The moduli of abelian varieties and its compactification

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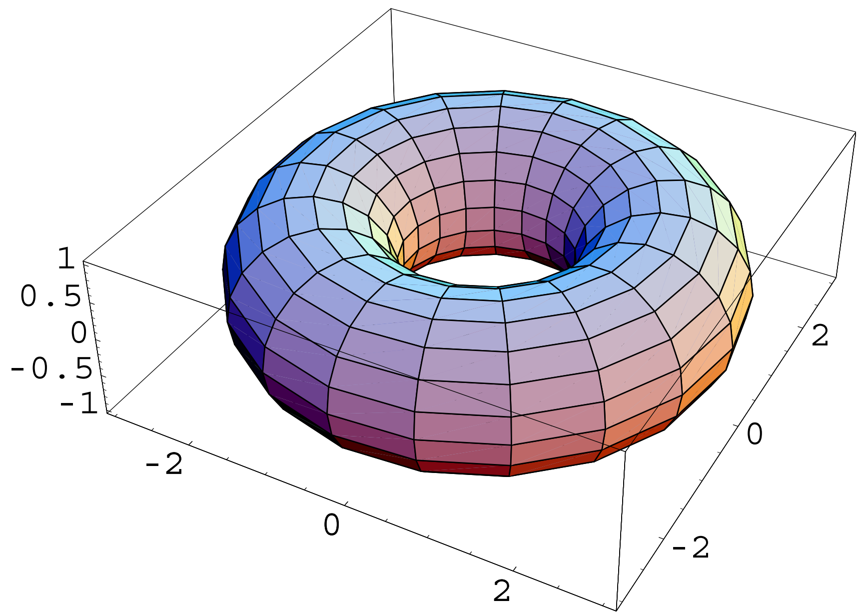
1 Hesse cubic curves

$$C(\mu) : x_0^3 + x_1^3 + x_2^3 - 3\mu\, x_0 x_1 x_2 = 0$$

$$(\mu \in \mathrm{P}^1_{\mathrm{C}})$$



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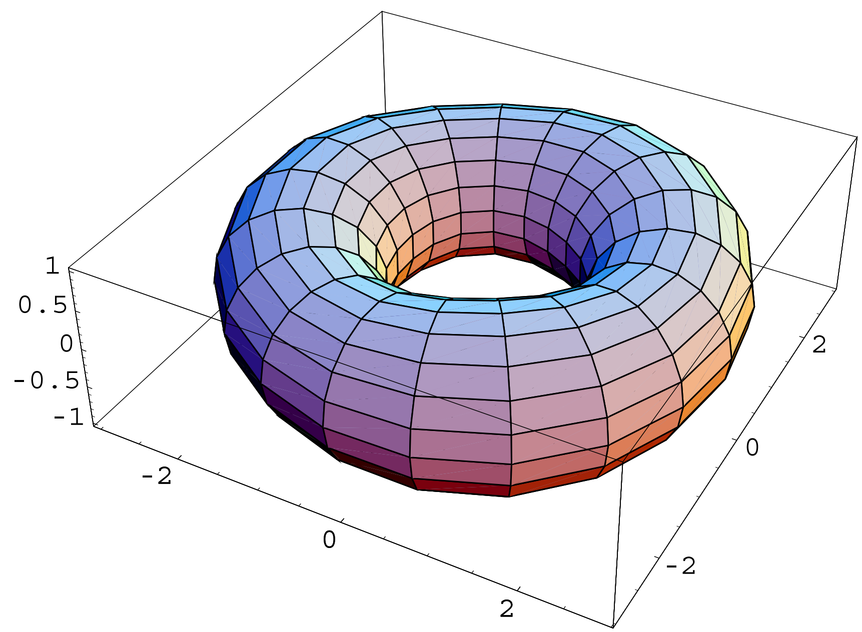
if μ gets close to ∞



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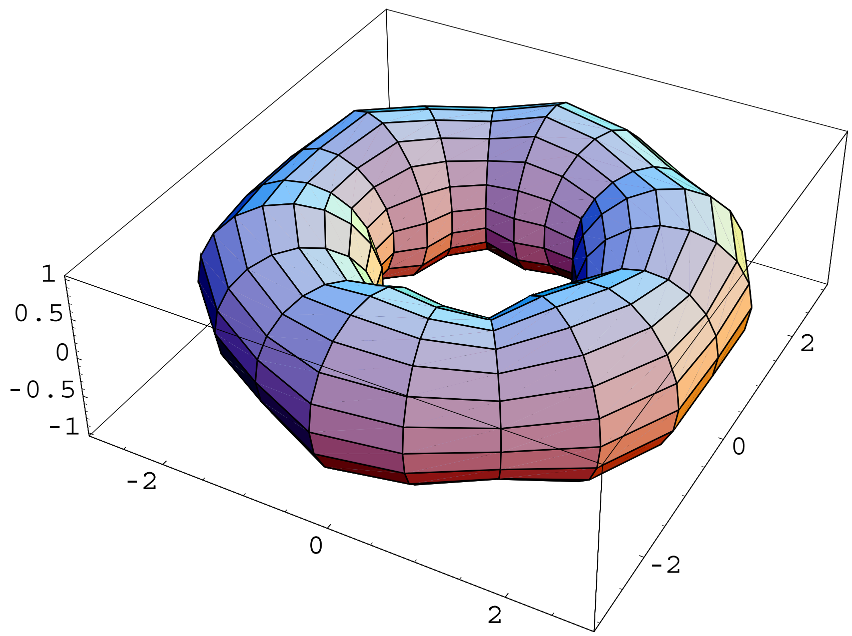
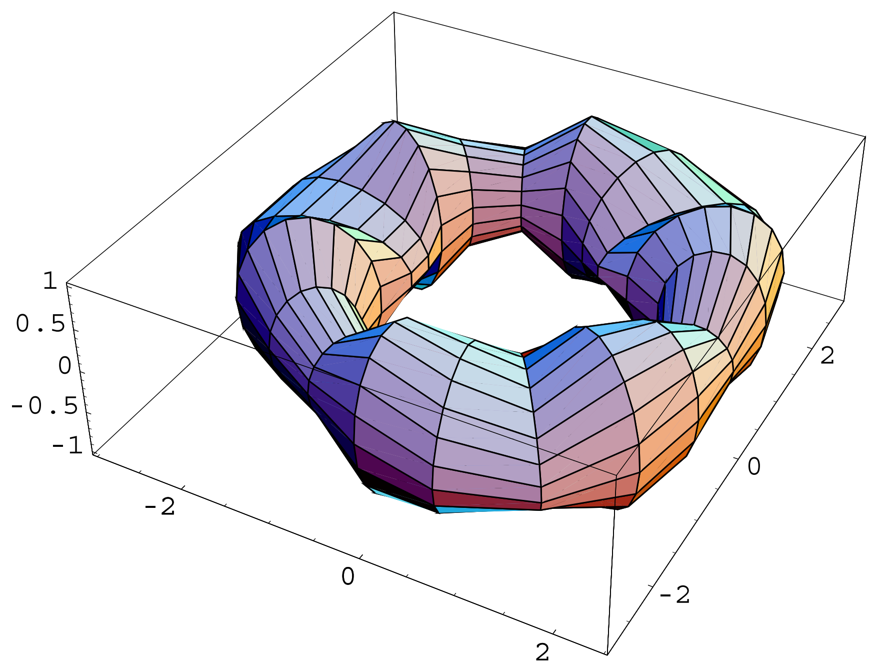


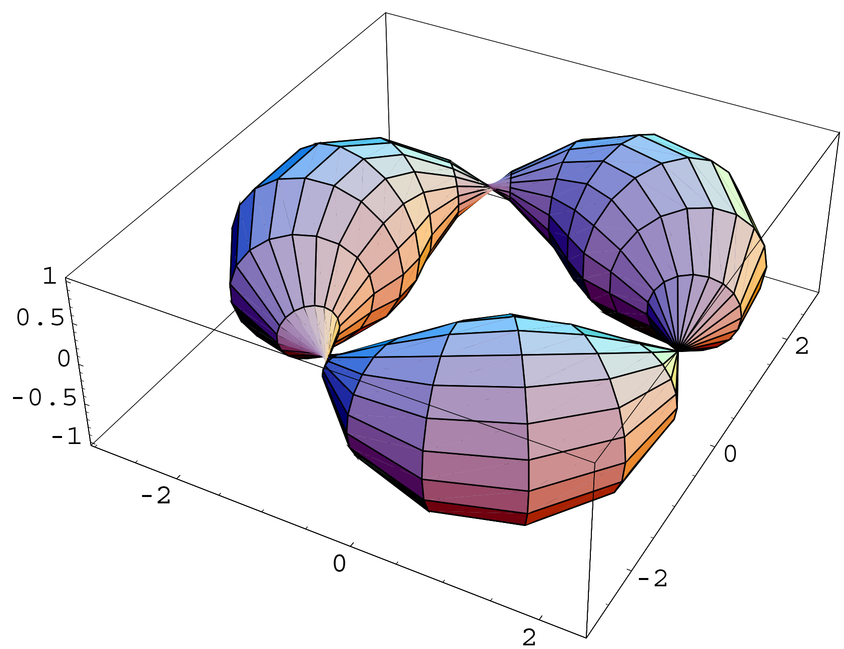
$$x_0^3 + x_1^3 + x_2^3 - 3\mu x_0 x_1 x_2 = 0 \ (\mu \in \mathrm{C})$$

if μ gets much closer to ∞



$$x_0^3 + x_1^3 + x_2^3 - 3\mu x_0 x_1 x_2 = 0 \ (\mu^3 = 1 \ \text{or} \ \infty)$$

It degenerates into 3 copies of P^1 ($= S^2$)



Thm 1 (Hesse 1849)

(0) $C(\mu)$ is nonsing. iff $\mu \neq 1, \zeta_3, \zeta_3^2, \infty$.

