Motivic Euler Product and Its Applications

Lin WENG

Kyushu University

Arithmetic and Algebraic Geometry
University of Tokyo January 30, 2014

Arithmetic Masses

Setting

- $X/\mathbb{F}_q$: irre. red. reg proj curve of genus g
- $\mathcal{M}_{X,r}^{\mathrm{tot}}(d)$: stack of rank r degree d bdls/X
- $\mathcal{M}_{X,r}^{\mathrm{ss}}(d)$: substack of rank r degree d semi-stable bdls/X
- Mass:

$$\beta_{X,r}^{\mathrm{tot}}(d) := \sum_{E\in\mathcal{M}_{X,r}^{\mathrm{tot}}(d)} \frac{1}{|\mathrm{Aut}(E)|}, \quad \beta_{X,r}^{\mathrm{ss}}(d) := \sum_{E\in\mathcal{M}_{X,r}^{\mathrm{ss}}(d)} \frac{1}{|\mathrm{Aut}(E)|}$$

Theorem (Harder-Narasimhan, Desale-Ramanan, Zagier)

$$\frac{\beta_{X,r}^{\mathrm{ss}}(0)}{q^{\frac{n(n-1)}{2}(g-1)}} = \sum_{k\geq 1}(-1)^{k-1} \sum_{\substack{n_1+\cdots+n_k=n \\ n_1,\ldots,n_k>0}} \frac{\prod_{j=1}^{k} \beta_{X,n_j}^{\mathrm{tot}}(0)}{\prod_{i=1}^{k-1}(q^{n_i+n_{i+1}}-1)}$$

Arithmetic versus Geometry

Arithmetic Story

- + Dwork & Grothendieck (Weil Conjecture)
- $\Longrightarrow$ Relations of Poincare Series for the related spaces

Geometric Story

- Atiyah-Bott (Morse Theory, Yang-Mills)
- $\Longrightarrow$ Relations Poincare Series for the related spaces/ Riemann surfaces

Geometric Tamagawa Measure?

Fundamental Question (Atiyah-Bott)

- Arithmetic Approach: based on Tamagawa number conjectures/reductive groups
- Atiyah-Bott: similar theory/Riemann surface EXISTS ???
  (History: [ADK] A. Asok, B. Doran, F. Kirwan:
  Yang-Mills theory and Tamagawa numbers)

Basic Difficulties

- over $\mathbb{F}_q$,
  - $\#\mathrm{Aut}(E) < \infty$
  - $\mathbb{F}_q(X)_P$ ($P \in X$ closed point): locally compact
  - $G(\mathbb{A})$: locally compact
- over general k, say $\mathbb{C}$,
  - $\#\mathrm{Aut}(E)$ does not makes sense
  - $\mathbb{F}_q(X)_P$, $G(\mathbb{A})$: lack nice topology

Our Theory

Our Solutions: different from [ADK], but all related to $\mathbb{A}^1$-homology

- go Motivic (existing) & use Ind-Pro Topology (existing)
- introduce Motivic Euler Product & define Adelic Motivic Measures (new)

What been Achived

- introduce Atiyah-Bott's conjectural Tamagawa Measures
- define Motivic Non-abelian Zetas and Group Zetas
- offer stable partition of $Z_{G^1(\mathbb{A})}G(F)\backslash G^1(\mathbb{A})/\mathbb{K}$
- expose relations on Special Uniformity of Zetas and "Parabolic Reduction, Stability & the Masses"

K-Ring for varieties $/k$

Definition

- k: field
- $K_0(\mathrm{var}/k)$: Motivic Grothendieck K-ring for varieties over k
- Generators: varieties X over $k \Rightarrow \mu(X) = [X] \in K_0(\mathrm{var}/k)$
- Relations: (i) $[X] = [X - Z] + [Z] \qquad \forall\, Z \hookrightarrow X$ closed
  (ii) $[X \times Y] = [X] \cdot [Y]$

Examples

- $\mathbb{L} := \mu(\mathbb{A}^1_k)$
- $\mu(\mathbb{P}^1_k) = \mathbb{L} + 1$
- $\mu(\mathbb{G}^1_{m,k}) = \mathbb{L} - 1$
- $\mu(\mathrm{GL}_n(k)) = (\mathbb{L}^n - 1)(\mathbb{L}^n - \mathbb{L}) \cdots (\mathbb{L}^n - \mathbb{L}^{n-1})$

