THE COMPLETE MODULI SPACES OF DEGENERATE ABELIAN VARIETIES

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Abstract. For any positive integers g, d, there is Alexeev's complete moduli $\overline{AP}_{g,d}$ of seminormal degenerate abelian varieties, each coupled with a semiabelian action and an ample divisor [A02], while there is our second geometric compactification $SQ^{\text{toric}}_{g,K}$ of the moduli of abelian varieties [N10] for any finite symplectic abelian group K. We prove that if $|K| = N^2 \geq 1$, there is a $(N-1)$-dimensional effective family of closed immersions of $SQ^{\text{toric}}_{g,K}$ into $\overline{AP}_{g,N}$. We also prove $SQ^{\text{toric}}_{g,1} \simeq \overline{AP}_{g,1}$.

1. Introduction

Let K be a finite abelian group with symplectic form e_K, and $\mathcal{G}(K)$ the nonabelian Heisenberg group associated with K. The polarized abelian varieties with classical level-K structure admit level-$\mathcal{G}(K)$ structure in the sense of [N99]. For K sufficiently large, the fine moduli $A_{g,K}$ of g-dimensional abelian varieties with level-K structure is compactified into $SQ_{g,K}$ over $\mathbf{Z}[\zeta_N, 1/N]$, the "fine" moduli of *GIT-stable degenerate abelian schemes* (called PSQASes) with level-$\mathcal{G}(K)$ structure [N99].

Another compactification $SQ^{\text{toric}}_{g,K}$ of $A_{g,K}$ is constructed in [N10] as the "coarse" moduli of *reduced degenerate abelian varieties* (called TSQASes) with level-$\mathcal{G}(K)$ structure. There is a bijective morphism $\mathrm{sq} : SQ^{\text{toric}}_{g,K} \to SQ_{g,K}$ by [N10], which induces an isomorphism between their normalizations. In this sense, $SQ^{\text{toric}}_{g,K}$ is quite similar to $SQ_{g,K}$.

Alexeev [A02] constructs a complete moduli $\overline{AP}_{g,d}$ of *seminormal degenerate abelian varieties*, each coupled with semiabelian group action and an ample divisor. It is the compactification of the coarse moduli $AP_{g,d}$ of pairs (A, D) with A a g-dimensional abelian variety, D an ample divisor with $h^0(A, D) = d$. We note that the dimension of $\overline{AP}_{g,d}$ is equal to $g(g+1)/2 + d - 1$, while the dimension of $SQ^{\text{toric}}_{g,K}$ is equal to $g(g+1)/2$.

The purpose of this article is to define morphisms from [N10] to [A02], and consequently to indirectly define maps from [N99] to [A02]. We prove

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Theorem 1.1. *Let K be a finite symplectic abelian group, and $N = \sqrt{|K|}$. Then there exists an $(N-1)$-dimensional family of closed immersions of $SQ_{g,K}^{\text{toric}}$ into $\overline{AP}_{g,N}$ parametrized by a nonempty open subset of $\mathbf{P}^{N-1}$.*

Corollary 1.2. $SQ_{g,1}^{\text{toric}} \simeq \overline{AP}_{g,1}$.

The present article is organized as follows. Section 2 reviews the functors $\mathcal{SQ}_{g,K}^{\text{toric}}$ and $\overline{\mathcal{AP}}_{g,d}$. Section 3 proves that any TSQAS over a scheme has a canonical semi-abelian action. Section 4 proves Theorem 4.14, a more precise form of Theorem 1.1. Section 5 discusses the one dimensional case.

2. The functors $\mathcal{SQ}_{g,K}^{\text{toric}}$ and $\overline{\mathcal{AP}}_{g,d}$

Definition 2.1. Let $H := H(e) := \oplus_{i=1}^{g}(\mathbf{Z}/e_i\mathbf{Z})$ $(e_i|e_{i+1})$ be a finite abelian group of order $|H| = N$, $K := K_H = H \oplus H^\vee$, $H^\vee$ the Cartier dual of H and $\mathcal{O}_N := \mathbf{Z}[\zeta_N, 1N]$, ζ_N a primitive N-th root of unity. We define central extensions $\mathcal{G}(K)$ (resp. $G(K)$) of K by $\mathbf{G}_m$ (resp. by μ_N) with product $\cdot$ and an alternating form e_K on $K \times K$ as follows:

$$\mathcal{G}(K) := \{(a, z, \alpha); a \in \mu_N, z \in H, \alpha \in H^\vee\},$$
$$G(K) := \{(a, z, \alpha); a \in \mu_N, z \in H, \alpha \in H^\vee\},$$
$$(a, z, \alpha) \cdot (b, w, \beta) = (ab\beta(z), z + w, \alpha + \beta),$$
$$e_K((z, \alpha), (w, \beta)) = \beta(z)\alpha(w)^{-1}.$$

In what follows we denote $(1, u)$ by $\omega(u)$ for $u \in K$. Therefore $(a, z, \alpha) = a \cdot \omega(\alpha) \cdot \omega(z)$. Let $V(K) := \mathcal{O}_N[H^\vee] = \mathcal{O}_N[v(\chi); \chi \in H^\vee]$ be the group algebra of $H^\vee$ over $\mathcal{O}_N$, on which $\mathcal{G}(K)$ acts by $U(K)$;

$$U(K)(a, z, \alpha)v(\chi) := a\chi(z)v(\chi + \alpha). \tag{1}$$

It is an irreducible module under both $\mathcal{G}(K)$ and $G(K)$ [N10, § 4]. We denote $\mathcal{G}(K)$ (resp. $G(K)$, $V(K)$, $U(K)$) by $\mathcal{G}_H$ (resp. G_H, V_H, U_H) to emphasize dependence on H. For any nonnegative integer m we define a $\mathcal{G}_H$-module V_m by $V_m = V_H$ as a set, and $U_m(a, z, \alpha)v(\chi) = a^{mN+1}\chi(z)v(\chi+\alpha)$. Over $\mathcal{O}_N$, V_m is an irreducible $\mathcal{G}_H$-module of weight $mN + 1$, unique up to isomorphism, and any $\mathcal{G}_H$-module of weight $mN + 1$ is a direct sum of V_m because $U_m = U_H$ on G_H.

Definition 2.2. Let $(Z, \mathcal{L})$ be a polarized T-scheme. The set of isomorphisms $\Phi := \{(T_g, \phi_g)\}_{g \in \mathcal{G}_H}$ is called a $\mathcal{G}_H$-linearization of $\mathcal{L}$ if

1. $T_g \in \mathrm{Aut}_T(Z)$ and $\phi_g : \mathcal{L} \simeq T_g^*(\mathcal{L})$ is a Z-isomorphism,
2. $T_g = \mathrm{id}_Z$ and ϕ_g is multiplication by g if $g \in \mu_N$,
3. $T_{gh} = T_g T_h$ and $\phi_{gh} = T_h^*\phi_g \cdot \phi_h$ $(\forall g, h \in \mathcal{G}_H)$.