$C(\mu)$ is a 3-gon iff $\mu = 1, \zeta_3, \zeta_3^2, \infty$.

(1) $C(\mu)$ has 9 inflection points $[1 : -\beta : 0]$, $[0 : 1 : -\beta]$, $[-\beta : 0 : 1]$, where $\beta^3 = 1$.

(2) Any nonsing. cubic curve is isom. to some $C(\mu)$.

(3) $\mu = \mu'$ if and only if $C(\mu)$ and $C(\mu')$ are isom. with 9 points preserved

2 Moduli of cubic curves

Thm 2 (classical form)

$$A_{1,3} := \{\text{nonsing. cubics with 9 inflection pts}\}/\text{ isom.}$$

$$\simeq \mathbf{C} \setminus \{1, \zeta_3, \zeta_3^2\} \simeq \mathbf{H}/\Gamma(3) \text{ } (\mathbf{H}: \text{upper half plane})$$

$$= \{(\mathbf{C}/(\mathbf{Z}+\mathbf{Z}\tau), (i+j\tau)/3), \tau \in \mathbf{H}\}/\text{ isom.}$$

$$\tau \mapsto \tau' = \frac{a\tau+b}{c\tau+d}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mod 3$$

$$\overline{A_{1,3}} := \{\text{stable cubics with 9 inflection pts}\}/\text{ isom.}$$

$$= \{\text{Hesse cubics}\}/\text{isom=id}$$

$$= A_{1,3} \cup \left\{ C(\mu); \mu^3 = 1 \text{ or } \infty \right\} \simeq \mathbf{P}^1.$$

We wish to extend this to aribitrary dimension

1. **over $\mathbf{Z}[\zeta_N, 1/N]$ or over $\mathbf{Z}$ (including bad primes)**
2. **to define a representable functor $F := SQ_{g,K}$ (fine moduli) of compact obj.**
3. **to relate to GIT stability, that is,**

   to aim at $F(k) =$GIT stable objects for k alg. closed

   (This is missing in any other theory such as Alexeev, Olsson, Faltings-Chai) This is very difficult in general because it classifies stable objects completely.

There are difficulties never seen in dimension one

- Classical level structure = base of n-divison points,
- Since singular limits of Abelian varieties are very reducible in general, level structure may cause non-separatedness of the moduli
- That is, we need to prove in any dimension,

Lemma. (Valuative Lemma for Separatedness)

Let R be a DVR with frac. fld K, $X, Y \in F(R)$.

If $X_K \simeq Y_K$, then $X \simeq Y$. Or rather,

if isom over K, then isom over R.

- separated = Hausdorff, (e.g. if X projective, then separated)
- X: non-separated = non Hausdorff,
- If non-Hausdorff, then $\exists\ P_n \in X\ (n = 1, 2, \cdots),$ $P = \lim P_n$, $Q = \lim P_n$. But $P \neq Q$
- This really happens in geometry.

Example R : DVR, q : uniformizer of R, $K = R[1/q]$,

E, E' : elliptic curves over R

$$\mathrm{E} : y^2 = x^3 - q^6, \quad \mathrm{E}' : Y^2 = X^3 - 1,$$

$$P := \mathrm{E}_0 : y^2 = x^3, \quad Q := \mathrm{E}'_0 : Y^2 = X^3 - 1,$$

$$\mathrm{E}_K : (y/q^3)^2 = (x/q^2)^3 - 1, \quad \mathrm{E}'_K : Y^2 = X^3 - 1$$

Hence $P_n := \mathrm{E}_K \simeq \mathrm{E}'_K$, $P = \lim \mathrm{E}_K$, $Q = \lim \mathrm{E}'_K$, But $P \neq Q$

To overcome the difficulty of level str. we do as follows:

- New level structure = Framing of irreducible reps.
- Use the action of Heisenberg gp instead of n-div. pts
- To prove Val. Lemma for Separatedness, we use

Schur's Lemma over R.

Let R any ring over $\mathbf{Z}[\zeta_N, 1/N]$, G : Heisenberg gp $\subset GL(V \otimes R)$, V irr. rep. of G, $|G| = N$.

Let $h \in GL(V \otimes R)$. If $gh = hg$ for $\forall\ g \in G$, then h is scalar.

Schur's Lemma over R. Let $h \in \mathrm{GL}(V \otimes R)$.
If $gh = hg$ for any $g \in G$, then h is scalar.

Let R : DVR, q : unif. of R, $K = R[1/q]$.

1. Given a pair of SQASes X , Y over R s.t. $X_K \simeq Y_K$
2. If h is isom. of SQASes X_K , Y_K over K, then $gh = hg$ for any $g \in G$, and $h \in GL(V \otimes K)$.
3. By Schur's Lemma. Then $h = c\,\mathrm{id}_{V\otimes K}$.
4. Hence $h = \mathrm{id}_{\mathrm{P}(V\otimes K)}$, hence h extends to $\mathrm{id}_{\mathrm{P}(V\otimes R)}$,
5. hence $X \simeq Y$ over R, This proves Valuative Lemma.

Next Representability of the functor (page 10).

Case $g = 1$, $X_0(N)$ the integral model of $\overline{\mathrm{H}/\Gamma_0(N)}$.

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathrm{Z}); c \equiv 0 \mod N \right\}$$

Thm 3 (Mazur) $X_0(N)(\mathrm{Q}) =$ cusps for N large.

Corollary Let g be an autom. of an elliptic curve. $\mathrm{ord}(g) \neq 11$, but ≤ 12.

$\boldsymbol{F(S) = SQ_{g,K}(S)}$ **for any** $\boldsymbol{S}$ **over** $\mathbf{Z}[\zeta_N, 1/N]$**, hence in particular,** $\boldsymbol{F(\mathrm{Q}(\zeta_N)) = SQ_{g,K}(\mathrm{Q}(\zeta_N))}$**.**

Conjecture $\boldsymbol{SQ_{g,K}(\mathrm{Q}(\zeta_N)) \subset}$ **Boundary for** $\boldsymbol{N}$ **large.**

We re-start with

Thm 4 (classical form)

$$A_{1,3} := \{\text{nonsing. cubics with 9 inflection pts}\}/\ \text{isom.}$$

$$\overline{A_{1,3}} := \{\text{stable cubics with 9 inflection pts}\}/\ \text{isom.}$$

$$= \{\text{Hesse cubics}\}/\text{isom=id}$$

$$= A_{1,3} \cup \left\{C(\mu); \mu^3 = 1 \,\text{or}\, \infty\right\} \simeq \mathrm{P}^1.$$

We convert it into $G(3)$-equivariant theory

$G(3)$: Heisenberg group of level 3

the Heisenberg group $G = G(3)$ of level 3

$G = \langle \sigma, \tau \rangle$ acts on V, order $|G| = 27$,

$$V = \mathrm{C}x_0 + \mathrm{C}x_1 + \mathrm{C}x_2,$$

$$\sigma(x_i) = \zeta_3^i x_i, \quad \tau(x_i) = x_{i+1} \quad (i \in \mathrm{Z}/3\mathrm{Z})$$

ζ_3 is a primitive cube root of 1, **We will see later** **(page 22/23)**

$x_0^3 + x_1^3 + x_2^3$, $x_0x_1x_2 \in S^3V$ only are G-invariant

$$x_0^3 + x_1^3 + x_2^3 - 3\mu x_0x_1x_2 = 0 \ (\mu \in \mathrm{C}),$$

"Hesse cubic curves" in P^2

$\Downarrow$ **generalized**

A compactification of moduli of abelian varieties

3 Theta functions

Why-How does $G(3)$ get involved ?, $E(\tau)$: an elliptic curve $/\mathbb{C}$

$$E(\tau) = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau) = \mathbb{C}^*/w \mapsto wq^6, \quad q = e^{2\pi i\tau/6}$$

Def 5 Theta functions ($k = 0, 1, 2$)

$$\theta_k(\tau, z) = \sum_{m\in\mathbb{Z}} q^{(k+3m)^2} w^{k+3m} = \sum_{k+y\in k+Y} q^{(k+y)^2} w^{k+y}$$

where $w = e^{2\pi i z}$, $Y = 3\mathbb{Z} \subset X = \mathbb{Z}$, $k \in X/Y = \mathbb{Z}/3\mathbb{Z}$.