Motivic class of Reductive Group

Reductive Group

- G/k: split reductive group,
- B, T: Borel and maximal torus
- $R_u(B)$: unipotent radical of B
- $X_*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$: space of characters on T
- W: Weyl group
- $\left(\mathrm{Sym}\, X_*(T)_{\mathbb{Q}}\right)^W =: \oplus_d V_d$: canonical motive of G
- $\mu(G) = \mathbb{L}^{\dim G} \cdot \prod_d (1 - \mathbb{L}^{-d})^{\dim V_d}$
- Reasons: (i) $\mu(G) = \mu(G/B) \cdot \mu(R_u(B)) \cdot \mu(T)$
  (ii) $\mu(R_u(B)) = \mathbb{L}^{\dim R_u(B)}$, $\mu(T) = (\mathbb{L} - 1)^r$
  (iii) $\mu(G/B) = \sum_{w \in W} \mathbb{L}^{l(w)}$ Burhat decomp

(see e.g., [Behrend-Dhillon])

K-Ring for nice stacks $/k$

Definition

- (nice) stacks over k: stacks with linear stabilizer $\overset{\text{Kresch}}{\Longrightarrow}$ stratified by global quotients
- $K_0(\mathrm{sta}/k)$: Motivic Grothendieck K-ring for nice k-stacks
- Generators: stacks X over $k \Rightarrow \mu(X) = [X] \in K_0(\mathrm{sta}/k)$
- Relations: (i) $[X] = [X - Z] + [Z] \qquad \forall\, Z \hookrightarrow X$ closed
  (ii) $[X \times Y] = [X] \cdot [Y]$
  (iii) $[X/G] = [X]/[G]$

Fact

$$K_0(\mathrm{sta}/k) = K_0(\widehat{\mathrm{var}/k)}[1/\mathbb{L}]$$

(see: any standard reference for motivic K)

Motivic Zeta

Definition (Kapranov)

- X/k: variety
- $\mathrm{Sym}^{(m)}X$: m-th symmetric product of X
- Motivic zeta for X

$$Z_X(u) := \sum_{m \geq 0} \mu(\mathrm{Sym}^{(m)}X) \cdot u^m.$$

Facts

- If $k = \mathbb{F}_q$, $u = q^{-s}$, $Z_X(u)$ coincides with Artin-Weil zeta
- $$Z_X(u) = Z_{X-Y}(u) \cdot Z_Y(u) \qquad \forall\, Y \hookrightarrow X \quad \text{closed}$$

Motivic Euler Product

Definition (Weng)

- X/k: variety, $x \in X$: closed point, $k(x)$: residue field
- $\deg(x) := [k(x) : k]$, $u_x := u^{\deg(x)}$
- Motivic Euler Product for X/k

$$\prod\nolimits_{x\in X}^{\omega} \frac{1}{1-u_x} := \sum_{m\geq 0} \mu(\mathrm{Sym}^{(m)} X) \cdot u^m$$

Basic properties

$$\prod\nolimits_{x\in X}^{\omega} \frac{1}{1-u_x} = \Big(\prod\nolimits_{x\in X-Y}^{\omega} \frac{1}{1-u_x}\Big) \cdot \Big(\prod\nolimits_{x\in Y}^{\omega} \frac{1}{1-u_x}\Big) \quad Y : \text{closed}$$

In particular, $\prod\nolimits_{x\in X}^{\omega} f(x) = \prod\nolimits_{x\in X} f(x) \qquad \forall' x,\ f(x) = 1$

Motivic Measures

Local Theory

- X/k: irre, red, reg proj curve of genus g, F: functional field
- $x \in X$: closed point, $(F_x, \mathcal{O}_x, \mathfrak{m}_x, \pi_x)$: its local field
- $|\omega_x|$: motivic measure on F_x s.t.

$$\mu\left(\mathcal{O}_x = \lim_{\leftarrow n} \mathcal{O}_x/\mathfrak{m}_x^n\right) = \int_{\mathcal{O}_x} |\omega_x| = 1$$

