PRODUCT FORMULAS FOR THE MILNOR CLASS

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ABSTRACT. We give a general product formula for the Milnor class. Using this formula we obtain a Parusiński-Pragacz-type formula for the Milnor class of a finite Cartesian product of hypersurfaces and we show that in general it is necessary to also consider the Chern classes of subbundles of the total vector bundle to obtain a Parusiński-Pragacz-type formula for the Milnor class of general local complete intersections. Furthermore, we also give another kind of product formula for the Milnor class of singular hypersurfaces by using the so-called Thom-Sebastiani construction.

Dedicated to Professor Takuo Fukuda on the occasion of his 60-th birthday

INTRODUCTION

A generalization of the Milnor number of higher dimensional singular locus has been tried by many people and furthermore the Milnor number has been recently extended to the so-called "Milnor class" of a local complete intersection variety, which is supported on its singular locus. The Milnor class is by definition, up to sign, the difference between the virtual class [4, 9] and the Chern-Schwartz-MacPherson class [6, 13, 17, 26]. One of the main problems is to describe this class in terms of some invariants of the singular locus (e.g., see [1, 2, 3, 4, 7, 23, 28, 31, 32, 33]). For hypersurfaces Aluffi [3] described the Milnor class using his μ-class [2] and Parusiński and Pragacz [23] described it using the Chern-Schwartz-MacPherson class of the closures of Whitney strata (cf. [22]). For local complete intersections with isolated singularities Suwa [28] showed that the Milnor class is the sum of the Milnor numbers. More generally, for local complete intersections with arbitrary singularities, Brasselet, Lehmann, Seade and Suwa [4] described the Milnor class in terms of the localized Milnor classes, and in a special case when the singular locus is smooth they described the Milnor class more explicitly, using the Chern class of the locus and some extra cohomology classes "ω", precise geometric meanings of which have not been clarified yet.

In this paper we will give some simple observations and examples on the Milnor class of local complete intersections with arbitrary singularities, which will provide a hint for a general formula which one has been looking for. In §2 and §3, we will be concerned with a simple formula on the Milnor class of a finite Cartesian product of local complete intersections with arbitrary singularities. By this formula, it turns out that in order to describe the Milnor class of a local complete intersection with arbitrary singularities

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it is in general not sufficient to consider just the Chern classes of the vector bundle associated to the normal sheaf, but it requires to also consider the Chern classes of certain subbundles of the total bundle. This observation gives an explanation for why some extra cohomology class "ω" should be involved in the Brasselet-Lehmann-Seade-Suwa's formula [4, Theorem 7.13 and Corollary 7.18]. In the last section, we will give another kind of product formula for the Milnor class of singular hypersurfaces, which should be called "a Thom-Sebastiani type formula". To obtain this, we will use some results on the local geometry of function-germs, the proofs of which will be given in Appendix.

1. Milnor classes

Definition (1.1). ([3, 4, 23, 31]) Let M be an $(n+k)$-dimensional compact complex analytic manifold, and let E be a rank k holomorphic vector bundle over M. Let s be a regular holomorphic section of E, and set $X := s^{-1}(0)$, which is an n-dimensional local complete intersection. Then the following class is called *the Milnor class of Z*:

$$\mathcal{M}(X) := (-1)^n \left(C^{vir}(X) - C_*(X)\right),$$

where $C^{vir}(X)$ is the virtual homology class of X defined by

$$C^{vir}(X) := c(TM|_X - E|_X) \cap [X] = \frac{c(TM|_X)}{c(E|_X)} \cap [X],$$

and $C_*(X)$ is the Chern-Schwart-MacPherson class of X.

In particular, when X is nonsingular, it holds that $C^{vir}(X) = c(TX) \cap [X] = C_*(X)$, thus $\mathcal{M}(X) = 0$.

Definition (1.2). ([20], [22]) Let the situation be as in Definition (1.1). Then the *Parusiński's generalized Milnor number* $\mu(X)$ of X is defined by:

$$\mu(X) := (-1)^n \left(\chi(M|E) - \chi(X)\right),$$

where $\chi(X)$ is the topological Euler-Poincaré characteristic of X and $\chi(M|E)$ is defined by

$$\chi(M|E) := \int_M c(E)^{-1} c_{\mathrm{rank}E}(E) c(TM) \cap [M].$$

Here $\int_M$ means taking the sum of the 0-th part of the homology class, in other words, taking the image of the homology class by the homomorphism $H_*(M;\mathbb{Z}) \to H_*(\text{one point};\mathbb{Z}) = \mathbb{Z}$ induced by a constant map.