Then we say that $\mathcal{L}$ is *$\mathcal{G}_H$-linearized by* Φ. If $\mathcal{L}$ is $\mathcal{G}_H$-linearized, then $\mathcal{L}$ is $\mathcal{G}_{H'}$-linearized for any subgroup H' of H. We say that $\mathcal{L}$ is *strictly $\mathcal{G}_H$-linearized* if there is no group H'' such that $H \subset H''$, $H \neq H''$ and $\mathcal{L}$ is $\mathcal{G}_{H''}$-linearized. In what follows, we simply say that $\mathcal{L}$ is *$\mathcal{G}_H$-linearized* instead of *strictly $\mathcal{G}_H$-linearized* if no confusion is possible.

Definition 2.3. For a $\mathcal{G}_H$-linearization Φ of $\mathcal{L}$, we define the maps $\tau := \tau_\Phi$, $\tau^{ab} := \tau^{ab}_\Phi$ and $\rho := \rho_\Phi : \mathcal{G}_H \to \mathrm{End}\,(\pi_*(\mathcal{L}))$ by

$$\begin{aligned}\tau(g)(x,\zeta) &:= (T_g(x), \phi_g(x)\zeta) \in \mathcal{L},\ \tau^{ab}(g)(x) := T_g(x),\\ \rho(g)(\theta) &:= T^*_{g^{-1}}(\phi_g(\theta)), \quad (x \in Z, \zeta \in \mathcal{L}_x, \theta \in \pi_*(\mathcal{L}), g \in \mathcal{G}_H).\end{aligned}$$

We see that τ, τ^{ab} and ρ are group scheme morphisms. We note $\tau(g) \in \mathrm{Aut}_T(\mathcal{L}/Z)$ is a scheme automorphism of $\mathcal{L}$. Conversely if we are given a group T-scheme morphism $\tau : \mathcal{G}_H \to \mathrm{Aut}_T(\mathcal{L}/Z)$, then $\mathcal{L}$ is $\mathcal{G}_H$-linearized. See Lemma 3.6 for $\mathrm{Aut}_T(\mathcal{L}/Z)$.

Definition 2.4. Let k be an algebraically closed field over $\mathcal{O}_N$. A triple (P_0, ϕ, τ) or $(P_0, \mathcal{L}_0, \phi, \tau)$ is a k-TSQAS with *rigid level-$\mathcal{G}_H$ structure* (or abbr. *a rigid-$\mathcal{G}_H$ k-TSQAS*) if

1. $\mathcal{L}_0$ is an ample line bundle, $\mathcal{G}_H$-linearized by $\Phi = \{(T_g, \phi_g)\}_{g\in\mathcal{G}_H}$,
2. $\tau := \tau_\Phi : \mathcal{G}_H \to \mathcal{G}(P_0, \mathcal{L}_0)$ is an isomorphism, where $(P_0, \mathcal{L}_0)$ is the closed fiber of a proper flat family $(P, \mathcal{L})$ over a complete discrete valuation ring with generic fiber an abelian variety [N99, pp. 669-681], [N10, pp. 74,78,79]
3. $\phi : P_0 \to \mathbf{P}(V_H)$ is a rational map such that $\phi^* : V_H \otimes_{O_N} k \simeq H^0(P_0, \mathcal{L}_0)$ is a $\mathcal{G}_H$-isomorphism via τ,
4. $\rho(\phi, \tau) = U_H \otimes_{O_N} k$, where $\rho(\phi, \tau)(g) := (\phi^*)^{-1}\rho_\Phi(g)\phi^*$ $(\forall g \in \mathcal{G}_H)$.

It is clear from (2.4.2) that $\tau^{ab}(\mathcal{G}_H) = K(P_0, \mathcal{L}_0) \simeq K$.

Definition 2.5. Let T be any scheme over $\mathcal{O}_N$. The triple $(P \stackrel{\pi}{\to} T, \mathcal{L}, \phi, \tau)$ is a T-TSQAS with rigid level-$\mathcal{G}_H$ structure [N10, 5.3 (ii)] (or abbr. *a rigid-$\mathcal{G}_H$ T-TSQAS*) if

1. π is flat with $\mathcal{L}$ π-ample and $\mathcal{G}_H$-linearized by $\Phi = \{(T_g, \phi_g)\}_{g\in\mathcal{G}_H}$,
2. $\tau := \tau_\Phi : (\mathcal{G}_H)_T \to \mathrm{Aut}_T(\mathcal{L}/P)$ is a closed T-immersion,
3. $\phi : P \to \mathbf{P}(V_H)_T$ is a rational map such that $\phi^* : V_H \otimes_{O_N} \mathcal{M} \simeq \pi_*(\mathcal{L})$ is a $(\mathcal{G}_H)_T$-isomorphism for some trivial $(\mathcal{G}_H)_T$-module $\mathcal{M} \in \mathrm{Pic}(T)$,
4. $\rho(\phi, \tau) := (\phi^*)^{-1}\rho_\Phi\phi^* = U_H \otimes_{O_N} O_T$,
5. any geometric fiber $(P_s, \mathcal{L}_s, \phi_s, \tau_s)$ is a rigid-$\mathcal{G}_H$ $k(s)$-TSQAS.

Remark 2.6. For a T-TSQAS $(P, \mathcal{L})$ with $\mathcal{L}$ $\mathcal{G}_H$-linearized, $\mathcal{L}$ is *strictly $\mathcal{G}_H$-linearized* iff $h^0(P_s, \mathcal{L}_s) = \sqrt{|K|}$ for any geometric fiber $(P_s, \mathcal{L}_s)$.