- Filtration with graded quotients $k(x)$:

$$\{0\} \subset \mathfrak{m}_x^{n-1}/\mathfrak{m}_x^n \subset \cdots \subset \mathfrak{m}_x^1/\mathfrak{m}_x^n \subset \mathcal{O}_x/\mathfrak{m}_x^n$$

$$\Rightarrow \qquad \mu(\mathcal{O}_x) = \mu(\mathfrak{m}_x^n) \cdot \mu(\mathcal{O}_x/\mathfrak{m}_x^n) = \mu(\mathfrak{m}_x^n) \cdot \mathbb{L}_x^n$$

$$\Rightarrow \qquad \mu(\mathfrak{m}_x^n) = \mathbb{L}_x^{-n} \qquad \text{where} \qquad \mathbb{L}_x = \mathbb{L}^{\deg(x)}$$

Global Theory: Topology

Adelic Space and its ind-pro topopology

- $\mathbb{A} := \mathbb{A}_F = \prod_{x\in X}^{\omega'}(F_x, \mathcal{O}_x)$: adelic ring of X
- $D = \sum_x n_x \cdot x$: divisor on X
- $\mathbb{A}(D) := \{(a_x) \in \mathbb{A} : \mathrm{ord}_x(a_x) + n_x \geq 0\ \forall x\}$

$$\mathbb{A} = \lim_{\to D_1} \lim_{\leftarrow D_2 : D_2 \leq D_1} \mathbb{A}(D_1)/\mathbb{A}(D_2)$$

Facts on locally linearly compact topology

- $\mathbb{A}(D)$ is open and linearly compact
- $\mathbb{A}(D_1)/\mathbb{A}(D_2)$: linearly compact and discrete
  $\Rightarrow$ finite dimensional k-vector space

Riemann-Roch Theorem

Riemann-Roch Theorem

- $\mathbb{A}(0) = \prod_{x \in X}^{\omega} \mathcal{O}_x$
- $\mathbb{A}(D)/\mathbb{A}(0) \simeq \prod_{x \in X}^{\omega} k(x)^{n_x} \simeq k^{\deg(D)}$ for $D \geq 0$
- $H^0(X, D) = \mathbb{A}(D) \cap F, \qquad H^1(X, D) = \mathbb{A}/(\mathbb{A}(D) + F)$
- (Riemann-Roch) $h^0(X, D) - h^1(X, D) = \deg(D) - (g - 1)$

Nine Diagram: exact columns and exact rows

$$\begin{array}{ccccc}
F \cap \mathbb{A}(D) & \hookrightarrow & \mathbb{A}(D) & \twoheadrightarrow & \mathbb{A}(D)/F \cap \mathbb{A}(D) \\
\downarrow & & \downarrow & & \downarrow \quad \text{all inj} \\
F & \hookrightarrow & \mathbb{A} & \twoheadrightarrow & \mathbb{A}/F \\
\downarrow & & \downarrow & & \downarrow \quad \text{all sur} \\
F/F \cap \mathbb{A}(D) & \hookrightarrow & \mathbb{A}/\mathbb{A}(D) & \twoheadrightarrow & \mathbb{A}/(\mathbb{A}(D) + F)
\end{array}$$

Global Theory: Motivic Measure

Definition (Weil, Weng)

- Motivic measure on $\mathbb{A}$: $|\omega_{\mathbb{A}}| := \prod_{x\in X}^{\omega}\omega_x$ s.t.

$$\mu\Big(\prod_{x\in X}^{\omega}\mathcal{O}_x\Big) = \prod_{x\in X}^{\omega}\mu(\mathcal{O}_x) = \prod_{x\in X}^{\omega}1 = 1$$

Theorem (Chevally, Tate, Weil, Weng)

$$\mu(\mathbb{A}/F) = \mathbb{L}^{g-1}$$

Proof.

- Nine diagram $\Rightarrow \mu(\mathbb{A}/F) = \mu(\mathbb{A}(D))\cdot\mathbb{L}^{-\chi(X,0)}$
- For $D\geq 0$, $\mu(\mathbb{A}(D)) = \mu(\mathbb{A}(D)/\mathbb{A}(0)) = \prod_{x\in X}^{\omega}\mu(\mathfrak{m}_x^{-n_x})$
$$= \prod_{x\in X}^{\omega}\mathbb{L}^{n_x} = \mathbb{L}^{\sum_x n_x \deg(x)} = \mathbb{L}^{\deg(D)}.$$
- Riemann-Roch