The Milnor class $\mathcal{M}(X)$ can be viewd certainly as a class version of the Parusiński's generalized Milnor number $\mu(X)$. In fact, it is clear that

$$c(E)^{-1} c_{\mathrm{rank}E}(E) c(TM) \cap [M] = c(E)^{-1} c(TM) \cap i_*[X] = i_* C^{vir}(X)$$

and $\int_M i_* C^{vir}(X) = \int_X C^{vir}(X)$, where $i : X \to M$ is the inclusion, and hence we have

$$\mu(X) = (-1)^n \left(\int_X C^{vir}(X) - \int_X C_*(X)\right) = \int_X \mathcal{M}(X).$$

Now let us consider the case when X is a singular hypersurface, that is, X is given by a holomorphic section s of a holomorphic line bundle L over an $(n+1)$ dimensional manifold M. Then the Milnor class $\mathcal{M}(X)$ can be expressed by using a Whitney startification of X and certain Chern-Schwarz-MacPherson classes, that is a recent result due to Parusiński and Pragacz [23]. Consider the function $\chi : X \to \mathbb{Z}$ defined by, for $x \in X$, $\chi(x) := \chi(F_x)$ the topological Euler-Poincaré characteristic of the Milnor fiber F_x at x. Then, since the number $(-1)^n(\chi(F_x) - 1)$ is nothing but the Milnor number of X at x when x is an isolated singularity, the function $\mu := (-1)^n(\chi - 1\!\!1_X)$ shall be provisionally called the “Milnor number function” (or the function of “vanishing Euler charcteristics”). It turns out that for any Whitney stratification of X the above function μ is constant along each strata [8, 21], therefore the function μ is a constructible function ; thus it shall be provisionally called the *Milnor constructible function*, abusing words. Let us denote the value of the Milnor constructible function μ on the stratum S by μ_S.

Theorem (1.3). *([23, Theorem 0.2]) Let the situation be as above. Then we have*

$$\mathcal{M}(X) = c(L_{|X})^{-1} \cap \sum_{S \in \mathcal{X}} \alpha(S)(i_{\overline{S},X})_* C_*(\overline{S}),$$

where $i_{\overline{S},X} : \overline{S} \to X$ *is the inclusion and* $\alpha(S) = \mu_S - \sum_{S' \neq S, \overline{S}' \supset S} \alpha(S')$.

The above theorem simply says that:

$$\mathcal{M}(X) = c(L_{|X})^{-1} \cap C_*(\mu).$$

In fact, the way Parusiński and Pragacz proved the theorem is to compare the three characteristic cycles (i.e., the Lagrangian cycles) associated to the three distinguished constructible functions: the characteristic function $1\!\!1_X$, the Euler constructible funcion χ and the Milnor constructible function μ. They use the work of Briançon-Maisonobe-Merle [8] , Lê-Mebkhout [17] and Sabbah [24]. For more details see [23].

2. Product formulas

M. Kwieciński [12] (cf. [14]) has proved that the cross product formula holds for the Chern-Schwartz-MacPherson class, using the resolution of singularities and by induction of dimension, i.e.,

Theorem (2.1). *([12]) Let* α *and* β *be constructible functions on* X *and* Y*, respectively. The exterior product* $\alpha \otimes \beta \in \mathcal{F}(X \times Y)$ *is defined to be* $(\alpha \otimes \beta)(x, y) := \alpha(x)\beta(y)$*. Then we have*

$$C_*(\alpha \otimes \beta) = C_*(\alpha) \times C_*(\beta) \in H_*(X \times Y),$$

where $\times$ *is the homology cross product. In particular, we have the product formula for the Chern-Schwartz-MacPherson class and the product formula for the Chern-Mather class:*

$$C_*(X \times Y) = C_*(X) \times C_*(Y), \tag{2.1.1}$$

$$C^M(X \times Y) = C^M(X) \times C^M(Y). \tag{2.1.2}$$

(2.1.1) and (2.1.2) follow from the fact that $1_X \otimes 1_Y = 1_{X\times Y}$ and $Eu_X \otimes Eu_Y = Eu_{X\times Y}$ [17], respectively.

Let $i = 1, 2$. Let M_i be an $(n_i + k_i)$-dimensional compact complex analytic manifold, and let E_i be a rank k_i holomorphic vector bundle over M_i. Let s_i be a regular holomorphic section of E_i, and set $X_i := s_i^{-1}(0)$, which is an n_i-dimension local complete intersection. Let $p_i : M_1 \times M_2 \to M_i$. Then we have the holomorphic (exterior product) section $s_1 \oplus s_2 : M_1 \times M_2 \to p_1^*E_1 \oplus p_2^*E_2$, which is defined by $(s_1 \oplus s_2)(x, y) = (s_1(x), s_2(y))$. Then $X_1 \times X_2 = (s_1 \oplus s_2)^{-1}(0)$. Let $e_i : X_i \to M_i$ be the inclusion $(i = 1, 2)$. It is then easy to see the following.