Definition 2.7. We define the functor $\mathcal{SQ}^{\mathrm{toric}}_{g,K}$ from $\mathcal{O}_N$-schemes to sets by

$$\begin{aligned}\mathcal{SQ}^{\mathrm{toric}}_{g,K}(T) = {}&\text{the set of } T\text{-TSQASes } (P, \phi, \tau) \text{ of relative dimension } g\\ &\text{with } \textit{rigid level-}\ \mathcal{G}_H\textit{-structure} \text{ modulo } T\text{-isomorphism}\end{aligned}$$

See [N10, 5.11, (i)-(iii)] for T-isomorphism between (P, ϕ_i, τ_i). The condition (ii) in [*ibid.*] is replaced here by $\phi_1^* = f^*\phi_2^*$. See also [N99, 9.17]

Theorem 2.8. *$\mathcal{SQ}^{\mathrm{toric}}_{g,K}$ has a separated reduced-coarse moduli algebraic space over $\mathcal{O}_N$, which we denote by $SQ^{\mathrm{toric}}_{g,K}$.*

Proof. See [N10, 11.4] for reduced-coarse moduli. We note that for any fixed nonnegative integer m, any $\mathcal{G}_H$-module of weight $mN+1$ is a direct sum of a fixed $\mathcal{G}_H$-module V_m of the same weight. See Definition 2.1. Hence we can apply [N10, Sections 5-11] to prove Theorem 2.8 without any restriction on elementary divisors of K. The properness of the action of $\mathrm{PGL}\times\mathrm{PGL}$ [N10, p. 123] is proved by reducing to the case where every elementary divisor of K is at least 3. For this it suffices to prove the following

Claim 2.8.1. (cf. [N10, Lemma 6.7]) *Let R be a complete discrete valuation ring, $k(\eta)$ the fraction field of R and $S := \operatorname{Spec} R$. Let (Z_i, ϕ_i, τ_i) be rigid-$\mathcal{G}_H$ S-TSQASes whose generic fibers are abelian varieties. If (Z_i, ϕ_i, τ_i) are $k(\eta)$-isomorphic, then they are S-isomorphic.*

Claim 2.8.1 follows from the following Claim 2.8.2 :

Claim 2.8.2. *With the same notation as above, let $(P, \mathcal{L})$ be an S-TSQAS with generic fiber $(P_\eta, \mathcal{L}_\eta)$ an abelian variety. Then $(P, \mathcal{L})$ is the normalization of a modified Mumford family for the generic fiber $(P_\eta, \mathcal{L}_\eta)$.*

Proof of Claim 2.8.2. Let P_{for} be the formal completion of P along P_0. Since P_0 is reduced, by [SGA1, Corollaire 8.4], there is a category equivalence between étale coverings of P_0 and étale coverings of P_{for}. Let n be a positive integer prime to the characateristic of $k(0)$ and $|H|$. Then it is easy to see that there exists an étale $H^\dagger/H \simeq (\mathbf{Z}/n\mathbf{Z})^g$-covering $(P_0^\dagger, \mathcal{L}_0^\dagger)$ of $(P_0, \mathcal{L}_0)$ such that $K(P_0^\dagger, \mathcal{L}_0^\dagger) = H^\dagger \oplus (H^\dagger)^\vee$. Hence there exists a formal scheme $(P_{\mathrm{for}}^\dagger, \mathcal{L}_{\mathrm{for}}^\dagger)$ which is an étale $(\mathbf{Z}/n\mathbf{Z})^g$-covering of $(P_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}})$. Then there exists a projective S-scheme $(P^\dagger, \mathcal{L}^\dagger)$ algebrizing $(P_{\mathrm{for}}^\dagger, \mathcal{L}_{\mathrm{for}}^\dagger)$ which is an étale $(\mathbf{Z}/n\mathbf{Z})^g$-covering of $(P, \mathcal{L})$ with $\mathcal{L}^\dagger$ the pull back of $\mathcal{L}$. It follows that $(P_\eta^\dagger, \mathcal{L}_\eta^\dagger)$ is a polarized abelian variety, $(P_0^\dagger, \mathcal{L}_0^\dagger)$ is a reduced $k(0)$-TSQAS and $P^\dagger$ is normal by [N10, 10.2]. Since $n \geq 3$, by [N10, 10.4] $(P^\dagger, \mathcal{L}^\dagger)$ is the normalization of a modified Mumford family for the generic fiber $(P_\eta^\dagger, \mathcal{L}_\eta^\dagger)$. Hence the quotient $(P,\mathcal{L})$ of $(P^\dagger, \mathcal{L}^\dagger)$ by $(\mathbf{Z}/n\mathbf{Z})^g$ is also the normalization of a modified Mumford family for the generic fiber $(P_\eta, \mathcal{L}_\eta)$.

This completes the proof of Theorem 2.8. □

Definition 2.9. [A02] Let k be an algebraically closed field. A g-dimensional semiabelic k-pair of degree d is a quadruple $(G, P, \mathcal{L}, \Theta)$ such that

1. P is a connected seminormal *complete* k-variety, and any irreducible component of P is g-dimensional,
2. G is a semi-abelian k-scheme acting on P,
3. there are only finitely many G-orbits,
4. the stabilizer subgroup of every point of P is connected, reduced and lies in the torus part of G,
5. $\mathcal{L}$ is an ample line bundle on P with $h^0(P, \mathcal{L}) = d$,
6. Θ is an effective Cartier divisor of P with $\mathcal{L} = O_P(\Theta)$ which does not contain any G-orbits.

Recall that a variety Z is said to be *seminormal* if any bijective morphism $f : W \to Z$ with W reduced is an isomorphism.

Definition 2.10. Let T be a scheme. A g-dimensional semiabelic T-pair of degree d is a quadruple $(G, P \xrightarrow{\pi} T, \mathcal{L}, \Theta)$ such that

1. G is a semi-abelian group T-scheme of relative dimension g,
2. P is a proper flat T-scheme, on which G acts,
3. $\mathcal{L}$ is a π-ample line bundle on P with $\pi_*(\mathcal{L})$ locally free of rank d,
4. any geometric fiber $(G_s, P_s, \mathcal{L}_s, \Theta_s)$ $(s \in T)$ is a stable semiabelic pair.

Definition 2.11. We define the functor $\mathcal{M}_{g,d}$ from schemes to sets by

$$\mathcal{M}_{g,d}(T) = \text{the set of } g\text{-dimensional semiabelic } T\text{-pairs of degree } d/T\text{-isom.}$$

The functor $\overline{\mathcal{AP}}_{g,d}$ is a subfunctor of $\mathcal{M}_{g,d}$ of semiabelic T-pairs with any generic fibers $P_\eta = G_\eta$ abelian varieties. $\overline{\mathcal{AP}}_{g,d}$ has a coarse moduli algebraic space $\overline{AP}_{g,d}$ over $\mathbf{Z}$ by [A02, 5.10.1].

3. The semi-abelian group action on a T-TSQAS

The purpose of this section to construct a semiabelian group action on any T-TSQAS. We freely use the notation in [N99, Sections 1-3].