Motivic Measures: Reductive Groups

Local Definition

- $\mathcal{G}/X$: reductive group scheme
- η: generic point of X
- $G = \mathcal{G}_\eta/F$: split w/ B, T: split Borel and maximal torus $/F$
- ω: gauge form, non-zero element of $\Lambda^{\mathrm{top}}\,\mathrm{Lie}\,G$
  $\Rightarrow$ for $x \in X$,
  $$\omega_x : T_e(\mathcal{G}(\mathcal{O}_x)) \to F_x$$
  $\Rightarrow$ fractional ideal $\mathrm{Im}(\omega_x) =: <\pi_x^{\mathrm{ord}_x(\omega_x)}>$
- $$\int_{\mathcal{G}(\mathcal{O}_x)} |\omega_x| := \mathbb{L}_x^{-\mathrm{ord}_x(\omega_x)} \cdot \mathbb{L}_x^{-\dim G} \cdot \mu(G(k(x)))$$

Motivic Measures: Reductive Groups

Definition (Weil, Weng)

- Motivic adelic measure on $G(\mathbb{A})$:

$$|\omega_{\mathbb{A}}| := \mathbb{L}^{(1-g)\dim G} \cdot \prod_{x\in X}^{\omega} |\omega_x|$$

- First factor: make it comptible with Weil restriction $R_{K/F}$
- Similar as that for $\mathbb{A}$: $G = \mathbb{G}_a$, not reductive

Conjecture (Weil, Weng)

- For connected semi-simple G

$$\int_{G(F)\backslash G^1(\mathbb{A})} |\omega_{\mathbb{A}}| = 1.$$

- $k = \mathbb{F}_q$: Tamagawa number. See [Weil, Harder, Lurie]

Motivic Measure of $\mathbb{K}$

Theorem (Weng)

Let $\mathbb{K} := \prod_{x\in X}^{\omega} \mathcal{G}(\mathcal{O}_x)$,

$$\mu(\mathbb{K}) := \int_{\mathbb{K}} |\omega_{\mathbb{A}}| = \mathbb{L}^{(1-g)\dim G} \cdot \prod_{d\geq 1} Z_X(\mathbb{L}^{-d})^{-\dim V_d}.$$

Proof (Application of Motivic Euler Product)

- $$\mathbb{L}^{(g-1)\dim G} \cdot \int_{\mathbb{K}} |\omega_{\mathbb{A}}| = \prod_{x\in X}^{\omega} \int_{\mathcal{G}(\mathcal{O}_x)} |\omega_x|$$
$$= \prod_{x\in X}^{\omega} \Big(\mathbb{L}_x^{-\mathrm{ord}_x(\omega_x)} \cdot \mathbb{L}_x^{-\dim G} \cdot \mu(G(k(x))) \Big)$$
$$= \mathbb{L}^{-\deg(\omega)} \cdot \prod_{x\in X}^{\omega} \Big(\prod_d (1 - \mathbb{L}_x^{-d})^{\dim V_d} \Big)$$
- $\deg(\mathrm{Lie}(G)) = 0$ for reductive G

Non-Abelian Motivic Zeta Function

Definition (Weng)

- X/k: irre. red. reg. proj. curve of genus g
- $\mathcal{M}_{X,n}(d)$: moduli stack of s. stable bundles of rk n and deg d
- $d\mu$: motivic measure induced from the adelic one
- Non-Abelian Motivic Zeta Function of X/k:

$$\widehat{Z}_{X,n}(u) := \sum_{m\geq 0} \int_{V\in\mathcal{M}_{X,n}(mn)}^{\omega} \frac{\mu(H^0(X,V)-\{0\})}{\mu(\mathrm{Aut}(V))} d\mu \cdot u^{\chi(X,V)}$$

$\int^{\omega}$: motivic integration, well-defined
for filtered constant function over filtered stacks
i.e., constant over each open filtered strata.

Non-Abelian Motivic Zeta Function

Non-Abelian Motivic Invariants

- α-invariants: $\alpha_{X,n}(d) := \int^{\omega}_{V\in\mathcal{M}_{X,n}(d)} \frac{\mu(H^0(X,V)-\{0\})}{\mu(\mathrm{Aut}(V))}\, d\mu$
- β-invariants: $\beta_{X,n}(d) := \int^{\omega}_{V\in\mathcal{M}_{X,n}(d)} \frac{1}{\mu(\mathrm{Aut}(V))}\, d\mu$

Basic Properties

- $\beta_{X,n}(d) = \beta_{X,n}(0)$.