Proposition (2.2).

$$C^{vir}(X_1 \times X_2) = C^{vir}(X_1) \times C^{vir}(X_2).$$

In particular,

$$\chi(M_1 \times M_2 | p_1^*E_1 \oplus p_2^*E_2) = \chi(M_1|E_1)\chi(M_2|E_2).$$

It hence follows that

Theorem (2.3). *Let $\{(M_i, E_i, s_i, X_i)\}_{1\le i\le r} (r \ge 2)$ be a finite system of compact complex analytic manifolds M_i of dimension $n_i + k_i$, holomorphic vector bundles E_i of rank k_i over M_i, regular holomorphic sections $s_i : M_i \to E_i$ and the n_i-dimensional local complete intersections X_i which are the zeros of the holomorphic section s_i. Then we have*

$$\mathcal{M}(X_1 \times \cdots \times X_r) = \sum_{\substack{P_i = C_* \text{ or } \mathcal{M} \\ (P_1,\cdots,P_r) \ne (C_*,\cdots,C_*)}} (-1)^{n_1\varepsilon_1 + \cdots + n_r\varepsilon_r} P_1(X_1) \times \cdots \times P_r(X),$$

where

$$\varepsilon_i = \begin{cases} 1, & \text{if } P_i = C_* \\ 0, & \text{if } P_i = \mathcal{M}. \end{cases}$$

In particular, we have

$$\mu(X_1 \times \cdots \times X_r) = \sum_{\substack{p_i = \chi \text{ or } \mu \\ (p_1,\cdots,p_r) \ne (\chi,\cdots,\chi)}} (-1)^{n_1\varepsilon_1 + \cdots + n_r\varepsilon_r} p_1(X_1) \cdots p_r(X),$$

where

$$\varepsilon_i = \begin{cases} 1, & \text{if } p_i = \chi \\ 0, & \text{if } p_i = \mu. \end{cases}$$

In the above product formula the distinguished part $\mathcal{M}(X_1) \times \cdots \times \mathcal{M}(X_r)$ seems an interesting object. A study of this class in connection with the Thom-Sebastiani operation will be done in §§4 and 5.

3. Parusiński-Pragacz-type formulas

Instead of bundles E_i, let us take line bundles L_i in the above situation. Let $\mathcal{S}_{X_i} = \{S_{i,j_i}\}$ be a Whitney stratification of X_i such that the smooth part X_i^o is the top dimensional stratum, and let $\alpha(S_{i,j_i})$ be the number defined in the formula of Parusiński-Pragacz in §2. Then we have

Corollary (3.1).

$$\mathcal{M}(X_1\times\cdots\times X_r) = \sum_{(S_{1,j_1},\cdots,S_{r,j_r})\neq(X_1^o,\cdots,X_r^o)} (-1)^{n_1\varepsilon_{1,j_1}+\cdots+n_r\varepsilon_{r,j_r}}\alpha(S_{1,j_1})^{1-\varepsilon_{1,j_1}}\cdots\alpha(S_{r,j_r})^{1-\varepsilon_{r,j_r}} \frac{c(L_1)^{\varepsilon_{1,j_1}}\cdots c(L_r)^{\varepsilon_{r,j_r}}}{c(L_1\oplus\cdots\oplus L_r)}\cap C_*(\overline{S_{1,j_1}}\times\cdots\times\overline{S_{r,j_r}}).$$

where

$$\varepsilon_{i,j_i}=\begin{cases}1, & S_{i,j_i}=X_i^o,\\ 0, & \dim S_{i,j_i}<n_i.\end{cases}$$

By this corollary we can see that in general the formula for the Milnor class via Chern cohomology classes of the bundle and the Chern-Schwartz-MacPherson classes of the closures of Whitney strata, which we are looking for, is not of the following form:

$$\mathcal{M}(X)=Q(c(E))\cap\sum_{S\in\mathcal{X}}\beta(S)C_*(\overline{S}),$$

where $Q(c(E))$ is a polynomial of the Chern cohomology classes $c_1(E),\cdots,c_{\operatorname{rank}E}(E)$ of the vector bundle E and $\beta(S)$ is some kind of number attached to each stratum like the number $\alpha(S)$ appearing in Theorem (1.3). It seems reasonable to speculate that

$$\mathcal{M}(X)=\sum Q_S(\widetilde{c(E)})\cap C_*(\overline{S}),$$

where, if E has the decomposition or splitting $E=E_1\oplus\cdots\oplus E_k$, for each stratum S $Q_S(\widetilde{c(E)})$ would be a polynomial of the Chern classes $c_1(E_j),\cdots,c_{\operatorname{rank}E_j}(E_j)$ $(1\le j\le k)$. Here we put the extra notation $\widetilde{\ }$ to emphasize that $\widetilde{c(E)}$ indicates the Chern cohomology classes of the factor vector bundles E_j, not just the total bundle E. It should be noted that in general the cohomology class part $Q_S(\widetilde{c(E)})$ cannot be expressed as a polynomial in the Chern classes of the bundle E, as the following examples show.