Define a map $\Theta : E(\tau) \to \mathbb{P}^2_{\mathbb{C}}$ as

$$z \;\mapsto\; [\theta_0(\tau, z), \theta_1(\tau, z), \theta_2(\tau, z)]$$

This is a closed immersion, Idenitify $\theta_k = x_k$

$G(3)$ get involved as follows :

Recall again

$$\theta_k(\tau, z + \frac{1}{3}) = \zeta_3^k \theta_k(\tau, z),$$
$$\theta_k(\tau, z + \frac{\tau}{3}) = q^{-1} w^{-1} \theta_{k+1}(\tau, z),$$
$$[\theta_0, \theta_1, \theta_2](\tau, z + \frac{\tau}{3}) = [\theta_1, \theta_2, \theta_0](\tau, z)$$

where $w = e^{2\pi i z}$, $q = e^{2\pi i \tau/6}$

σ, τ are liftings of these to GL(3):

$$z \mapsto z + \tfrac{1}{3} \text{ is lifted to } \sigma(\theta_k) = \zeta_3^k \theta_k$$
$$z \mapsto z + \tfrac{\tau}{3} \text{ is lifted to } \tau(\theta_k) = \theta_{k+1}$$

(To be more precise, we need to consider contragredient rep.)

Then $G(3) :=$ the group $\langle \sigma, \tau \rangle$

Let $V = Rx_0 + Rx_1 + Rx_2$, char.$R \neq 3$, R any ring,

Define $\sigma, \tau \in \mathrm{End}(V)$, and $G(3) :=$ the group $\langle \sigma, \tau \rangle$

$$\sigma(x_k) = \zeta_3^k x_k, \quad \tau(x_k) = x_{k+1}$$

Then $[\sigma, \tau] := \sigma\tau\sigma^{-1}\tau^{-1} = (\zeta_3 \cdot \mathrm{id_V})$ Thus $G(3)$ is of order 27.

Lemma 6 **For R any ring with $1/3 \in R$, V is $G(3)$-irreducible, that is, it has no proper $G(3)$-subspace except IV, I any ideal of R.**

Schur's lemma follows, Hence the base x_j are unique up to simulataneous constant multiple.

Thus $G(3)$ determines x_j "uniquely"

x_j is viewed as an algebraic theta function.

We recall Formulae:

$$\theta_k(\tau, z + \frac{1}{3}) = \zeta_3^k \theta_k(\tau, z),$$
$$\theta_k(\tau, z + \frac{\tau}{3}) = q^{-1} w^{-1} \theta_{k+1}(\tau, z)$$

Define a map $\Theta : E(\tau) \to \mathrm{P}^2_{\mathrm{C}}$ **as**

$$z \ \mapsto \ [\theta_0(\tau, z), \theta_1(\tau, z), \theta_2(\tau, z)]$$

This is a closed immersion, Identify $\theta_k = x_k$

The cubic curve $\Theta(E(\tau))$ **is** $G(3)$**-invariant,**

It is a Hesse cubic curve. Why ? (Compare page 18)

As a G(3)-module,

$$S^3V = 2\cdot 1_0 \oplus \bigoplus_{j=1}^{8}(1_j)$$

where

$$2\cdot 1_0 = \{x_0^3 + x_1^3 + x_2^3,\ x_0x_1x_2\},$$

$$1_j = \{x_0^3 + \zeta_3^j x_1^3 + \zeta_3^{2j} x_2^3\} \quad (j = 1, 2)$$

$$1_k = \{x_0^2x_1 + \zeta_3 x_1^2 x_2 + \zeta_3^2 x_2^2 x_0\} \quad (k \geq 3)$$

$2\cdot 1_0$ gives the equation of $\Theta(E(\tau))$ (Compare page 18)

$$x_0^3 + x_1^3 + x_2^3 - 3\mu(\tau)x_0x_1x_2 = 0$$

because Hesse cubics form a one parameter family.

4 Stability for compactification

moduli = the set of isomorphism classes,

roughly, "moduli"$=X/G$, where G: algebraic group

Comparison Table

GIT	Geometry
X	the set of geometric objects
G	the group of isomorphisms
x, x' are isom.	G-orbits are the same $O(x) = O(x')$
X_{ps}	stable objects
X_{ss}	semistable objects
X_{ps}/G	"moduli"
$X_{ss}//G$	"compactification" of moduli

- **A lot of compactif. of the moduli of abelian varieties are known.**

  Satake, Baily-Borel, Mumford, Namikawa (/C), Faltings-Chai,

- **What is nice? What is natural?**

- **Naively wish "to classify the isomorphism classes by invariants"**

> **(algebraic) moduli = the set of isom. classes distinguished (or identified) by the invariants**

- **But it is difficult to investigate by the invariants.**

- **it is easier to investigate geometrically.**

- **Consider only those geometric objects (= semi-stable objects) with their invariants well-defined**

- **(algebraic) moduli = the set of isom. classes distinguished (or identified) by the invariants =: the set of semi-stable objects**
- **Thus Stability and Semistability (Mumford:GIT)**

5 The space of closed orbits

X	the set of geometric objects
G	the group of isomorphisms
x, x' are isom.	G-orbits are the same $O(x) = O(x')$
X_{ps}	the set of properly-stable objects
X_{ss}	the set of semistable objects
$X_{ss}//G$	"compact moduli"

Rem

$$\text{stability} \Longrightarrow \text{closed orbits} \Longrightarrow \text{semistability}$$

Exam 1 Action on C^2 of $G = \mathrm{G}_m (= \mathrm{C}^*)$,

$$\mathrm{C}^2 \ni (x, y) \mapsto (\alpha x, \alpha^{-1} y) \quad (\alpha \in \mathrm{G}_m)$$

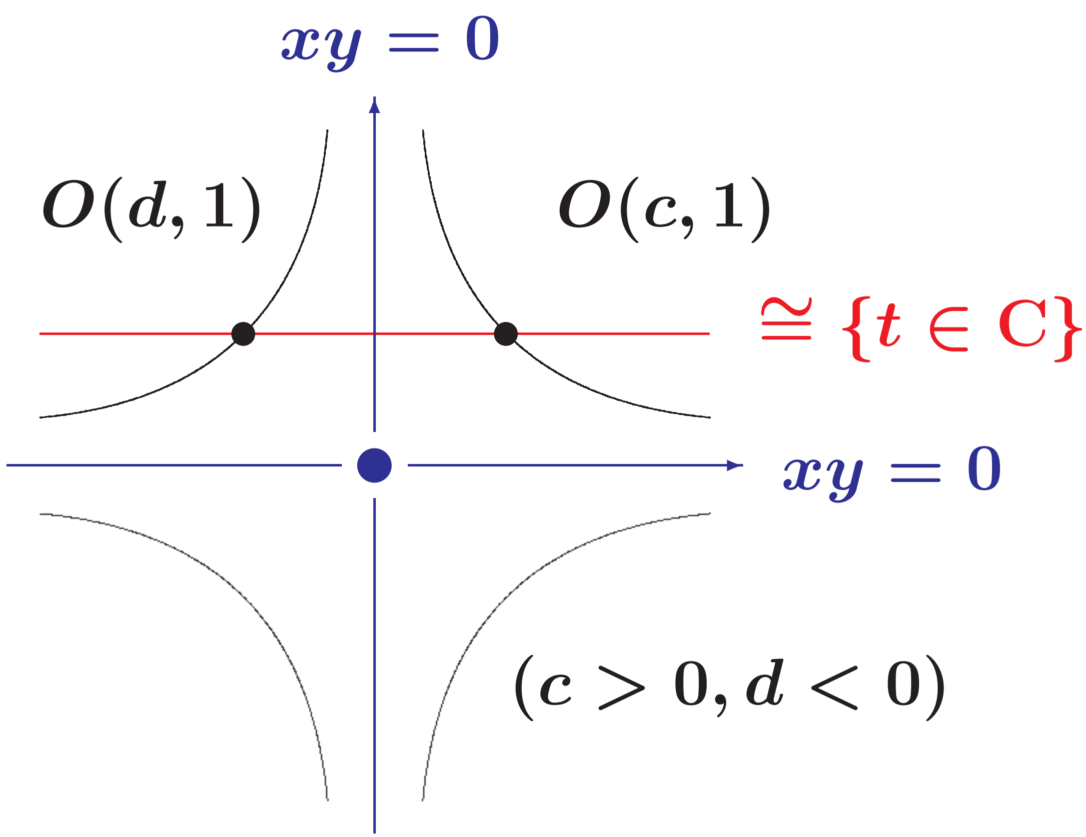
What is the quotient of C^2 by G ?

- Simple answer : the set of G-orbits (×)
- Answer : Spec(the ring of all G-invariant poly.)(○)
- $t := xy$ (and its polynomials) is the unique G-invariant !

$$\mathrm{C}^2 /\!/ G := \operatorname{Spec} \mathrm{C}[t] = \{t \in \mathrm{C}\}$$

But this is different from ”the set of G-orbits”.