3.1. **Notation.** Let R be a complete discrete valuation ring with q uniformizer, $k(0) := R/qR$ and $k(\eta)$ the fraction field. Let $(P, \mathcal{L})$ the one-parameter family of TSQASes over R such that the generic fiber P_η is an abelian variety, and the closed fiber P_0 of P is a TSQAS. Let A_0 the abelian variety part of P_0, T_0 the torus part of P_0, $X = \mathrm{Hom}_k(T_0, \mathbf{G}_m)$, $g' = \dim T_0$, $g'' = \dim A_0$, $g = g' + g''$ and $\mathrm{Del} = \mathrm{Del}_B$ the Delaunay decomposition of P_0 on the lattice X of rank g' and B the integral positive bilinear form on $X \times X$ associated with P_0, which we abbreviate as $(x, y) := B(x, y)$. By choosing $q^{r(x)}w^x$ for w^x by taking a finite base change of Spec R in [N99, p. 671] we may assume that B is even, and $r(x) = 0$ for any $x \in X$. This implies that P_0 is reduced. Let $T_0^t := T^t \otimes k(0)$ be the dual torus of T_0, and $Y = \mathrm{Hom}_k(T_0^t, \mathbf{G}_m)$ [*ibid.*, p. 666].

Lemma 3.2. *Let $\tau \in \mathrm{Del}(0)$ and $C(0, \tau)$ the closed cone over $\mathbf{R}_0$ generated by τ. Let $X^C(\tau)$ be the sublattice of X generated by $C(0, \tau) \cap X$. Then $X/X^C(\tau)$ is torsion-free. In particular, $X^C(\sigma) = X$ if $\sigma \in \mathrm{Del}^{(g')}(0)$.*

Proof. It suffices to prove $X^C(\tau)_{\mathbf{R}} \cap X = X^C(\tau)$. We suffice to prove $X^C(\tau)_{\mathbf{R}} \cap X \subset X^C(\tau)$ because the converse inclusion is clear. Let $f \in X^C(\tau)_{\mathbf{R}} \cap X$. Then there exists $x \in C(0, \tau) \cap X$ such that $x + f \in C(0, \tau) \cap X$. Hence $f = (x + f) - x$ with $x + f$, $x \in C(0, \tau) \cap X$. Hence $f \in X^C(\tau)$, hence $X^C(\tau)_{\mathbf{R}} \cap X = X^C(\tau)$. □

Lemma 3.3. *Let $\tau \in \mathrm{Del}^{(g'-1)}(c)$, $\sigma_i \in \mathrm{Del}^{(g')}(c)$ $(i = 1, 2)$ with $\tau = \sigma_1 \cap \sigma_2$ and $Z(\sigma_i) = \overline{O(\sigma_i)}$ the irreducible component of P_0 associated with σ_i. Then*

1. *$O(\tau)$ is a Cartier divisor of $Z(\sigma_i)$ defined by a single equation $\zeta_{x_i,c}=0$ for some generator $x_i\in C(c,-c+\sigma_i)$ of $X/X^C(\tau)$,*
2. *P_0 is, along $O(\tau)$, defined by the single equation $\zeta_{x_1,c}\zeta_{x_2,c}=0$.*

Proof. By [N99, 4.9], O_{P_0} is isomorphic to

$$O_{P_0,O(\tau)} := O_{A_0}[\zeta_{x,c},\zeta^{\pm}_{y,c}]_{\ x\,\in\,C(0,\,-c\,+\,\sigma_1\,\cup\,\sigma_2)\,\cap\,X,\ y\,\in\,X^C(\tau)}\ .$$

Since $X/X^C(\tau)$ is torsion free in view of Lemma 3.2, $X/X^C(\tau)$ is infinite cyclic. Since the subset $C(0,\sigma_i)+X^C(\tau)$ is a closed half space of $X_{\mathbf{R}}$, we can choose an element $x_i\in C(0,\sigma_i)\cap X$ such that $X/X^C(\tau)=\mathbf{Z}x_i\simeq\mathbf{Z}$. By choosing in addition a $\mathbf{Z}$-basis y_j $(2\le j\le g)$ of $X^C(\tau)$, we may assume

(i) x_i generates $X/X^C(\tau)=X^C(\sigma_1)/X^C(\tau)=X^C(\sigma_2)/X^C(\tau)$,
(ii) x_1 (resp. x_2) and y_j $(2\le j\le g)$ is a $\mathbf{Z}$-basis of X.

Let $M=\sum_{i=1,2}(\alpha(\sigma_i)-\alpha(\tau),x_i)$. Then $M\in\mathbf{Z}$ from our assumption. We prove $M>0$. It follows from (i) that $x_1+x_2\in X^C(\tau)_{\mathbf{R}}\cap X$, hence $x_1+x_2\in X^C(\tau)$ by Lemma 3.2. Since $x_i\in C(0,-c+\sigma_i)$, there exists $r_{i,\lambda}>0$ and $z_{i,\lambda}\in(-c+\sigma_i)\cap X$ such that $x_i=\sum_\lambda r_{i,\lambda}z_{i,\lambda}$. For each λ,

$$(\alpha(\sigma_i),z_{i\lambda})\ge(z_{i\lambda},z_{i\lambda})/2\ge(\alpha(\sigma_i),z_{i\lambda})$$

by [N99, 1.3]. Hence $(\alpha(\sigma_i),x_i)\ge(\alpha(\tau),x_i)$ where equality holds iff any $z_{i\lambda}\in\tau$. Since x_i is a generator of $X/X^C(\tau)$, there is at least one $z_{i\lambda}$ such that $z_{i\lambda}\notin\tau$. Hence $M>0$ and $\zeta_{x_1,c}\zeta_{x_2,c}=q^M\zeta_{x_1+x_2,c}=0$ in $O_{P_0,O(\tau)}$.

For any $w_i\in C(0,-c+\sigma_i)\cap X$ with $w_i\notin C(0,-c+\tau)$, there are a positive integer n_i and $y_i\in X^C(\tau)$ such that $w_i=n_ix_i+y_i$, hence $\zeta_{w_i,c}=\zeta^{n_i}_{x_i,c}\zeta_{y_i,c}\in O_{P_0,O(\tau)}$. Thus $\zeta_{x_i,c}=0$ (resp. $\zeta_{x_1,c}\zeta_{x_2,c}=0$) is a defning equation of $O(\tau)$ in $Z(\sigma_i)$ (resp. a defining equation of P_0). □

Definition 3.4. Let $\mathrm{Sing}\,(P_0)$ be the singular locus of P_0. Let $\Omega^1_{P_0}$ be the sheaf of germs of regular one-forms over P_0, and $\Theta_{P_0}:=\mathcal{H}om_{O_{P_0}}(\Omega^1_{P_0},O_{P_0})=\mathcal{D}er(O_{P_0})$. Then we define $\widetilde{\Omega}_{P_0}$ to be the sheaf of germs of rational one forms ϕ over P_0 such that

1. ϕ is regular outside $\mathrm{Sing}\,(P_0)$, and it has log poles along the codimension-one singularities (We say ϕ has log poles on P_0 for simplicity),
2. the sum of the residues of ϕ along any of Weil divisors of $\mathrm{Sing}\,(P_0)$ is equal to zero. (These conditions makes sense by Lemma 3.3.)