Example (3.2). For $i=1,2$ let M_i be a compact complex analytic manifold of dimension n_i+1 with $n_1\ge 2$, L_i be a holomorphic line bundle over M_i, s_i a holomorphic section of L_i and X_i the zero of the holomorphic section s_i. And assume that the singular locus Γ of X_1 is a smooth curve so that $\{X_1-\Gamma,\Gamma\}$ is a Whitney stratification and that X_2 is smooth. (Note (as pointed out by P. Aluffi) that in general even if the reduced scheme of the singular locus is smooth, $\{X_1-\Gamma,\Gamma\}$ is not necessarily a Whitney stratification.) Then by Theorem (2.3) we have

$$\mathcal{M}(X_1\times X_2)=(-1)^{n_2}\mathcal{M}(X_1)\times C_*(X_2).$$

Hence, by by Parusiński-Pragacz's theorem [23, Theorem 0.2] or Brasselet-Lehmann-Seade-Suwa's theorem [4, Corollary 7.18] we have

$$\begin{aligned}\mathcal{M}(X_1\times X_2)&=(-1)^{n_2}\left(\frac{\mu_\Gamma}{c(L_1)}\cap C_*(\Gamma)\right)\times C_*(X_2),\\ &=(-1)^{n_2}\frac{\mu_\Gamma c(L_2)}{c(L_1\oplus L_2)}\cap C_*(\Gamma\times X_2).\end{aligned}$$

For this example our claim is that the cohomology class part $\frac{c(L_2)}{c(L_1 \oplus L_2)}$ cannot be replaced by any polynomial of the Chern cohomolgy classes $c_1(L_1 \oplus L_2), c_2(L_1 \oplus L_2)$ of the total vector bundle $L_1 \oplus L_2$. Let $c(L_1) = 1 + a$ and $c(L_2) = 1 + b$. Then, since $\frac{1}{c(L_1)} = \frac{1}{1+a} = 1 - a + a^2 + \cdots$ and Γ is a curve, by the dimension reason we have

$$\left(\frac{1}{c(L_1)} \cap C_*(\Gamma)\right) \times C_*(X_2) = (C_*(\Gamma) - a \cap C_*(\Gamma)) \times C_*(X_2).$$

On the other hand if we assume that $Q_\Gamma(c(L_1 \oplus L_2))$ is a polynomial of the Chern cohomolgy classes $c_1(L_1 \oplus L_2), c_2(L_1 \oplus L_2)$, then we have

$$\begin{aligned} Q_\Gamma(c(L_1 \oplus L_2)) &= p_0 + p_1 c_1(L_1 \oplus L_2) + p_{11} c_1(L_1 \oplus L_2)^2 + p_2 c_2(L_1 \oplus L_2) + \cdots \\ &= p_0 + p_1(a+b) + p_{11}(a+b)^2 + p_2 ab + \cdots \\ &= p_0 + p_1(a+b) + p_{11}(a^2+b^2) + (2p_{11} + p_2)ab + \cdots \end{aligned}$$

If the following equality holds

$$\left(\frac{1}{c(L_1)} \cap C_*(\Gamma)\right) \times C_*(X_2) = Q_\Gamma\left(c(L_1 \oplus L_2)\right) \cap \left(C_*(\Gamma) \times C_*(X_2)\right), \tag{3.2.1}$$

i.e.,

$$\begin{aligned} &((1-a) \cap C_*(\Gamma)) \times C_*(X_2) \\ &= C_*(\Gamma) \times C_*(X_2) - a \cap C_*(\Gamma) \times C_*(X_2) \\ &= \big(p_0 + p_1(a+b) + p_{11}(a^2+b^2) + (2p_{11}+p_2)ab + \cdots\big) \cap \big(C_*(\Gamma) \times C_*(X_2)\big) \\ &= p_0 C_*(\Gamma) \times C_*(X_2) + (p_1 a \cap C_*(\Gamma)) \times C_*(X_2) + p_1 C_*(\Gamma) \times (b \cap C_*(X_2)) + \cdots, \end{aligned}$$

then we have to have that $p_0 = 1$ and $p_1 = -1$. However, the extra term $-C_*(\Gamma) \times (b \cap C_*(X_2))$ is not in the Milnor class. Therefore (3.2.1) does not hold. Thus we can conclude the above claim.

Remark (3.3). The above simple example gives an explanation about the "ω" appearing in the Brasselet-Lehmann-Seade-Suwa's formula [4, Theorem 7.13 and Corollary 7.18]. Indeed, in the above example the Milnor class $\mathcal{M}(X_1 \times X_2)$ is expressed as follows:

$$\begin{aligned} &\mathcal{M}(X_1 \times X_2) = (-1)^{n_2} \frac{\mu_\Gamma}{c(L_1 \oplus L_2)} \cap C_*(\Gamma \times X_2) + \\ &(-1)^{n_2} \frac{\mu_\Gamma c_1(L_1 \oplus L_2)}{c(L_1 \oplus L_2)} \cap C_*(\Gamma \times X_2) - (-1)^{n_2} \frac{\mu_\Gamma c_1(L_1)}{c(L_1 \oplus L_2)} \cap C_*(\Gamma \times X_2). \end{aligned}$$

Then $c_1(L_1)$ appearing in the third term of the right-hande side of the above equality is nothing but the "ω^1" in the sense of [4]. Thus we can see that the "ω" appearing in [4, Theorem 7.13] is some necessary and inevitable ingredient to get a general formula for the Milnor class in the case of local complete intersections with higher dimensional singular locus when one tries to express the Milnor class in terms of the Chern classes of the total vector bundle E. It seems that "ω" of [4] is some kind of geometric invariant related to the splitting or decomposition of the total vector bundle E, which remains to be seen.