$$\widehat{Z}_{X,n}(u) = \sum_{m=0}^{(g-1)-1} \alpha_{X,n}(mn)\Big[\Big(\frac{1}{u^n}\Big)^{(g-1)-m} + (\mathbb{L}^n u^n)^{(g-1)-m}\Big]$$

$$+\alpha_{X,n}(n(g-1)) + \beta_{X,n}(0)\cdot\frac{\mathbb{L}^n\cdot(u^n)^g}{(1-u^n)(1-\mathbb{L}^n u^n)}.$$

Counting Miracle and the Riemann Hypothesis

Theorem ($g = 1$: Weng-Zagier; in general, Sugahara)

- Counting Miracle: $\alpha_{X,n}(0) = \mathbb{L}^{n(g-1)}\beta_{X,n}(0)$

see also [Mozgovoy-Reineke]

Theorem (Weng-Zagier)

When $X = E$ and elliptic curve $/\mathbb{F}_q$,

- Multiplicative structure of *beta*-invariants:

$$\sum_{n\geq 1}\beta_{E,n}(0)q^{-ns} = \prod_{k\geq 1}\zeta_E(s+k) \qquad \mathrm{Re}(s) > 0$$

- Riemann Hypothesis:

$$\zeta_{E,n}(s) = 0 \qquad \Rightarrow \qquad \mathrm{Re}(s) = \frac{1}{2}$$

Motivic Periods for SL_n

Definition (Weng)

- $(V, \Phi^+ = (\alpha_{ij} = e_i - e_j, i < j), \Delta = (\alpha_{i(i+1)}), W = S_n)$: root system for A_{n-1}, $\rho := \frac{1}{2}\sum_{\alpha>0}\alpha$
- $\varpi := (\varpi_i)$: fundamental weight so that $\langle \alpha_i, \varpi_j \rangle = \delta_{ij}$
- $\lambda := \sum_{j=1}^{n-1}(1 - s_j)\varpi_j = \rho - \sum_{j=1}^{n-1} s_j\varpi_j$: coordinate system
- $u_j := \mathbb{L}^{-s_j}$ and $u = \mathbb{L}^{-s}$,
- Period of SL_n:

$$\Omega_X^{SL_n}(\lambda) := \sum_{w\in W} \frac{1}{\prod_{\alpha\in\Delta}(1 - \mathbb{L}^{-\langle w\lambda-\rho,\alpha^\vee\rangle})} \cdot \prod_{\alpha>0, w\alpha<0} \frac{\widehat{Z}_X(\mathbb{L}^{-\langle\lambda,\alpha^\vee\rangle})}{\widehat{Z}_X(\mathbb{L}^{-\langle\lambda,\alpha^\vee\rangle-1})}$$

Motivic Periods for SL_n

Definition: Continued

- Period for $(SL_n, P_{n-1,1})$:

$$\Omega_X^{SL_n,P_{n-1,1}}(s) := \mathrm{Res}_{\substack{\langle \lambda-\rho,\alpha_j^\vee\rangle=0 \\ j=1,2,\dots,n-2}} \Omega_X^{SL_n}(\lambda)$$

- $m(i) := \#\{\alpha > 0 : \langle \rho, \alpha \rangle = i\}, \quad n(i) = m(i) - m(i-1)$
- Motivic SL_n-Zeta Function:

$$\widehat{Z}_X^{SL_n}(u) := \mathbb{L}^{\frac{n(n-1)}{2}(g-1)} \cdot \Big(\prod_{i\geq 2} \widehat{Z}_X(\mathbb{L}^{-i})^{-n(i)}\Big) \cdot \widehat{Z}_X(u^{-n}) \cdot \Omega_X^{SL_n,P_{n-1,1}}(u^{-n}).$$

Special Uniformity of Zetas

Conjecture (Weng)

- Special Uniformity of Zetas:

$$\widehat{Z}_{X,n}(u) = \widehat{Z}_X^{SL_n}(u)$$

- Set $\widehat{v}_{X,1} = \frac{\mu(\mathrm{Pic}^0(X))}{\mathbb{L}-1}$, $\widehat{v}_{X,n+1} = \widehat{Z}_X(\mathbb{L}^{-(n+1)}) \cdot \widehat{v}_{X,n} \quad \forall n \geq 1$

$$\frac{\beta_{X,n}(0)}{\mathbb{L}^{\frac{n(n-1)}{2}(g-1)}} = \sum_{k\geq 1}(-1)^{k-1} \sum_{\substack{n_1+\cdots+n_k=n \\ n_1,\ldots,n_k\geq 1}} \frac{\prod_{i=1}^{k} \widehat{v}_{X,n_i}}{\prod_{j=1}^{k-1}(\mathbb{L}^{n_j+n_{j+1}}-1)}$$