- $\mathrm{C}^2 /\!/ G = \{t \in \mathrm{C}\}$ is the set of all closed orbits !!



- $t = 0$ is a point of $\mathrm{C}^2//G$.
- But $\{xy = 0\}$ consists of three G-orbits

$$\mathrm{C}^* \times \{0\}, \quad \{0\} \times \mathrm{C}^*, \quad \{(0,0)\}$$

- $\{(0,0)\}$ is the only closed orbit in $\{xy = 0\}$

Thm 7 $\mathrm{C}^2//G = \{t \in \mathrm{C}\}$ $(t = xy)$ is the set of all closed orbits.

Proof of Thm 7 : (Compare page 41)

- $O(t) = \{(x, y); xy = t\}$ is a closed orbit for any $t \neq 0$.
- For $t = 0$, $\{(0, 0)\}$ is the only closed orbit in $\{xy = 0\}$
- Any $t \in \mathrm{C}$ corresp. to a unique closed orbit in $\{xy = t\}$ $\square$

Thm 8 (Seshadri,Mumford) G : reductive, acting on a scheme X, (e.g. $G = \mathrm{G}_m$). Let X_{ss} = the set of semistable points. Then

$$X_{ss}//G := \mathrm{Spec}(\text{all } G\text{-invariants})$$

$$= \text{the set of closed orbits.}$$

Closed means that the orbit is closed in X_{ss}.

Thm 9 (Seshadri-Mumford) Let X be a projective scheme over a closed field k, G a reductive algebraic k-group acting on X. Let X_{ss} be an open subscheme of all semistable points in X, Then $\exists$ (cat.) quotient $Y = X_{ss}//G$. To be more precise,

(0) $\exists$ a proj. k-scheme Y and a G-invariant $\pi : X_{ss} \longrightarrow Y$ such that

(1) π is universal

(2) For $a, b \in X_{ss}$, $\pi(a) = \pi(b)$ iff $\overline{O(a)} \cap \overline{O(b)} \neq \emptyset$

where the closure is taken in X_{ss},

(3) $Y(k) =$ the set of G-orbits closed in X_{ss}.

Def 10 We keep the same notation as in Theorem 9 (Seshadri-Mumford). Let $p \in X$.

(1) the point p is said to be *semistable* if there exists a G-invariant homogeneous polynomial F on X such that $F(p) \neq 0$,

(2) the point p is said to be *Kempf-stable* if the orbit $O(p)$ is closed in X_{ss},

(3) the point p is said to be *properly-stable* if p is Kempf-stable and the stabilizer subgroup of p in G is finite.

We note that if $a, b \in X_{ps}$, (or if a, b Kempf-stable)

$$
\begin{aligned}
\pi(a) = \pi(b) &\iff \overline{O(a)} \cap \overline{O(b)} \neq \emptyset \\
&\iff O(a) \cap O(b) \neq \emptyset \\
&\iff O(a) = O(b) \\
&\iff \text{ } a \text{ and } b \text{ are isomorphic.}
\end{aligned}
$$

1. Each point of X_{ps} gives a closed orbit and
2. the first moduli $X_{ps}//G = X_{ps}/G$ (just the orbit space),
3. Moreover $X_{ps}//G$ is compactified by $X_{ss}//G$.

This is currently one of the most powerful methods for compactifying moduli spaces.

Thus we consider only those objects with closed orbits

As its consequence we will see

- **Abelian varieties have closed orbits (Kempf), and**
- **our SQASes have closed orbits,**
- **Conversely, any degenerate abelian scheme with closed orbit is one of our SQASes**
- **There is a simple characterization of our SQASes,**
- **This characterization enables us to compactify of the moduli of abelian varieties.**

6 GIT-stability and stable critical points

Recall

- **Definition of GIT-stability (born in 1965) has nothing to do with stable critical points**
- **But it has to do with stable critical points.**

Let V : vector space, G : reductive group acting on V,

K : a max. compact subgp of G,

$\|\cdot\|$: K-inv. metric

$$p_v(g) := \|g \cdot v\| \ (v \in V)$$

Thm 11 (Kempf-Ness 1979) The following are equivalent

(1) the orbit $O(v)$ is closed (= GIT-stable)

(2) p_v attains a minimum on $O(v)$

(3) p_v has a (stable) critical point on $O(v)$

Exam 2 Let $G = \mathbb{C}^*$, $K = S^1$, $V = \mathbb{C}^2$,

$$\mathbb{C}^2 \ni (x, y) \mapsto (tx, t^{-1}y) \quad (t \in G)$$

$$p_v(g) := \|(x, y)\|^2 = |x|^2 + |y|^2, \quad v = (x, y)$$

- If $v = (x, y)$ and $xy = t \neq 0$,

  then p_v attains the min. when $|tx| = |t^{-1}y|$

  because $|tx|^2 + |t^{-1}y|^2 \geq 2|tx \cdot t^{-1}y| = 2|xy|$.

- If $xy = 0$, then p_v attains min. at $(0, 0)$.

- When $xy = 0$, p_v has no min. on $\mathbb{C}^* \times \{0\}$, $\{0\} \times \mathbb{C}^*$

  where $\{xy = 0\} = \{(0, 0)\} \cup \mathbb{C}^* \times \{0\} \cup \{0\} \times \mathbb{C}^*$

7 Stable curves of Deligne-Mumford

Def 12 C is a stable curve of a genus g if

(0) it is a connected projective reduced curve

(1) with finite automorphism group,

(2) the singularities of C are like $xy = 0$

(3) $\dim H^1(O_C) = g$

Thm 13 (Deligne-Mumford 1969+ Knudsen)

Let $\overline{M_g}$: moduli of stable curves of genus g,

M_g : moduli of nonsing. curves of genus g.

Then $\overline{M_g}$ is projective (compact),

M_g is a Zariski open subset of $\overline{M_g}$.

Caution: Definition of stable curves is irrelevant to GIT stability

Nevertheless we have

Thm 14 **The following are equivalent**

(1) C is a stable curve (moduli-stable)

(2) any Hilbert point of $\Phi_{|mK|}(C)$ is GIT-stable (GIT-stable)

(3) any Chow point of $\Phi_{|mK|}(C)$ is GIT-stable (GIT-stable)

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(1)$\Leftrightarrow$(2) Gieseker 1982 (actually done before Mumford's work)

(1)$\Leftrightarrow$(3) Mumford 1977 (suggested by Gieseker's work)

8 Stability of cubic curves

CUBIC CURVES	STABILITY	STAB GP.
smooth elliptic	stable	finite
3-gon	closed orbits	2-dim
a line+a conic (transv.)	semistable	1-dim
irred. with node	semistable	finite
others	unstable	1-dim

Thm 15 For a cubic C, the following cond. are equiv.

(1) C has a closed SL(3)-orbit in $(S^3V)_{ss}$

(2) C is a Hesse cubic curve, that is, $G(3)$-invariant

(3) C is either smooth elliptic or a 3-gon

Exam 3

$$C_{a,b,c} \ : \ ax_0^3 + bx_1^3 + cx_2^3 - x_0x_1x_2 = 0. \tag{1}$$

The diagonal subgroup $G \simeq (\mathrm{G}_m)^2$ of SL(3) on the parameter space Spec $k[a, b, c]$ acts by

$$(a, b, c) \mapsto (sa, tb, uc) \tag{2}$$

where $stu = 1$, and $s, t, u \in \mathrm{G}_m$. We also see

(i) $(\mathrm{G}_m)^2$-Kempf-stable points are $abc \neq 0$ or $(a, b, c) = (0, 0, 0)$,

(ii) $(\mathrm{G}_m)^2$-semistable points which are not $(\mathrm{G}_m)^2$-Kempf-stable are $abc = 0$ except $(0, 0, 0)$. (Compare page 30)

9 Stability in higher-dim.

Thm 16 (N.1999) k is alg. closed, char.k and $|K|$ are coprime

K ($\cong H \oplus H^{\vee}$) : a finite symplectic abelian group, large enough

$G(K)$: Heisenberg gp assoc. to K, $V = k[H^{\vee}]$: gp ring of $H^{\vee}$

Assume X is a limit of abelian varieties with K-torsions (Here K large enough implies $X \subset \mathrm{P}(V)$)

Then the following are equivalent:

(1) X has a closed $\mathrm{SL}(V)$-orbit in Hilb_{ss} (GIT-stable)

(2) X is invariant under $G(K)$ ($G(K)$-stable)

(3) X is one of our SQASes (moduli-stable)

Thm 17 For cubics the following are equiv:

(1) it has a closed SL(3)-orbit (GIT-stable)

(2) it is a Hesse cubic, that is, $G(3)$-invariant ($G(3)$-stable)

(3) it is smooth elliptic or a 3-gon. (moduli-stable)

This is generalized into

Thm 18 Let X be a degenerate abelian variety (posssibly nonsingular). The following are equivalent under natural assump.:

(1) it has a closed SL(V)-orbit (GIT-stable)

(2) X is invariant under $G(K)$ ($G(K)$-stable)

(3) it is one of our SQASes (moduli-stable)

10 Moduli over $\mathbf{Z}[\zeta_N, 1/N]$

Thm 19 (a new version of the theorem of Hesse)

$$SQ_{1,3} = \mathbf{P}^1_{\mathbf{Z}[\zeta_3, 1/3]},$$

the projective fine moduli

(1) The universal cubic curve

$$\mu_0(x_0^3 + x_1^3 + x_2^3) - 3\mu_1 x_0 x_1 x_2 = 0$$

where $(\mu_0, \mu_1) \in SQ_{1,3} = \mathbf{P}^1$.