In the above example we assume that X_2 is non-singular. In the next example we assume that X_2 is also singular.

Example (3.4). For $i = 1,2$ let M_i be a compact complex analytic manifold of dimension $n_i + 1$ with $n_2 > n_1 \geq 2$, L_i be a holomorphic line bundle over M_i, s_i a holomorphic section of L_i and X_i the zero of the holomorphic section s_i. And assume that the singular locus Γ of X_1 is a smooth curve so that $\{X_1 - \Gamma, \Gamma\}$ is a Whitney stratification and that X_2 has only one isolated singularity x_2. Then by Theorem (2.3) we have

$$\begin{aligned}
&\mathcal{M}(X_1 \times X_2) \\
&= \mathcal{M}(X_1) \times \mathcal{M}(X_2) + (-1)^{n_1} C_*(X_1) \times \mathcal{M}(X_2) + (-1)^{n_2} \mathcal{M}(X_1) \times C_*(X_2) \\
&= \left(\frac{\mu_\Gamma}{c(L_1)} \cap C_*(\Gamma) \right) \times \left(\frac{\mu_{x_2}}{c(L_2)} \cap [x_2] \right) + (-1)^{n_1} C_*(X_1) \times \left(\frac{\mu_{x_2}}{c(L_2)} \cap [x_2] \right) \\
&\qquad\qquad + (-1)^{n_2} \left(\frac{\mu_\Gamma}{c(L_1)} \cap C_*(\Gamma) \right) \times C_*(X_2) \\
&= \frac{\mu_\Gamma \mu_{x_2}}{c(L_1 \oplus L_2)} \cap C_*(\Gamma \times x_2) + (-1)^{n_1} C_*(X_1 \times x_2) + (-1)^{n_2} \frac{\mu_\Gamma c(L_2)}{c(L_1 \oplus L_2)} \cap C_*(\Gamma \times X_2)
\end{aligned}$$

Then by the dimension reason as in Example (3.2) we can see that the cohomology class part $\frac{c(L_2)}{c(L_1 \oplus L_2)}$ in the third term of the last line cannot be replaced by any polynomial of the Chern cohomolgy classes $c_1(L_1 \oplus L_2), c_2(L_1 \oplus L_2)$ of the total vector bundle $L_1 \oplus L_2$.

§4 A Thom-Sebastiani type formula

In the previous sections we studied the Milnor class $\mathcal{M}(X_1 \times \cdots \times X_r)$ of a finite Cartesian product of local complet intersections $X_1, \cdots, X_r$. One of our formulas says that $\mathcal{M}(X_1 \times \cdots \times X_r) = \mathcal{M}(X_1) \times \cdots \times \mathcal{M}(X_r) + \text{other classes}$. In this section we focus our attention on the distinguished part $\mathcal{M}(X_1) \times \cdots \times \mathcal{M}(X_r)$ for hypersurfaces X_i's, especially in the case of $r = 2$. We will see that this term can be expressed by another kind of Milnor class (via a deformation) of a certain variety, denoted by $X_1 \perp \cdots \perp X_r$, which is related to the "Thom-Sebastiani construction" [19, 25, 27, 29].

First of all, we recall that the Thom-Sebastiani construction from given two function singularities is the sum of these functions within different valuables, and also that in the case of isolated hypersurface singularities, the Milnor number behaves well under this operation, that is, the Milnor number of $f + g$ is equal to the product of the Milnor number of f and the one of g (cf. [27]). Now we will describe such a formula in a bit more general setup, which will be needed later.

Let $(\mathcal{X}_1, x_0) \subset (\mathbb{C}^{m_1}, x_0)$ and $(\mathcal{X}_2, x_0) \subset (\mathbb{C}^{m_2}, y_0)$ be germs of analytic spaces. Let $f : (\mathcal{X}_1, x_0) \to (\mathbb{C}, 0)$ and $g : (\mathcal{X}_2, y_0) \to (\mathbb{C}, 0)$ be germs of holomorphic functions, and then we define

$$f + g : (\mathcal{X}_1 \times \mathcal{X}_2, (x_0, y_0)) \to (\mathbb{C}, 0)$$

by $(f + g)(x, y) := f(x) + g(y)$. Applying Milnor-Lê's fibration theorem [15] (see Appendix) to representatives of f, g and $f + g$, we get Milnor fibrations , so we let F_f, F_g and F_{f+g} denote the Milnor fibers corresponding to f, g and $f + g$, respectively. We can show the following proposition:

Proposition (4.1). *(Join Theorem) Let* $f : (\mathcal{X}_1, x_0) \to (\mathbb{C}, 0)$ *and* $g : (\mathcal{X}_2, y_0) \to (\mathbb{C}, 0)$ *be germs of non-constant holomorphic functions. Then the join* $F_f * F_g$ *and* F_{f+g} *have the same homotopy type.*

This theorem was firstly proved by Thom-Sebastiani [27] in the case of isolated singularities f and g defined over non-singular $\mathcal{X}_i (= \mathbb{C}^{m_i})$, and proved by Sakamoto [25] in the case of possibly non-isolated singularities of functions over $\mathbb{C}^{m_i}$ (also see Oka [19] for singularities of weighted homogeneous polynomial). The proof in [25] basically works also in our general case, i.e., the case of arbitrary singularities of fuctions over arbitrary varieties. The proof will be sketched in Appendix. Another generalized Thom-Sebastiani formula via derived category has been recently obtained by D. Massey [18].

For germs $f : (\mathcal{X}, x_0) \to (\mathbb{C}, 0)$, we set

$$\mu(f, x_0) := (-1)^{\dim F_f}(\chi(F_f) - 1).$$

By a standard argument (see also Appendix), it turns out that the number $\mu(f, x_0)$ enjoys the following good property under the Thom-Sebastiani construction.

Corollary (4.2). *Let* $f : (\mathcal{X}_1, x_0) \to (\mathbb{C}, 0)$ *and* $g : (\mathcal{X}_2, y_0) \to (\mathbb{C}, 0)$ *be non-constant holomorphic germs. Then it holds that*

$$\mu\left(f + g, (x_0, y_0)\right) = \mu(f, x_0)\mu(g, y_0).$$

The next step is concering the way to produce a global invariant from the local invariant – the vanishing Euler characteristics $\mu(f, x_0)$. In the workshop "Classes de Milnor", Marseille, February 1999, Jean-Paul Brasselet and Jose Seade [5] introduced the notion of *the Milnor class* $\mathcal{M}(f, X)$ *of* X *associated to a deformation* $f : \mathcal{X} \to \operatorname{Int} D_\eta \subset \mathbb{C}$ of X $(= f^{-1}(0))$ (Here a deformation is simply meant to be a proper analytic function over an analytic varieity $\mathcal{X}$). Let $\mu_f : X \to \mathbb{Z}$ be the constructible function over the special fiber X defined by

$$\mu_f(x) := (-1)^{\dim \mathcal{X} - 1}(\chi(F_{f,x}) - 1)$$

where $F_{f,x}$ is the Milnor fiber of f at $x \in X$. Then

Definition (4.3). The following class is called *the Milnor class of* X *associated to the deformation* f :

$$\mathcal{M}(f, X) := C_*(\mu_f) \in H_*(X; \mathbb{Z}).$$

Remark (4.4). (1) We can say that the class $\mathcal{M}(f, X)$ is defined rather for the germ of f over the set-germ $(\mathcal{X}, X)$. (2) By using Verider's specialization of constructible functions $\sigma_{\mathcal{F}} : \mathcal{F}(\mathcal{X}) \to \mathcal{F}(X)$ (for the definition, see [23, 29]), the function μ_f is rewritten in the following form :

$$\mu_f = (-1)^{\dim \mathcal{X} - 1}(\sigma_{\mathcal{F}} 1\!\!1_{X_t} - 1\!\!1_X),$$

where $X_t = \mathcal{X} \cap f^{-1}(t)$ and $t \in \operatorname{Int} D_\eta$ is sufficiently near the origin. Furthermore, we should note that originally Brasselet and Seade ([5]) introduced *the Milnor class* $\mathcal{M}(\alpha; f, X)$ *associated to a deformation* f *of* X *and a constructible function* $\alpha \in \mathcal{F}(\mathcal{X})$, which is defined by replacing $\sigma_{\mathcal{F}} 1\!\!1_{X_t} - 1\!\!1_X$ by $\sigma_{\mathcal{F}}(\alpha|_{X_t}) - \alpha|_X$ in the above definition of μ_f. Hence $\mathcal{M}(f, X) = \mathcal{M}(1\!\!1_{\mathcal{X}}; f, X)$.

Remark (4.5). In Parusiński-Pragacz's paper [23], a prototype of the Milnor class of a deformation as just described is used in an alternative proof (by specialization argument) of their main theorem which we quoted as Theorem (1.3). Let X be a singular hypersurface given by a section s of a holomorphic line bundle L over a smooth manifold M. And assume that we are given a generic section s' of L so that s' is transverse to the zero section of L. Take a small disk $D \subset \mathbb{C}$ centered at the origin such that for each $t \in D - \{0\}$, the section $(1-t)s + ts'$ is generic, i.e., transverse to the zero section. Set $\mathcal{X} := \{ (x,t) \in M \times D \,|\, (1-t)s(x) + ts'(x) = 0 \}$ and $p : \mathcal{X} \to D$ to be the canonical projection to the second factor. In this case, it is clear that $\mathcal{M}(p, X) = \mathcal{M}(X)$ since $C_*(\sigma_{\mathcal{F}} 1\!\!1_{X_t})$ coincides with the virtual class of the hypersurface X.