Theorem ($k = \mathbb{F}_q$)

- (Harder-Narasimhan, Zagier) Formula for $\beta_{X,n}(0)$ holds.
- (Mozgovoy-Reineke-Weng-Zagier) $\widehat{Z}_{X,n}(u) = \widehat{Z}_X^{SL_n}(u)$

Motivic Eisenstein Series

Current proof of Special Uniformity

- Mozgovoy-Reineke: wall-crossing for $\widehat{Z}_{X,n}(u)$
- Weng-Zagier: Lie theory and combinatorics for $\widehat{Z}_X^{SL_n}(u)$
- MR=WZ

Special Uniformity and Motivic Eisenstein Periods

- Number fields: proved using Eisenstein periods
- Over $\mathbb{F}_q$: OK if Mellin transform applies for functional field.
  this is supposed to be very easy
- To prove the conjecture:
  (i) study motivic Eisenstein series
  or (ii) Try MR's wall-crossing approach for any base field

Total & Stable Motivic Masses

Definition

- X/k: irre. red. reg. proj curve of genus g
- (G, B, T): conn, split red group over $F = k(X)$, $\lambda \in \mathfrak{a}$
- $\mathcal{M}_{X,G}(\lambda)$: moduli stack of G-bundles of slope λ on X
- Total Motivic Mass:

$$\beta_{X,G}^{\omega;\,\mathrm{tot}}(\lambda) := \int_{\mathcal{M}_{X,G}(\lambda)}^{\omega} \frac{1}{[\mathrm{Aut}(\mathcal{E})]}\, d\mu$$

- $\mathcal{M}_{X,G}^{\mathrm{ss}}(\lambda)$: moduli stack of s. stable G-bundles of slope λ
- Stable Motivic Mass:

$$\beta_{X,G}^{\omega;\,\mathrm{ss}}(\lambda) := \beta_{X,G}(\lambda) := \int_{\mathcal{M}_{X,G}^{\mathrm{ss}}(\lambda)}^{\omega} \frac{1}{[\mathrm{Aut}(\mathcal{E})]}\, d\mu$$

Harder-Narasimhan Filtration

Notations

- $\mathcal{P}$: collection of standard parabolic subgroups of G
- $P = M_P N_P,\ Q = M_Q N_Q \in \mathcal{P}, \quad P \subset Q$
- A'_P: maximal quotient split torus of M_P^{ab}, $\qquad A'_P \twoheadrightarrow A'_Q$
- $X_*(A'_P)$: 1 PS of A'_P; $\qquad X^*(A'_P)$: characters of A'_P
- $\mathfrak{a}_P := X_*(A_P)_{\mathbb{R}} = X_*(A'_P)_{\mathbb{R}}, \quad \mathfrak{a}_Q \hookrightarrow \mathfrak{a}_P \twoheadrightarrow \mathfrak{a}_Q$
- $(\mathfrak{a}_P = \mathfrak{a}_P^Q \oplus \mathfrak{a}_Q,\ \mathfrak{a}_P^*;\ \ \lambda = [\lambda]_P^Q + [\lambda]_Q;\ \ \Phi_P^+, \quad \Delta_P)$
- $\alpha^\vee$: coroot for α; $\quad \{\varpi_\alpha^G\}$: dual of $\{\alpha^\vee : \alpha \in \Delta_P\}$
- $\rho_P = \frac{1}{2} \sum_{\alpha \in \Phi_P^+} \alpha$
- $\Lambda_P^Q := X_*(A'_P) \Big/ \sum_{\alpha \in \Delta_P^Q} \mathbb{Z}\alpha^\vee, \ \ \Pi_P^Q := X_*(A'_P) \Big/ \sum_{\alpha \in \Delta_P^Q} \mathbb{Z}\varpi_Q^\vee$

Canonical Filtration, Total Mass

Theorem (HN, Behrend, Ramanathan)

$\forall \mathcal{E}$: G-bundle of slope λ

$\Rightarrow\ \exists! :\quad P \in \mathcal{P},\ \ \lambda \in \mathfrak{a}_P$, s.stable P-bundle $\mathcal{E}_P$ of slope λ_P satisfying

$$[\lambda_P]^G \in \mathfrak{a}_P^{G+} \qquad \text{and} \qquad \phi : \mathcal{E}_P \times^P G \simeq \mathcal{E}.$$