(2) when k is alg. closed and char. $k \neq 3$

$$\boldsymbol{SQ_{1,3}(k)} = \left\{ \begin{array}{l} \textbf{closed orbit cubic curves } /\boldsymbol{k} \\ \textbf{with level 3-structure} \end{array} \right\} /\textbf{isom.}$$

$$= \left\{ \begin{array}{l} \textbf{Hesse cubics } /\boldsymbol{k} \\ \textbf{with level 3-structure} \end{array} \right\} /\textbf{isom.=id.}$$

$$\boldsymbol{A_{1,3}(k)} = \left\{ \begin{array}{l} \textbf{closed orbit nonsingular cubic curves } /\boldsymbol{k} \\ \textbf{with level 3-structure} \end{array} \right\} /\textbf{isom.}$$

$$= \left\{ \begin{array}{l} \textbf{nonsingular Hesse cubics } /\boldsymbol{k} \\ \textbf{with level 3-structure} \end{array} \right\} /\textbf{isom.=id.}$$

Thm 20 (N. 1999) There exists the fine moduli $SQ_{g,K}$ projective over $\mathbf{Z}[\zeta_N, 1/N]$, $N = \sqrt{|K|}$

For k : alg. closed, if char.k and $N = \sqrt{|K|}$ are coprime

$$SQ_{g,K}(k) = \left\{ \begin{array}{l} \text{degenerate abelian schemes } /k \\ \text{with level } G(K)\text{-structure} \\ \text{and a closed SL-orbit} \end{array} \right\} /\text{isom.}$$

$$= \left\{ \begin{array}{l} G(K)\text{-invariant degenerate} \\ \text{abelian schemes } /k \\ \text{with level } G(K)\text{-structure} \end{array} \right\}$$

$$= \left\{ \begin{array}{l} G(K)\text{-invariant SQAS } /k \\ \text{with level } G(K)\text{-structure} \end{array} \right\}$$

$$\mathcal{A}_{g,K}(k) = \left\{ \begin{array}{l} \text{(nonsingular) abelian schemes } /k \\ \text{with level } G(K)\text{-structure} \\ \text{and a closed SL-orbit} \end{array} \right\} /\text{isom.}$$

$$= \left\{ \begin{array}{l} G(K)\text{-invariant (nonsingular)} \\ \text{abelian schemes } /k \\ \text{with level } G(K)\text{-structure} \end{array} \right\}$$

$$= \left\{ \begin{array}{l} G(K)\text{-invariant nonsingular SQAS } /k \\ \text{with level } G(K)\text{-structure} \end{array} \right\}$$

Compare page 78.

11 Tate curve and SQAS

SQAS : a generalization of Tate curve, R:DVR

Tate curve : $\mathbf{G}_m(R)/w \mapsto qw$

Hesse cubics at ∞ : $\mathbf{G}_m(R)/w \mapsto q^3 w$

Rewrite Tate curve as : $\mathbf{G}_m(R)/w^n \mapsto q^{mn} w^n (m \in \mathbf{Z})$

Hesse cubics at ∞ : $\mathbf{G}_m(R)/w^n \mapsto q^{3mn} w^n (m \in \mathbf{Z})$

The general case : B pos. def. symmetric

$$\mathbf{G}_m(R)^g/w^x \mapsto q^{B(x,y)} b(x,y) w^x, \quad b(x,y) \in R^\times \ \ (x \in X, y \in Y)$$

”natural limit as $q \to 0$” $\Longrightarrow$

3-gon and SQAS are born

12 Faltings-Chai degeneration data

R : a discrete valuation ring R, m the max. ideal of R,

$k(0) = R/m$, $k(\eta)$: the fraction field of R

Let (G, L) a quasi abelian scheme over R,

That is, (G_η, L_η) : abelian variety over $k(\eta)$

and suppose that G_0 is a split torus over $k(0)$,

$({}^tG, {}^tL)$: the (connected) Neron model of $({}^tG_\eta, {}^tL_\eta)$

May then suppose that $({}^tG_0, {}^tL_0)$ is a split torus over $k(0)$

Then we have a Faltings Chai degeneration data ass. to (G, L)

$$\text{Let } X = \mathrm{Hom}(G_0, \mathrm{G}_m), \quad Y = \mathrm{Hom}({}^tG_0, \mathrm{G}_m).$$

Hence $X \simeq \mathrm{Z}^g$, $Y \simeq \mathrm{Z}^g$, Y : a sublattice of X of finite index.

BECAUSE $\exists$ a natural surjective morphism $G \longrightarrow {}^t G$,

$\exists$ a surjective morphism $G_0 \longrightarrow {}^t G_0$,

$\exists$ $\mathrm{Hom}({}^tG_0, \mathrm{G}_m) \longrightarrow \mathrm{Hom}(G_0, \mathrm{G}_m)$,

Hence $\exists$ an injective homom. $Y \rightarrow X$ $\square$

Consider always over $\mathbf{Z}[\zeta_N, 1/N]$,

Let $K = X/Y \oplus (X/Y)^\vee$, $G(K)$: Heisenberg group

$$1 \to \mu_N \to G(K) \to K \to 0(\text{exact})$$

$$R[X/Y] = \oplus_{x \in X/Y} R\, v(x) \quad (\text{the group algebra of } X/Y)$$

$$(a, z, \alpha) \cdot v(x) = a\alpha(x)v(z + x)$$

$H^0(G, L)$: $G(K)$-irreducible $\simeq R[X/Y]$

$\Rightarrow$ a unique basis $v(x) = \theta_x \in H^0(G, L)$ (theta functions)

Let G_{for} : the formal completion of G along G_0

$$\boxed{G_{\text{for}} \simeq (\mathbf{G}^g_{m,R})_{\text{for}}}$$

θ_x $(x \in X/Y)$ are expanded on G_{for} as

$$\theta_x = \sum_{y \in Y} a(x+y) w^{x+y}$$

These $a(x)$ satisfy the conditions:

(1) $a(0) = 1$, $a(x) \in k(\eta)^\times \quad (\forall x \in X)$,

(2) $b(x,y) := a(x+y)a(x)^{-1}a(y)^{-1}$ is bilinear $(x, y \in X)$

(3) $B(x,y) := \mathrm{val}_q(a(x+y)a(x)^{-1}a(y)^{-1})$ is positive definite

$(x, y \in X)$

These $a(x)$ are called a degeneration data of (G, L)

Exam 4 If $g = 1, N = 3$, then theta functions $(k = 0, 1, 2)$

$$\theta_k = \theta_k(\tau, z) = \sum_{m \in \mathrm{Z}} q^{(3m+k)^2} w^{3m+k} = \sum_{3m \in Y} a(3m+k) w^{3m+k}$$

where $w \in \mathrm{G}_m$, $a(x) = q^{x^2}$, $X = \mathrm{Z}$ and $Y = 3\mathrm{Z}$, $B(x,y) = 2xy$.

Def 21

$$\widetilde{R} := R[a(x)w^x\vartheta, x \in X]$$

Define an action of Y on $\widetilde{R}$ by

$$S_y(a(x)w^x\vartheta) = a(x+y)w^{x+y}\vartheta$$

$\mathrm{Proj}(\widetilde{R})$: locally of finite type over R

$\mathcal{X}$: the formal completion of $\mathrm{Proj}(\widetilde{R})$

$\mathcal{X}/Y$: the top. quot. of $\mathcal{X}$ by Y

$O_{\mathcal{X}}(1)$ descends to $\mathcal{X}/Y$: ample

Grothendieck (EGA) guarantees

$\exists$ a projective R-scheme $(Z, O_Z(1))$

s.t. the formal completion Z_{for} of Z

$$Z_{\text{for}} \simeq \mathcal{X}/Y$$

$$(Z_\eta, O_{Z_\eta}(1)) \simeq (G_\eta, L_\eta)$$

(the stable reduction theorem)

The central fiber $(Z_0, O_{Z_0}(1))$ is our (P)SQAS.

If we take the normalization Z^{norm} of Z with Z_0^{norm} reduced, we get a bit different central fiber $(Z_0^{\text{norm}}, O_{Z_0^{\text{norm}}}(1))$, we call it TSQAS.