Given two deformations $f : \mathcal{X}_1 \to \operatorname{Int} D_\eta$ and $g : \mathcal{X}_2 \to \operatorname{Int} D_{\eta'}$ of analytic varieties X_1 and X_2, respectively, we define

$$f + g : \mathcal{X}_1 \times \mathcal{X}_2 \to \operatorname{Int} D_{\eta''}, \text{ by } (f+g)(x,y) = f(x) + g(y).$$

Let $X_1 \perp X_2$ denote the special fiber $(f+g)^{-1}(0)$, following Teissier [29], and $i : X_1 \times X_2 \to X_1 \perp X_2$ be the inclusion. Then we obtain the following *Thom-Sebastiani type formula* :

Theorem (4.6). *It holds that*

$$\mathcal{M}(f+g, X_1 \perp X_2) = i_* \left(\mathcal{M}(f, X_1) \times \mathcal{M}(g, X_2) \right).$$

Note that $X_1 \perp X_2$ is a non-compact variety, but the LHS of the above fomula makes sense by using Borel-Moore homology, i.e., homology with closed supports (e.g., see [10] and also [17]).

Proof of Theorem (4.6). It follows from Corollary (4.2) that $\mu_{f+g} = i_*(\mu_f \otimes \mu_g)$. Note that since i is proper, we can use $C_* i_* = i_* C_*$. Then by using the cross product formula due to M. Kwieciński (Theorem (2.1)), we can see that

$$\begin{aligned}
\mathcal{M}(f+g, X_1 \perp X_2) &= C_*(\mu_{f_1+f_2}) \\
&= C_*(i_*(\mu_{f_1} \otimes \mu_{f_2})) \\
&= i_*(C_*(\mu_{f_1} \otimes \mu_{f_2})) \\
&= i_*(C_*(\mu_{f_1}) \times C_*(\mu_{f_2})) \\
&= i_*(\mathcal{M}(f_1, X_1) \times \mathcal{M}(f_2, X_2))
\end{aligned}$$

□

Let $i = 1, 2$ and let $p_i : \mathcal{X}_i \to D_i$ be the deformation of a singular hypersurface X_i in M_i as described in Remark (4.5). Then we can see that the cross product $\mathcal{M}(X_1) \times \mathcal{M}(X_2)$ of Milnor classes is related to the Thom-Sebastiani construction $X_1 \perp X_2$ as follows :

Corollary (4.7). *It holds that*

$$\mathcal{M}(p_1 + p_2, X_1 \perp X_2) = i_* \left(\mathcal{M}(X_1) \times \mathcal{M}(X_2) \right).$$

Remark (4.8). The difference between the LHS of the above formula and the Milnor class $\mathcal{M}(X_1 \perp X_2)$ is still unclear so far. In fact, the variety $X_1 \perp X_2$ lives in $M_1 \times M_2 \times D$ with codimension 2. Thus, this is related to the problem of finding general formulas for Milnor classes of local complete intersection varieties of codimention greater than one, which we mentioned in §3.

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Appendix

Here we give the proofs of Proposition (4.1) and Corollary (4.2). First of all, we recall Lê's Fibration Theorem.

Theorem. *(Lê [15, Theorem (1.1)]) Let $\mathcal{X}$ be an analytic space in a neighborhood U of $x_0 \in \mathbb{C}^m$, and Let $f : \mathcal{X} \to \mathbb{C}$ be a holomorphic function with $f(x_0) = 0$. Then there exist $\epsilon > 0$ and $\eta > 0$ such that the restriction map induced by f*

$$f^{-1}(Int\, D_\eta - \{0\}) \cap \mathcal{X} \cap B_\epsilon \to Int\, D_\eta - \{0\}$$

is a topological fibration, where B_ϵ is a closed ball cetered at x_0 with radius ϵ and $Int\, D_\eta$ is an open disc centered at the origin of $\mathbb{C}$ with radius η.

Note that as in [15, Remark (1.3)], the small ball B_ϵ in the above theorem can be replaced by a sufficiently small polydisc P_ϵ.

Proof of Proposition (4.1). As written in Lê's paper [15], by means of Hironaka's theorem [11], there are Whitney stratifications $\mathcal{S}_i$ of $\mathcal{X}_i$ $(i = 1, 2)$ so that they satisfy Thom condition (i.e., a_f-condition) respect to f and g, and also so that the special fiber is a union of strata (i.e., they are usually called *good stratifications*), by which we get the Milnor-Lê fibrations for f and g. Set

$$f \times g : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathrm{Int}\, D_{\eta_1} \times \mathrm{Int}\, D_{\eta_2}, \quad (f \times g)(x, y) := (f(x), g(y))$$