By [Rim72, p. 112] the tangent space of automorphism group $\mathrm{Aut}(P_0)$ is given by $H^0(P_0,\Theta_{P_0})$. We define $\Theta^\dagger_{P_0}$ and $\Omega^\dagger_{P_0}$ by

$$\Theta^\dagger_{P_0}:=\mathcal{H}om_{O_{P_0}}(\widetilde{\Omega}_{P_0},O_{P_0}),\quad \Omega^\dagger_{P_0}:=\mathcal{H}om_{O_{P_0}}(\Theta^\dagger_{P_0},O_{P_0}).$$

Lemma 3.5. *Let P_0 be a $k(0)$-TSQAS of dimension g, A_0 the abelian part of P_0, T_0 the torus part of P_0 and $X=\mathrm{Hom}(T_0,\mathbf{G}_{m,k(0)})$ the lattice of rank g'. Then*

1. $\Theta^\dagger_{P_0}\simeq O^{\oplus g}_{P_0}$, $\Omega^\dagger_{P_0}\simeq O^{\oplus g}_{P_0}$, *in particular if P_0 is totally degenerate, then* $\Theta^\dagger_{P_0}\simeq X\otimes_{\mathbf{Z}}O_{P_0}$, $\Omega^\dagger_{P_0}\simeq X^\vee\otimes_{\mathbf{Z}}O_{P_0}$,

2. $H^0(P_0, \Theta^{\dagger}_{P_0}) \simeq H^0(A_0, \Theta_{A_0}) \oplus X \otimes_{\mathbf{Z}} k(0)$, *which is the tangent space of the action of* $O(\sigma)$ *for any* $\sigma \in \mathrm{Del}^{(g')}(P_0)$.

Proof. Let $k = k(0)$. First we consider the case where P_0 is totally degenerate, $g = g'$. There is an exact sequence $0 \to \Omega^1_{P_0} \to \widetilde{\Omega}_{P_0} \to \mathcal{A} \to 0$ for some sheaf $\mathcal{A}$ with $\mathrm{Supp}\,(\mathcal{A})$ one-codimensional. The sheaf O_{P_0} is torsion free because P_0 is reduced and Cohen-Macaulay by [AN99]. Hence $\mathrm{Hom}(\mathcal{A}, O_{P_0}) = 0$. Hence $\Theta^{\dagger}_{P_0}$ is a subsheaf of Θ_{P_0}. Let $\theta \in H^0(P_0, \Theta^{\dagger}_{P_0})$. Then $\theta \in H^0(P_0, \Theta_{P_0})$, which is a global infinitesimal automorphism of P_0.

Let $Z(\sigma)$ be the closure of $O(\sigma)$ in P_0 with reduced structure. Since each $Z(\sigma)$ ($\sigma \in \mathrm{Del}^{(g)}(P_0)$) contains the torus $O(\sigma) \simeq \mathbf{G}^{\oplus g}_{m,k} = \mathrm{Spec}\ k[\zeta^{\pm 1}_{e_\lambda,\sigma}]$, the restriction of θ to $O(\sigma)$ is of the form

$$\sum_{\lambda} a_{e_\lambda,\sigma} \zeta_{e_\lambda,\sigma} \frac{\partial}{\partial \zeta_{e_\lambda,\sigma}}$$

for some $a_{e_\lambda,\sigma} \in \Gamma(O(\sigma), O_{P_0})$, where e_λ is a basis of X.

We shall prove that the restriction to $O(\tau)$ $(a_{e_\lambda,\sigma})_{|O(\tau)}$ of $a_{e_\lambda,\sigma}$ is independent of $\sigma \in \mathrm{Del}^{(g)}$. To prove this, it suffices to prove $(a_{e_\lambda,\sigma_1})_{|O(\tau)} = (a_{e_\lambda,\sigma_2})_{|O(\tau)}$. For any element $\omega \in \widetilde{\Omega}_{P_0}$, and any pair $\sigma_1, \sigma_2 \in \mathrm{Del}^{(g)}$ with $\tau = \sigma_1 \cap \sigma_2 \in \mathrm{Del}^{(g-1)}$, we have $\mathrm{Res}_{Z(\tau)}(\omega_{|Z(\sigma_1)}) + \mathrm{Res}_{Z(\tau)}(\omega_{|Z(\sigma_2)}) = 0$. Since $\theta \in \Theta^{\dagger}_{P_0}$, we have

$$\theta_{|Z(\sigma_1)}(\omega_{|Z(\sigma_1)}) = \theta_{|Z(\sigma_2)}(\omega_{|Z(\sigma_2)}).$$

By Lemma 3.3 (2), we may assume x_j, e_λ $(2 \leq \lambda \leq g)$ is a basis of $X = X^C(\sigma_j)$, while e_λ $(2 \leq \lambda \leq g)$ is a basis of $X^C(\tau)$, where we may further assume $e_1 = x_1 = -x_2$. Hence $d\zeta_{e_\lambda,\sigma}/\zeta_{e_\lambda,\sigma} \in \widetilde{\Omega}_{P_0}$ for $2 \leq \lambda \leq g$. Hence we have $(a_{e_\lambda,\sigma_1})_{|O(\tau)} = (a_{e_\lambda,\sigma_2})_{|O(\tau)}$ for $2 \leq \lambda \leq g$. By (3.4.2), we choose $\omega := d\zeta_{x_1,\sigma_1}/\zeta_{x_1,\sigma_1} = -d\zeta_{x_2,\sigma_2}/\zeta_{x_2,\sigma_2} \in \widetilde{\Omega}_{P_0}$. Then we introduce a coordinate on $Z(\sigma_2)$ as $\zeta_{e_1,\sigma_2} := \zeta^{-1}_{x_2,\sigma_2}$ to infer

$$\omega = d\zeta_{e_1,\sigma_1}/\zeta_{e_1,\sigma_1} = d\zeta_{e_1,\sigma_2}/\zeta_{e_1,\sigma_2},$$

whence $(a_{e_1,\sigma_1})_{|O(\tau)} = (a_{e_1,\sigma_2})_{|O(\tau)}$, hence $(a_{e_\lambda,\sigma})_{|O(\tau)}$ is independent of σ.