Exam 5 $g = 1$, $X = \mathrm{Z}$, $Y = 3\mathrm{Z}$.

$$\mathcal{X} = \mathrm{Proj}(\widetilde{R}), \quad a(x) = q^{x^2}, \ (x \in X)$$

The scheme $\mathcal{X}$ is covered with affine

$$V_n = \mathrm{Spec}\, R[a(x)w^x/a(n)w^n, x \in X]$$

$$V_n \simeq \mathrm{Spec}\, R[x_n, y_n]/(x_n y_n - q^2) \quad (n \in \mathrm{Z})$$
$$x_n = q^{2n+1}w, \ y_n = q^{-2n+1}w^{-1}.$$

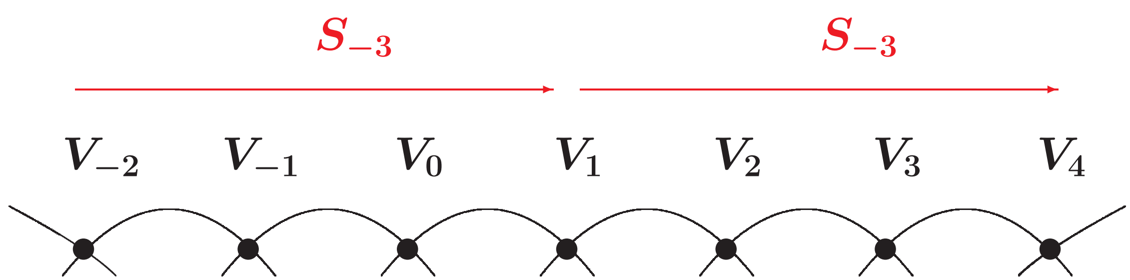
$$(V_n)_0 = \{(x_n, y_n) \in k(0)^2; x_n y_n = 0\}$$

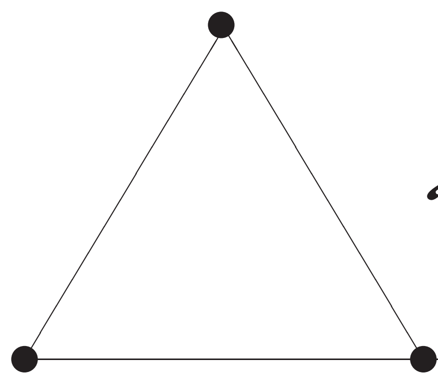
$\mathcal{X}_0$: a chain of infinitely many $\mathrm{P}^1_{k(0)}$

$$Y \text{ acts on } \mathcal{X}_0 \text{ as } V_n \xrightarrow{S_{-3}} V_{n+3},$$

$$(x_n, y_n) \overset{S_{-3}}{\longmapsto} (x_{n+3}, y_{n+3}) = (x_n, y_n)$$

$\mathcal{X}_0/Y$: a cycle of 3 $\mathrm{P}^1_{k(0)}$, $(\mathcal{X}/Y)^{\mathrm{alg}}_{\eta}$: a Hesse cubic over $k(\eta)$,





$\mathcal{X}_0/Y$

13 Limits of theta functions

$E(\tau)$ is embedded in P^2 by theta θ_k :

$$\theta_k(q,w) = \sum_{m\in \mathrm{Z}} q^{(3m+k)^2} w^{3m+k} \quad (k=0,1,2)$$

$$\theta_0^3 + \theta_1^3 + \theta_2^3 = 3\mu(q)\theta_0\theta_1\theta_2$$

Let R DVR, q uniformizer, $I = qR$, $w = q^{-1}u$

$$u \in R \setminus I,\ \overline{u} = u \mod I$$

$$\theta_k = \sum_{y\in Y} a(y+k) w^{y+k}$$

$$\theta_0(q, q^{-1}u) = \sum_{m\in\mathbb{Z}} q^{9m^2-3m}u^{3m}$$

$$= 1 + q^6u^3 + q^{12}u^{-3} + \cdots$$

$$\theta_1(q, q^{-1}u) = \sum_{m\in\mathbb{Z}} q^{(3m+1)^2-3m-1}u^{3m+1}$$

$$= u + q^6u^{-2} + q^{12}u^4 + \cdots$$

$$\theta_2(q, q^{-1}u) = \sum_{m\in\mathbb{Z}} q^{(3m+2)^2-3m-2}u^{3m+2}$$

$$= q^2\cdot(u^2 + u^{-1} + q^{18}u^5 + \cdots)$$

$$\Downarrow$$

$$\lim_{q\to 0}\,[\theta_k(q, q^{-1}u)] = [1, \overline{u}, 0] \in \mathbb{P}^2$$

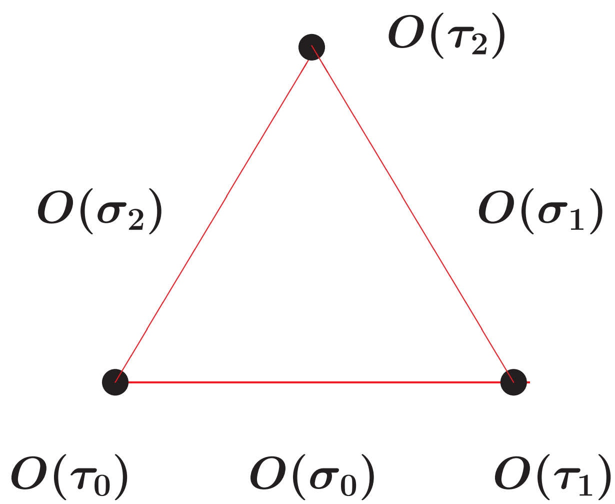
In $\mathbf{P}^2$

$$\lim_{q\to 0}\ [\theta_k(q, q^{-1}u)]_{k=0,1,2} = [1, \overline{u}, 0]$$

Similarly

$$\lim_{q\to 0}\ [\theta_k(q, q^{-3}u)]_{k=0,1,2} = [0, 1, \overline{u}]$$

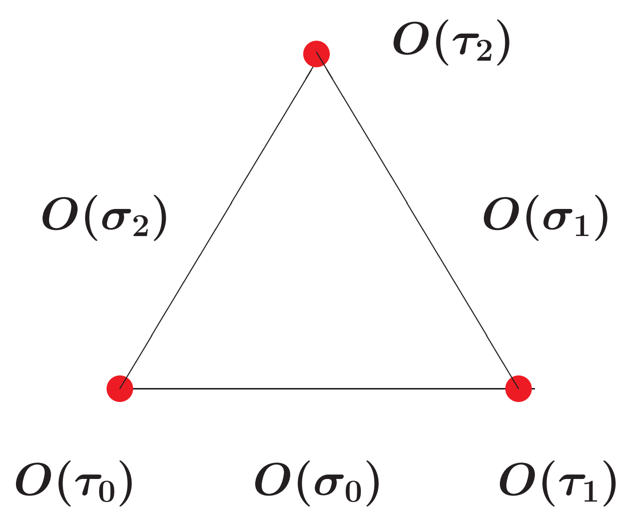
$$\lim_{q\to 0}\ [\theta_k(q, q^{-5}u)]_{k=0,1,2} = [\overline{u}, 0, 1]$$



$$w = q^{-2\lambda} u \text{ and } u \in R \setminus I.$$

$$\lim_{q \to 0} \left[\theta_k(q, q^{-2\lambda} u)\right] =$$

$$\begin{cases} [1,0,0] & (\text{if } -1/2 < \lambda < 1/2), \\ [0,1,0] & (\text{if } 1/2 < \lambda < 3/2), \\ [0,0,1] & (\text{if } 3/2 < \lambda < 5/2). \end{cases}$$



When λ ranges in R, the same limits repeat mod $Y = 3\mathrm{Z}$.

Thus $\lim_{\tau\to\infty} C(\mu(\tau))$ is the 3-gon $x_0x_1x_2 = 0$.

Def 22 For $\lambda \in X \otimes_{\mathrm{Z}} \mathrm{R}$ fixed

$$F_\lambda(x) = x^2 - 2\lambda x \quad (x \in X = \mathrm{Z})$$

Define $D(\lambda)$ (a Delaunay cell) by

the conv. closure of all $a \in X$ s.t. $F_\lambda(a) = \min\{F_\lambda(x); x \in X\}$.

Exam 6 1-dim. $B(x, x) = x^2$.

14 The shape of SQAS

”Limits of theta functions are described by the Delaunay decomposition.”

SQAS is a geometric limit of theta functions

SQAS is a generalization of 3-gons.

which is described by the Delaunay decomposition.

