. Algebraic Geom. 10 (2001), no.1, 63–79.

[29] I. R. Porteous, *Simple singularities of maps*, Proceedings of Liverpool Singularities I, Lect. Notes Math., vol. 192, Springer (1971), 286–307.

[30] R. Rimányi, *Thom polynomials, symmetries and incidences of singularities*, Inv. Math. 143 (2001), 499-521

[31] J. Schürmann, *A generalized Verdier-type Riemann-Roch theorem for Chern-Schwartz-MacPherson classes*, preprint (2002), math.AG/0202175.

[32] M. H. Schwartz, *Classes caractéristiques définies par une stratification d'une variété analytique complexe*, C. R. Acad. Sci. Paris t.260 (1965), 3262-3264, 3535–3537

[33] T. Suwa *Classes de Chern des intersections complètes locales*, C. R. Acad. Sci. Paris, vol 324 (1996), 67–70.

[34] ———, *Indices of Vector Fields and Residues of Singular Holomorphic Foliations*, Actualités Mathématiques, Hermann (1998).

[35] B. Totaro, *The Chow Ring of a Classifying Space*, Proc. Symposia in Pure Math., vol. 67 (1999), 249–281.

[36] K. Wohlfahrt, *Über einen Satz von Dey und die Modulgruppe*, Arch. math. (Basel) 29, 1977, 455-457.

[37] S. Yokura, *On a Verdier-type Riemann-Roch for Chern-Schwartz-MacPherson class*, Topology and Its Application 94 (1999), 315–327

[38] ———, *Chern classes of proalgebraic varieties and motivic measures*, preprint (2003).

[39] Y. Yomdin, *The structure of strata $\mu = const$ in a critical set of a complete intersction singularity*, "Singularities", Arcata, Proc. Symp. Pure Math. AMS. Vol. 40, Part 2 (1983), 663–666.

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