Let Z be the union of all $O(\rho)$ $(\forall \rho \in \mathrm{Del}^{(k)}, \forall k \leq g-2)$. Then the above proves $\Theta^{\dagger}_{P_0 \setminus Z} \simeq X \otimes O_{P_0 \setminus Z}$. This implies that $\Theta^{\dagger}_{P_0} \simeq X \otimes O_{P_0}$. In fact, let $j : P_0 \setminus Z \subset P_0$ be the inlcusion, $\phi \in \Theta^{\dagger}_{P_0 \setminus Z} = \mathcal{H}om(\widetilde{\Omega}_{P_0 \setminus Z}, O_{P_0 \setminus Z})$ and $\omega \in \widetilde{\Omega}_{P_0}$. Then $\phi(\omega_{|P_0 \setminus Z}) \in O_{P_0 \setminus Z} \simeq j_*(O_{P_0 \setminus Z}) = O_{P_0}$ because P_0 is reduced, Cohen-Macaulay (depth g) and $\mathrm{codim}_{P_0}(Z) \geq 2$. Hence $\phi(\omega_{|P_0 \setminus Z})$ extends regularly to P_0, so that $\phi(\widetilde{\Omega}_{P_0}) \in O_{P_0}$, that is, $\phi \in \Theta^{\dagger}_{P_0}$. Since the extension of ϕ to P_0 is unique by $j_*(O_{P_0 \setminus Z}) = O_{P_0}$, we see

$$\Theta^{\dagger}_{P_0} \simeq j_*(\Theta^{\dagger}_{P_0 \setminus Z}) \simeq j_*(X \otimes O_{P_0 \setminus Z}) = X \otimes O_{P_0}.$$

This proves (1) in the totally degenerate case.

Next we consider the general case $g = g' + g''$, $g'' > 0$. See [N99, p. 678]. Let e_λ be a basis of X, $\sigma \in \mathrm{Del}^{(g')}$, and $O(\sigma)$ is a $T_0(\sigma)$-bundle over A_0, where $T_0(\sigma) = \mathrm{Spec}\ k[\zeta_{e_\lambda,\sigma}^{\pm 1}] \simeq \mathbf{G}_m^{g'}$. Let $\theta \in H^0(P_0, \Theta_{P_0}^\dagger)$. Then there exists a closed subscheme Z of P_0 of codimension two such that the restriction of θ to $O(\sigma)$ is of the form

$$\theta' + \sum_\lambda a_{e_\lambda,\sigma}\zeta_{e_\lambda,\sigma}\frac{\partial}{\partial\zeta_{e_\lambda,\sigma}},$$

where $\theta' \in H^0(\Theta_{A_0}) \otimes_k H^0(P_0 \setminus Z, O_{P_0})$, $\zeta_{\sigma,e_\lambda}\frac{\partial}{\partial\zeta_{e_\lambda,\sigma}}$ is a global log one form on P_0, hence $a_{e_\lambda,\sigma} \in H^0(P_0 \setminus Z, O_{P_0})$. Since P_0 is reduced Cohen-Macaulay, $H^0(P_0 \setminus Z, O_{P_0}) = H^0(P_0, O_{P_0}) = k$, hence we have (1) and (2). □

Lemma 3.6. *Let $\mathcal{L}$ be a line bundle on a T-scheme Z (viewed as a Z-scheme). Then $\mathrm{Aut}_T(\mathcal{L}/Z)$ is a group T-scheme over $\mathrm{Aut}_T(Z)$.*

Proof. Let $\mathbf{P}$ be a $\mathbf{P}^1$-bundle $\mathbf{P}(O_Z \oplus \mathcal{L})$ which compactifies $\mathcal{L}$ along infinity by $Z^\infty := \mathbf{P}(0 \oplus \mathcal{L}) \simeq Z$, $\pi : \mathcal{L} \to Z$ the projection. Let 0 be the zero section of $\mathcal{L}$, $\infty = Z^\infty$ the infinity section of $\mathbf{P}$. We recall $\mathcal{A}ut_T(\mathcal{L}/Z)$ is the functor from T-schemes to sets

$$\begin{aligned} U &\mapsto \mathcal{A}ut_T(\mathcal{L}/Z)(U)\\ &:= \left\{(g,\phi);\ \begin{array}{l} g \in \mathrm{Aut}_T(Z)(U) \text{ and } \phi(0) = 0\\ \phi : \mathcal{L}_U \simeq g^*(\mathcal{L}_U) \text{ fiberwise linear } Z_U\text{-isom.}\end{array}\right\}\\ &= \left\{(g,\phi);\ \begin{array}{c} g \in \mathrm{Aut}_T(Z)(U) \text{ and } \phi(0) = 0\\ \phi \in \mathrm{Aut}_T(\mathcal{L})(U)\ U\text{-isom. s.t. } \pi\phi = g\pi\\ \phi : \text{fiberwise linear over } Z_U\end{array}\right\}\end{aligned}$$

where the product $(g,\phi_1)\cdot(h,\phi_2)$ is defined by $(gh, h^*\phi_1 \circ \phi_2)$. See Definition 2.3. Since any automorphism of $\mathbf{P}^1$ which fixes 0 and ∞ is linear,

$$\mathcal{A}ut_T(\mathcal{L}/Z)(U) = \left\{(g,\psi);\ \begin{array}{l} g \in \mathrm{Aut}_T(Z)(U),\ \psi(0) = 0, \psi(\infty) = \infty\\ \psi \in \mathrm{Aut}_T(\mathbf{P})(U) \text{ s.t. } \pi\psi = g\pi\end{array}\right\}.$$

It follows that $\mathcal{A}ut_T(\mathcal{L}/Z)$ is representable by the closed subgroup T-scheme (denoted $\mathrm{Aut}_T(\mathcal{L}/Z)$) of $\mathrm{Aut}_T(Z) \times \mathrm{Aut}_T(\mathbf{P})$:

$$\mathrm{Aut}_T(\mathcal{L}/Z) = \{(g,\psi); \psi(0) = 0, \psi(\infty) = \infty, \pi\psi = g\pi\}.$$

This proves Corollary. □

Theorem 3.7. *Let S be a scheme, $(P \xrightarrow{\pi} S, \mathcal{L})$ an S-TSQAS. Let $\widetilde{\Omega}_{P/S}$ be the sheaf of germs over P of relative rational one forms with log poles (Definition 3.4), the sum of whose residues along any of one-codimensional singular loci of the fibers is equal to zero, $\Theta_{P/S}^\dagger$ the O_P-dual of $\widetilde{\Omega}_{P/S}$ and $\Omega_{P/S}^\dagger$ the O_P-dual of $\Theta_{P/S}^\dagger$. We define $\mathrm{Aut}_S^\dagger(P)$ to be the maximal closed subgroup S-scheme of $\mathrm{Aut}_S(P)$ which keep $\Omega_{P/S}^\dagger$ stable, and $\mathrm{Aut}_S^\dagger(P)^0$ (resp. $\mathrm{Aut}_S^{\dagger 0}(P)$) the identity component (resp. the fiberwise identity component, that is, the minimal open subgroup S-schme) of $\mathrm{Aut}_S^\dagger(P)$. Then*