Conjecture: Total Mass (Behrend-Dhillon)

$$\beta_{X,G}^{\omega;\,\mathrm{tot}}(\lambda) = \prod_{d\geq 1} \widehat{Z}_X(\mathbb{L}^{-d})^{\dim V_d}$$

Partition of Moduli Stack

Notations

- $\mathcal{M}^{\mathrm{tot}}_{X,G,P}(\lambda_P)$: sub moduli stack of $\mathcal{M}^{\mathrm{tot}}_{X,G}(\lambda)$ induced by P-bundles of slope λ_P
- $\mathcal{M}^{\mathrm{ss}}_{X,G,P}(\lambda_P)$: sub stack of $\mathcal{M}^{\mathrm{ss}}_{X,G}(\lambda)$ w/ can. type (P, λ_P)

Theorem (Behrend, Weng)

- Parabolic Partition

$$\mathcal{M}^{\mathrm{tot}}_{X,G,Q}(\lambda_Q) = \bigcup_{P\in\mathcal{P},P\subset Q} \bigcup_{\substack{\lambda_P\in X_*(A'_P)\\ [\lambda_P]_Q=\lambda_Q,[\lambda_P]^Q\in\mathfrak{a}_P^{Q+}}} \mathcal{M}^{\mathrm{ss}}_{X,G,P}(\lambda_P)$$

- $\mu(\mathcal{M}^{\mathrm{ss}}_{X,G,P}(\lambda_P)) = \mathbb{L}^{2\langle\rho_P^G,\lambda_P\rangle+\dim N_P(g-1)} \cdot \mu(\mathcal{M}^{\mathrm{ss}}_{X,M_P}(\lambda_P))$

Conjecture

Conjecture: Parabolic Reduction, Stability & the Masses

$$\beta^{\omega;\,\mathrm{tot}}_{X,M_Q}(\lambda_Q) = \sum_{P\in\mathcal{P},\,P\subset Q}\ \sum_{\overline{\lambda_P}\in(\Lambda_P^Q)^\perp} \beta^{\omega;\,\mathrm{ss}}_{X,M_P}(\overline{\lambda_P})$$

$$\times \sum_{\pi\in\Pi_P^Q,\,[\pi]_Q=\lambda_Q}\ \prod_{\alpha\in\Delta_P^Q} \frac{\mathbb{L}^{2\cdot\langle\rho_P^Q,\varpi_\alpha^{\vee Q}\rangle\cdot\{\alpha^{\vee Q}(\pi)\}}}{\mathbb{L}^{2\cdot\langle\rho_P^Q,\varpi_\alpha^{\vee Q}\rangle}-1}$$

$$\frac{\beta^{\omega;\,\mathrm{ss}}_{X,M_Q}(\lambda_Q)}{\mathbb{L}^{(\dim N_0-\dim N_Q)\cdot(g-1)}} = \sum_{P\in\mathcal{P},\,P\subset Q} (-1)^{\dim \mathfrak{a}_P^Q}\,\beta^{\omega;\,\mathrm{tot}}_{X,M_P}(0)$$

$$\times \sum_{\lambda\in\Lambda_P^Q,\,[\lambda]_Q=\lambda_Q}\ \prod_{\alpha\in\Delta_P^Q} \frac{\mathbb{L}^{2\cdot\langle\rho_P^Q,\alpha^\vee\rangle\cdot\{\varpi_\alpha^Q(\lambda)\}}}{\mathbb{L}^{2\cdot\langle\rho_P^Q,\alpha^\vee\rangle}-1}$$

w/ $a\in\mathbb{R}/\mathbb{Z}$: $\quad 0<\langle a\rangle\le 1,\ 0\le\{a\}=1-\langle a\rangle<1$

Over $\mathbb{F}_q$

Theorem (SL_n: HN, DR, Zagier; General: Weng)

For $X/\mathbb{F}_q$, Conjectures hold!

Key Ingredients:

- Partition of Adelic Space
  - Arthur's Analytic truncation
  - Lafforgue's arithmetic truncation
  - combinatorial Langlands lemma (Laumon-Rapoport)
  - Parabolic reduction
- Tamagawa Number: Weil, Ono, Harder, Oesterle, Lurie ...

Thank You

Thank You

Tokyo, 30, 01, 2014