SQAS : a generalization of Tate curve, R:DVR

Tate curve : $\mathrm{G}_m(R)/w \mapsto qw$

Hesse cubics at ∞ : $\mathrm{G}_m(R)/w \mapsto q^3 w$

Rewrite Tate curve as : $\mathrm{G}_m(R)/w^n \mapsto q^{mn} w^n \ (m \in \mathrm{Z})$

Hesse cubics at ∞ : $\mathrm{G}_m(R)/w^n \mapsto q^{3mn} w^n \ (m \in \mathrm{Z})$

The general case : B pos. def. symmetric

$$\mathrm{G}_m(R)^g/w^x \mapsto q^{B(x,y)} b(x,y) w^x, \quad b(x,y) \in R^\times \ \ (x \in X, y \in Y)$$

"natural limit as $q \to 0$" $\Longrightarrow$

3-gon and SQAS are born

Let $X = \mathbf{Z}^g$, B a positive symmetric on $X \times X$.

$$\|x\| = \sqrt{B(x,x)} \text{ : a distance of } X \otimes \mathbf{R} \text{ (fixed)}$$

Def 23 **Let $\alpha \in X_{\mathbf{R}}$. a Delaunay cell $D = D(\alpha)$ is defiend to be the convex closure of points of X closest to α.**

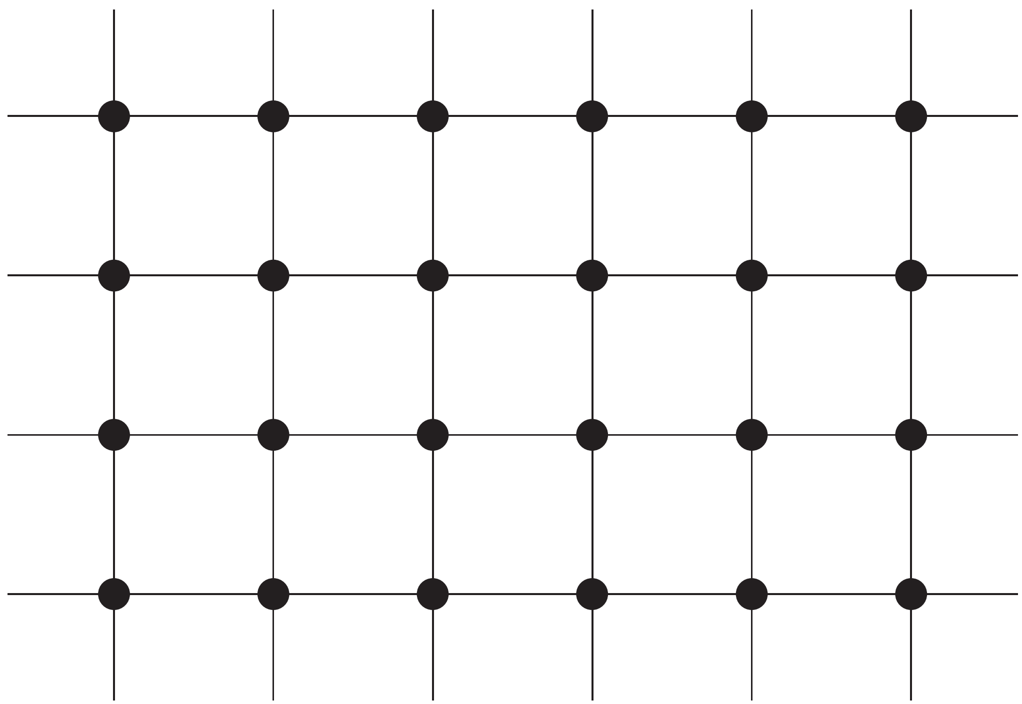
- **All Delaunay cells form a the Delaunay decomp. ass. to B**
- **Each SQAS (its scheme struture) and its decomposition into torus orbits (its stratification) are described by the Delaunay decomposition**
- **Each positive symmetric B defines a Delaunay decomp.**
- **Different B can yield the same Delaunay decomp. and the same SQAS.**

15 Delaunay decompositions

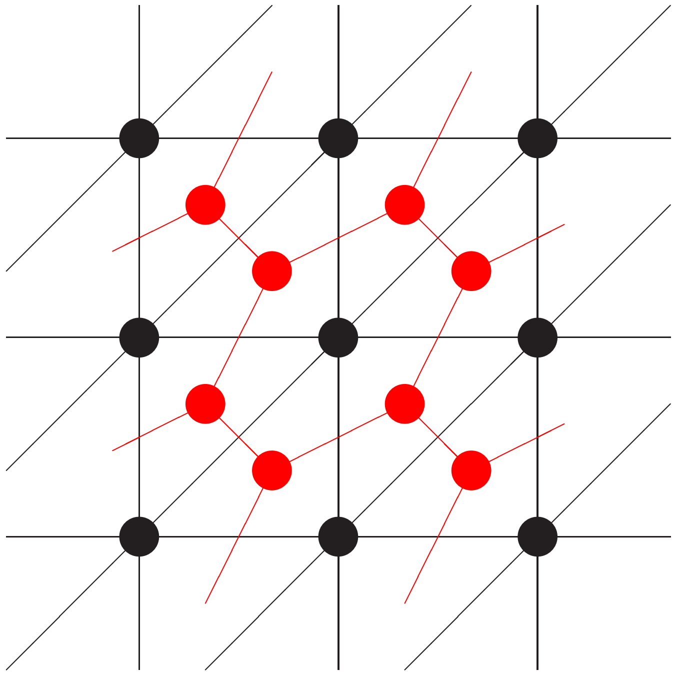
Exam 7 1-dim. $B(x,y)=2xy$, $X/Y=\mathbf{Z}/n\mathbf{Z}$,

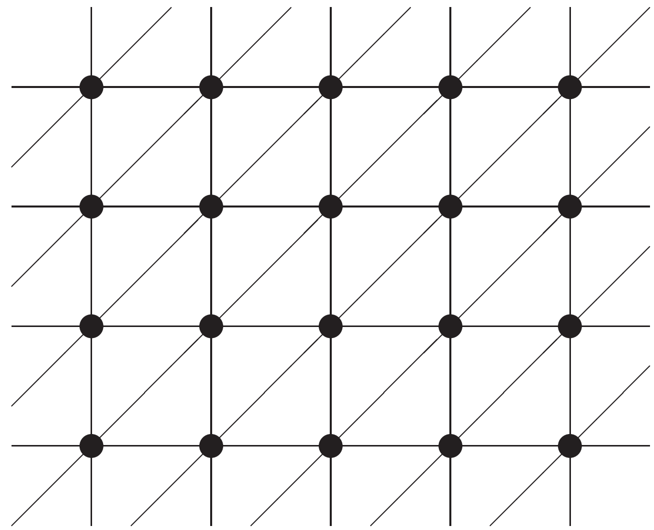
then SQAS Z_0 is an n-gon of $\mathbf{P}^1$

Exam 8 $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ **This (mod Y) is a union of $\mathbf{P}^1 \times \mathbf{P}^1$**



Exam 9 $B = \begin{pmatrix} \mathbf{2} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{2} \end{pmatrix}$





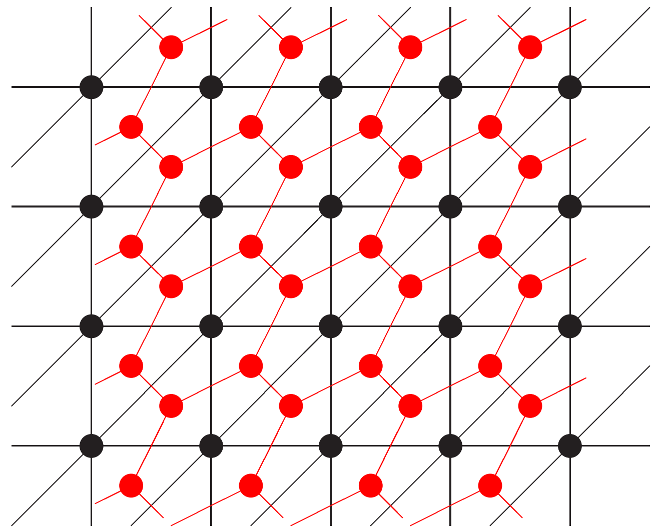
1. This (mod Y) is a SQAS.

   It is a union of P^2, each triangle denotes a P^2,

2. each line segment is a P^1

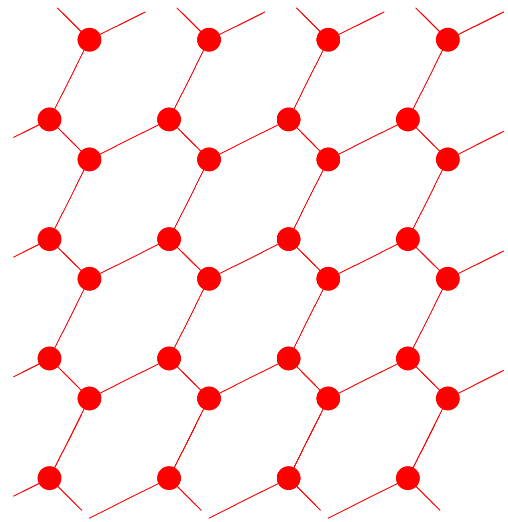
3. two P^2 intersect along P^1

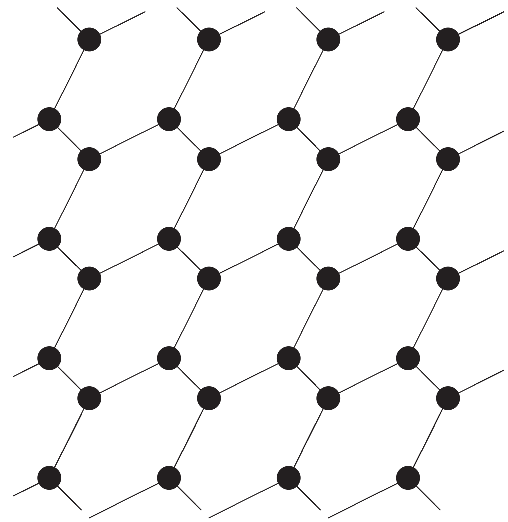
4. six P^2 meet at a point, locally $k[x_1, \cdots, x_6]/(x_i x_j, |i-j| \geq 2)$



Red one is the decomp. dual to the Delaunay decomp.

called Voronoi decomp.