1. $\mathrm{Aut}^{\dagger}_S(P)$ *is flat over* S*, and the fiber* $(\mathrm{Aut}^{\dagger}_S(P))_s$ *has the tangent space* $H^0(P_s, \Theta^{\dagger}_{P_s})$ *for any geometric point* s *of* S*,*
2. $\mathrm{Aut}^{\dagger 0}_S(P)$ *is a semi-abelian group scheme over* S*, flat over* S*, while* $\mathrm{Aut}^{\dagger}_S(P)^0$ *is a semi-abelian group scheme over* S*, flat over* S*, possibly with reducible geometric fibers.*

Proof. Let $s := \mathrm{Spec}\, k(s)$ be any geometric point of S. From its definition $\mathrm{Aut}^{\dagger}_S(P)$ is a closed subscheme of $\mathrm{Aut}_S(P)$, while $\mathrm{Aut}^{\dagger}_S(P)^0$ hence $\mathrm{Aut}^{\dagger 0}_S(P)$ is a closed subscheme of $\mathrm{Aut}_S(P)$ of finite type. Since $\mathrm{Aut}_S(P)$ commutes with base change (because $\mathrm{Aut}_S(P)$ represents the relative Aut functor), $\mathrm{Aut}_S(P)_s = \mathrm{Aut}_{k(s)}(P_s)$. Hence $(\mathrm{Aut}^{\dagger}_S(P))_s = \mathrm{Aut}^{\dagger}_{k(s)}(P_s)$ because $(\Omega^{\dagger}_{P/S})_s \simeq \Omega^{\dagger}_{P_s/k(s)}$. It follows that $(\mathrm{Aut}^{\dagger 0}_S(P)) \otimes k(s) = \mathrm{Aut}^{\dagger 0}_{k(s)}(P_s)$. The tangent space of $(\mathrm{Aut}^{\dagger}_S(P))_s$ equals $H^0(P_s, \Theta^{\dagger}_{P_s})$ by Lemma 3.5. $\pi_*\Theta^{\dagger}_{P/S}$ is a finite free O_P-module of rank g by Lemma 3.5. Hence $(\pi_*\Theta^{\dagger}_{P/S})_s \simeq H^0(\Theta^{\dagger}_{P_s/k(s)})$, hence $(\pi_*\Omega^{\dagger}_{P/S})_s \simeq H^0(\Omega^{\dagger}_{P_s/k(s)})$. Hence $(\mathrm{Aut}^{\dagger}_S(P))_s$ is smooth of dimension g, hence $\mathrm{Aut}^{\dagger}_S(P)$ is S_{red}-flat, hence S-flat because flatness is an open condition. This proves (1).

Since $\mathrm{Aut}^{\dagger}_S(P)$ is S-flat by (1), so are $\mathrm{Aut}^{\dagger 0}_S(P)$ and $\mathrm{Aut}^{\dagger}_S(P)^0$. In view of Lemma 3.5, $(\mathrm{Aut}^{\dagger 0}_S(P))_s = \mathrm{Aut}^{\dagger 0}_{k(s)}(P_s)$ coincides with the action of a semi-abelian scheme $O(\sigma)$ on P_s [N99, 4.12, p .680]. Hence $\mathrm{Aut}^{\dagger 0}_S(P)$ is a semi-abelian scheme over S, which proves (2). □

4. The closed immersions of $SQ^{\mathrm{toric}}_{g,K}$ into $\overline{AP}_{g,N}$

In this section we prove that there is a natural family of closed immersions of $SQ^{\mathrm{toric}}_{g,K}$ into $\overline{AP}_{g,N}$ parametrized by an open subset of $\mathbf{P}(V_H)$.

Definition 4.1. Let $H = H(e) := \oplus_{i=1}^{g}(\mathbf{Z}/e_i\mathbf{Z})$ $(e_i|e_{i+1})$ and let $K = H \oplus H^{\vee}$ be an abelian group with the symplectic form e_K in Section 2. $\mathrm{Aut}(K, e_K)$ is the group of automorphisms of K *keeping the symplectic form* e_K *invariant.* We call $g \in \mathrm{Aut}(K, e_K)$ a symplectic automorphism of K. Let $\overline{\mathrm{Aut}}(K, e_K) := \mathrm{Aut}(K, e_K)/\pm \mathrm{id}_K$.

Definition 4.2. We define $\mathrm{Aut}_c(\mathcal{G}_H)$ to be the group consisting of all automorphisms of $\mathcal{G}_H$ *which fix the center of* $\mathcal{G}_H$ *elementwise.*

Lemma 4.3. *Let* $\pi : \mathrm{Aut}_c(\mathcal{G}_H) \to \mathrm{Aut}(K, e_K)$ *be the natural homomorphism. Then the following are true :*

1. *there is an exact sequence over* $\mathcal{O}_{N^3}$
$$0 \to \ker(\pi) \to \mathrm{Aut}_c(\mathcal{G}_H) \xrightarrow{\pi} \mathrm{Aut}(K, e_K) \to 1,$$
2. $\ker(\pi) \simeq K^{\vee} = \mathrm{Hom}(K, \mathbf{G}_m)$*. This isomorphism is given explicitly as follows: for* $\gamma \in K^{\vee}$*, there exists* $t \in K$ *such that* $\gamma(s) = e_K(t, s)$ $(\forall s \in K)$*. Let* $\xi(\gamma)(g) := \omega(t)g\omega(t)^{-1}$*. Then* $\xi(\gamma) \in \ker(\pi)$ *and* $\xi(\gamma)(g) = [\omega(t), g]g$*,* $\xi(\gamma)(\omega(u)) = e_K(t, u)g$*. Moreover* $\xi(\gamma)\xi(\gamma') = \xi(\gamma + \gamma')$*.*

Proof. Since e_K is the commutator form of $\mathcal{G}_H$ with values in the center, it is invariant by $\mathrm{Aut}_c(\mathcal{G}_H)$. Hence any $\xi \in \mathrm{Aut}_c(\mathcal{G}_H)$ induces a symplectic automorphism $\pi(\xi)$ of K, which defines the natural homomorphism $\pi : \mathrm{Aut}_c(\mathcal{G}_H) \to \mathrm{Aut}(K, e_K)$. It is easy to see $\ker(\pi) \simeq K^\vee \simeq K$.