and take a stratification of $\mathrm{Int}\, D_{\eta_1} \times \mathrm{Int}\, D_{\eta_2}$ consisting of $\{(0, 0)\}$, $(\mathrm{Int}\, D_{\eta_1} - \{0\}) \times \{0\}$, $\{0\} \times (\mathrm{Int}\, D_{\eta_2} - \{0\})$ and thier complement. Then, the pair of $\mathcal{S}_1 \times \mathcal{S}_2$ and the stratification of $\mathrm{Int}\, D_{\eta_1} \times \mathrm{Int}\, D_{\eta_2}$ produces a Thom stratification of $f \times g$. By taking a suitable refiniment of $S_1 \times S_2$, we obtain a Whitney stratification $\mathcal{S}$ of $\mathcal{X}_1 \times \mathcal{X}_2$ such that S satisfies the Thom condition repsect to $f + g$ and the special fiber is a union of strata. This yields the Milnor-Lê fibration for $f + g$.

Let $\epsilon > 0$, $\eta > 0$ be sufficiently small numbers as appearing in the above fibration theorem for f, g and $f + g$. For small $t \in \mathbb{C}$, set $l_t = \{(z, w) \in \mathbb{C}^2, z + w = t, |z| < \eta_1, |w| < \eta_2\}$. Also set $P_\epsilon = B_\epsilon^{m_1} \times B_\epsilon^{m_2}$ and

$$F_f(t) = f^{-1}(t) \cap \mathcal{X}_1 \cap B_\epsilon^{m_1},$$
$$F_g(t) = g^{-1}(t) \cap \mathcal{X}_2 \cap B_\epsilon^{m_2},$$
$$F_{f+g}(t) = (f \times g)^{-1}(l_t) \cap (\mathcal{X}_1 \times \mathcal{X}_2) \cap P_\epsilon.$$

Fix a non-zero small complex number t so that $0 < |t| < \eta$ and take a closed real-line segment J in l_t connecting two points $(t, 0)$ and $(0, t)$. By standard argument, it can be shown that

(1) the map induced by $f \times g$

$$F_{f+g}(t) - (f \times g)^{-1}(\{(t,0),(0,t)\}) \to l_t - \{(t,0),(0,t)\}$$

is a topological fibration, the fiber of which is homeomorphic to $F_f(t) \times F_g(t)$;

(2) the inclusion map $(f \times g)^{-1}(J) \to F_{f+g}(t)$ is a homotopy equivalence ;

(3) $(f \times g)^{-1}(J)$ has the same homotopy type of the join $F_f(t) * F_g(t)$. This can be shown by (1) and the fact that $F_f(0)$ and $F_g(0)$ are contractible.

Thus, it follows that $F_{f+g}(t) \sim F_f(t) * F_g(t)$. $\square$

Proof of Corollary (4.2). In general there is a well-known formula, due to J. Milnor, concerning the reduced homology of the join $X * Y$ of topological spaces X and Y :

$$\tilde{H}_{r+1}(X * Y; \mathbb{Q}) \simeq \oplus_{i+j=r} \tilde{H}_i(X; \mathbb{Q}) \otimes \tilde{H}_j(Y; \mathbb{Q}).$$

Note that the Milnor fibers F_f, F_g and F_{f+g} are finite CW complexes of dimension n_1, n_2 and $n_1 + n_2 + 1$, respectively, and that F_{f+g} is always connected. In particular,

$$\mu(f, 0) = (-1)^{n_1}(\chi(F_f) - 1) = (-1)^{n_1} \sum_{i=0}^{n_1} (-1)^i \dim \tilde{H}_i(F_f; \mathbb{Q}).$$

It hence follows that

$$\begin{aligned}
\mu(f+g, 0) &= (-1)^{n_1+n_2+1} \sum_{k=1}^{n_1+n_2+1} (-1)^k \dim \tilde{H}_k(F_{f+g}; \mathbb{Q}) \\
&= (-1)^{n_1+n_2+1} \sum_{k=1}^{n_1+n_2+1} (-1)^k \dim \tilde{H}_k(F_f * F_g; \mathbb{Q}) \\
&= (-1)^{n_1+n_2+1} \sum_{k=1}^{n_1+n_2+1} (-1)^k \sum_{i+j=k-1} \dim \tilde{H}_i(F_f; \mathbb{Q}) \cdot \dim \tilde{H}_j(F_g; \mathbb{Q}) \\
&= (-1)^{n_1+n_2} \sum_{i+j=0}^{n_1+n_2} (-1)^i \dim \tilde{H}_i(F_f; \mathbb{Q}) \cdot (-1)^j \dim \tilde{H}_j(F_g; \mathbb{Q}) \\
&= \left((-1)^{n_1} \sum_{i=0}^{n_1} (-1)^i \dim \tilde{H}_i(F_f; \mathbb{Q}) \right) \cdot \left((-1)^{n_2} \sum_{j=0}^{n_2} (-1)^j \dim \tilde{H}_j(F_g; \mathbb{Q}) \right) \\
&= \mu(f, 0)\mu(g, 0)
\end{aligned}$$

This completes the proof. $\square$

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