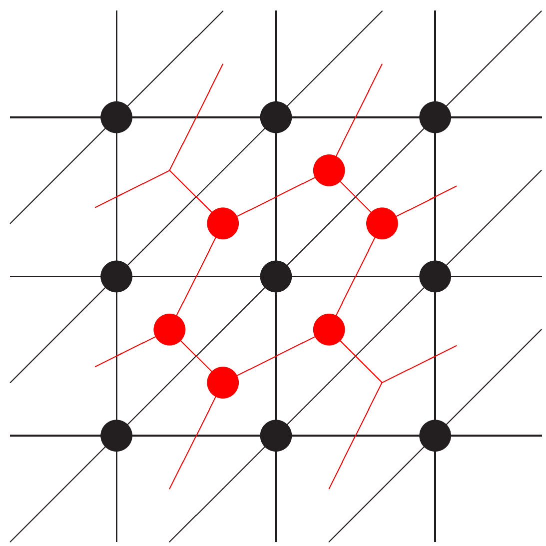
Voronoi decomposition

Def 24 D : for Delaunay cells

$$V(D) := \{\lambda \in X \otimes_{\mathbb{Z}} \mathbb{R}; D = D(\lambda)\}$$

We call it a Voronoi cell

$$\overline{V(0)} = \{\lambda \in X \otimes_{\mathbb{Z}} \mathbb{R}; \|\lambda\| \leqq \|\lambda - q\|, (\forall q \in X)\}$$



This is a crystal of mica.

$$\text{For } B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We get $\overline{V(0)}$, a cube (salt),

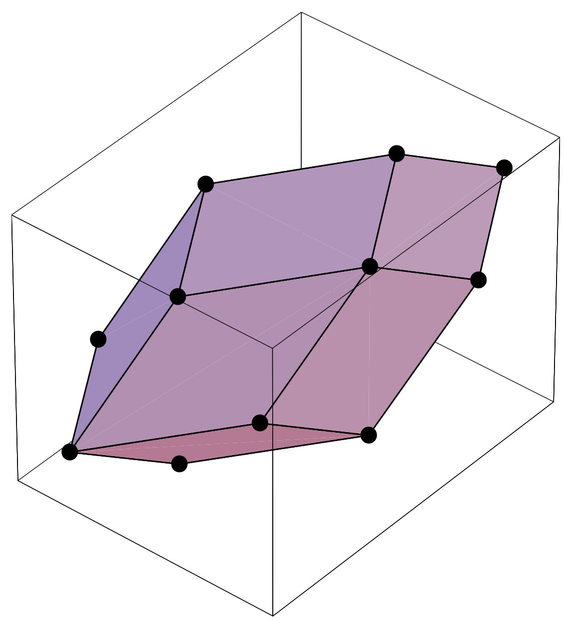
$$\text{For } B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

then we get a hexagonal pillar (calcite),

and then

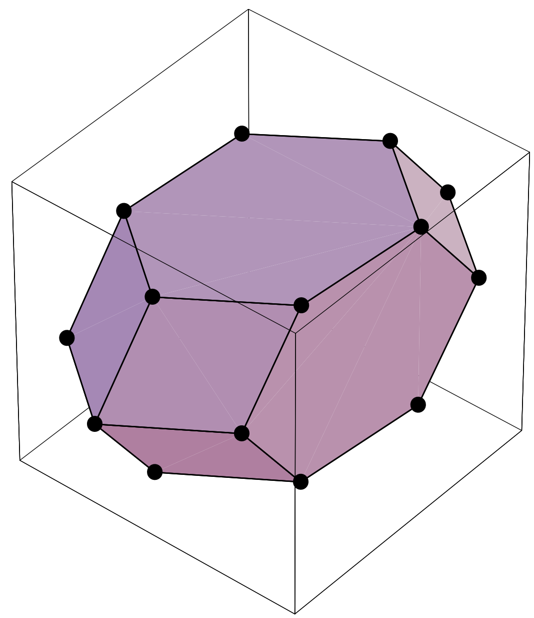
$$B = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

A Dodecahedron (Garnet)



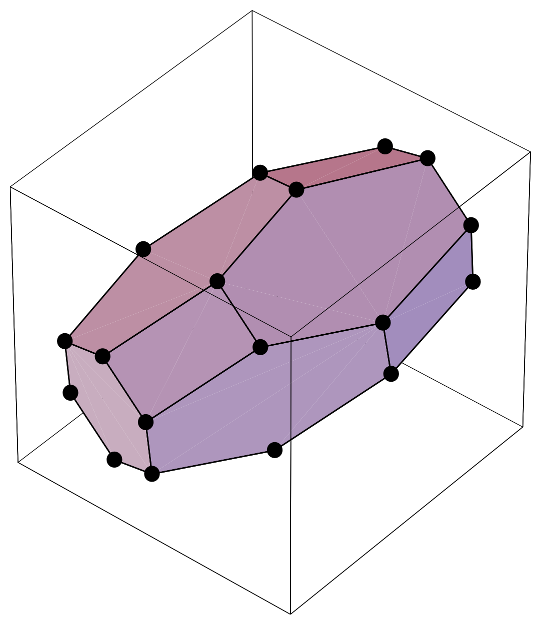
$$B = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Apophyllite $KCa_4(Si_4O_{10})_2F \cdot 8H_2O$



$$\boldsymbol{B} = \begin{pmatrix} \mathbf{3} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{3} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{-1} & \mathbf{3} \end{pmatrix}$$

A Trunc. Octahed. — Zinc Blende $\boldsymbol{ZnS}$



16 The Second Compactification over $\mathbf{Z}[\zeta_N, 1/N]$

Recall

Grothendieck (EGA) guarantees

$\exists$ a projective R-scheme $(Z, O_Z(1))$

s.t. the formal completion Z_{for} of Z

$$Z_{\text{for}} \simeq \mathcal{X}/Y, \quad (Z_\eta, O_{Z_\eta}(1)) \simeq (G_\eta, L_\eta)$$

(the stable reduction theorem)

The central fiber $(Z_0, O_{Z_0}(1))$ is our (P)SQAS.

The normalization Z^{norm} of Z with Z_0^{norm} reduced gives a bit different central fiber $(Z_0^{\text{norm}}, O_{Z_0^{\text{norm}}}(1))$, we call it TSQAS.

Thm 25 (N. 2010) $\exists$ a complete separated reduced-coarse moduli alg. space $SQ_{g,K}^{\mathrm{toric}}$ (Comapre page 46/47)

:moduli of TSQASes with level-$G(K)$ str. over $\mathbf{Z}[\zeta_N, 1/N]$.

Moreover, $\exists$ cano. bij. birat. morphism

$$\mathrm{sq} : SQ_{g,K}^{\mathrm{toric}} \to SQ_{g,K}$$

Corollay

The normalizations of $SQ_{g,K}^{\mathrm{toric}}$ and $SQ_{g,K}$ are isom.

Proof of Existence of $SQ_{g,K}^{\text{toric}}$.

1. Consider all TSQAS (X, L) with level $G(K)$. Then can embed (X, L) by L^n, any $n \equiv 1 \mod N$, $n \geq 2g+1$
2. $(X, L^n) \times (X, L^m) \in \text{Hilb} \times \text{Hilb}'$ for any rel. prime pair (n, m)
3. $H^0(X, L^n) \simeq V \otimes W_n$, $H^0(X, L^m) \simeq V \otimes W_m$ as $G(K)$-mod. where $V \simeq H^0(X, L)$
4. U a good reduced subsch. on which $GL(W_n) \times GL(W_m)$ acts
5. take quotient of U by $GL(W_n) \times GL(W_m)$ by Keel-Mori
6. $SQ_{g,K}^{\text{toric}} := U//GL(W_n) \times GL(W_m)$ is independent of n, m

Construction of a canonical morphism

1. For a given TSQAS over S with generic fibre AV, S any reduced scheme, we construct a PSQAS over S,
2. We can take U a subscheme of Hilb $\times$ Hilb$'$ over which universal TSQAS exists
3. (X, L) universal TSQAS, Then $|L|$ is base point free, we have a morphism $\Phi_{|L|} : X \to \mathrm{P}$
4. The image $\Phi_{|L|}(X)$ of (X, L) by $|L|$ is PSQAS.
5. Prove flatness of PSQAS
6. The map $X \mapsto \Phi_{|L|}(X)$ defines a morphism

$$\mathrm{sq} : SQ^{\mathrm{toric}}_{g,K} \to SQ_{g,K}.$$