We shall prove that π is surjective. For $\eta \in \mathrm{Aut}(K, e_K)$, we construct $\xi \in \mathrm{Aut}_c(\mathcal{G}_H)$ with $\pi(\xi) = \eta$ over $\mathcal{O}_{N^3}$. Let $s, t \in K$, $\omega(s) := (1, s) \in 1 \oplus K \subset \mathcal{G}_H$, and $\phi(s,t) := \omega(s+t)\omega(s)^{-1}\omega(t)^{-1}$ and $f(s,t) := \phi(\eta(s), \eta(t))/\phi(s,t)$. Then $\phi \in C^2(K, \mu_N)$, $f \in C^2(K, \mu_N)$ and $e_K(s,t) = \phi(s,t)/\phi(t,s)$ by [M12, p. 206, (d)]. Then ϕ and f belong to $H^2(K, \mu_N)$. Since $\eta \in \mathrm{Aut}(K, e_K)$, we have $e_K(s,t) = e_K(\eta(s), \eta(t))$, hence $f(s,t) = f(t,s)$.

Then we shall prove $f = 0$ in $H^2(K, \mu_{N^3})$. Now we choose a symplectic basis e_i, f_i of K such that $e_K(e_i, f_i) = \zeta_{\delta_i}$, $e_K(e_i, f_j) = 1$ $(i \neq j)$, $e_K(e_i, e_j) = e_K(f_i, f_j) = 1$ $(\forall i, j)$, where e_i and f_i are of order δ_i, $\sqrt{|K|} = N = \prod_{i=1}^g \delta_i$.

Then by the argument of [N99, 7.4, p.690], we can prove by the induction on the number of generators of K that there exists $\chi \in C^1(K, \mu_{N^3})$ such that $f = \delta(\chi)$, that is, $f(s,t) = \chi(s+t)\chi(s)^{-1}\chi(t)^{-1}$. In fact, in the proof of [*ibid.*] each time when the number of (symplectic) generators increases, we need to multiply the denominator of the cochain χ by the order (say δ_i) of the new generator, hence need to multiply the denominator of χ by $N^2 = (\prod_{i=1}^g \delta_i)^2$ in total to define χ, hence $\chi \in C^1(K, \mu_{N^3})$.

By using χ we define $\xi(a\omega(s)) = a\chi(s)\omega(\eta(s))$ $(a \in \mathbf{G}_m, s \in K)$. It follows from $\eta \in \mathrm{Aut}(K)$ that $\xi \in \mathrm{Aut}_c(\mathcal{G}_H \otimes \mathcal{O}_{N^3})$. The rest is easy. □

4.4. **The action of $\mathrm{Aut}_c(\mathcal{G}_H)$ on $SQ_{g,K}^{\mathrm{toric}}$.** Let $\xi \in \mathrm{Aut}_c(\mathcal{G}_H)$. Since $U_H \circ \xi$ is a representation of $\mathcal{G}_H$ of weight one over O_N, it is equivalent to U_H over $\mathcal{O}_N$ by [N10, p. 88]. It follows that there is $A(\xi) \in \mathrm{GL}(V_H)$, unique up to a constant multiple, such that

$$(U_H \circ \xi)A(\xi) = A(\xi)U_H, \quad \text{equivalently,} \tag{2}$$

$$U_H(\xi(a, z, \alpha))w(\beta) = a\beta(z)w(\alpha + \beta), \tag{3}$$

where $w(\beta) := A(\xi)v_H(\beta) =: \sum_\gamma a_{\beta,\gamma}(\xi)v_H(\gamma) \in V_H$. It is clear that $A(\xi\xi') = A(\xi)A(\xi')$ in $\mathrm{PGL}(V_H)$.

Let $p(\xi)$ be the automorphism of $\mathbf{P}(V_H)$ such that $p(\xi)^* = A(\xi)$. Let $\sigma := (P_0, \mathcal{L}_0, \phi, \tau)$ be any rigid-$\mathcal{G}_H$ T-TSQAS, $\phi(\xi) := p(\xi) \circ \phi$, and $\tau(\xi) := \tau \circ \xi$. Then $\sigma(\xi) := (P_0, \mathcal{L}_0, \phi(\xi), \tau(\xi))$ is a rigid-$\mathcal{G}_H$ T-TSQAS.

Lemma 4.5. *Let k be an algebraically closed field over $\mathcal{O}_N$, $\xi \in \mathrm{Aut}_c(\mathcal{G}_H)$ and $\sigma := (P_0, \mathcal{L}_0, \phi, \tau) \in SQ_{g,K}^{\mathrm{toric}}(k)$. Then the following are true :*

1. *for $\gamma \in K^\vee$, $\tau(h) : \sigma \to \sigma(\xi(\gamma))$ is an isomorphism for some $h \in \omega(K)$,*
2. *$\sigma \simeq \sigma(\xi(-\mathrm{id}_K))$, (see the proof below for $\xi(-\mathrm{id}_K)$)*
3. *Suppose $\sigma \in SQ_{g,K}^{\mathrm{toric}}(k)$ is generic. Then $\sigma \simeq \sigma(\xi)$ if and only if $\xi = \xi(\gamma)$ or $\xi = \xi(\gamma) \cdot \xi(-\mathrm{id}_K)$ for some $\gamma \in K^\vee$.*

Proof. First we shall prove (1). Let $\omega(s) = (1, s)$ for $s \in K$. For $\gamma \in K^\vee$, then there exists a unique $t \in K$ such that $\gamma(s) = e_K(t, s) = [\omega(t), \omega(s)]$.

Let $h = \omega(t)$. We define $\xi(\gamma) \in \mathrm{Aut}_c(\mathcal{G}_H)$ by $\xi(\gamma)(g) := hgh^{-1} = [\omega(t), g]g$ where $[\omega(t), g] \in \mathbf{G}_m$. Hence

$$U_H(\xi(\gamma)(g))U_H(h) = U_H(h)U_H(g),$$

hence we can identify $A(\xi(\gamma)) = U_H(h)$. In view of Definition 2.3, $U_H(h)$ on V_H induces the translation $T_{h^{-1}}$ of P_0. It follows that $\phi(\xi(\gamma))^* = \phi^*(p(\xi(\gamma)))^* = \phi^* U_H(h) = T^*_{h^{-1}}\phi_h\phi^*$, hence $\phi = \phi(\xi(\gamma)) \cdot T_h$ because both ϕ and $\phi(\xi(\gamma))$ are the maps from P_0 to $\mathbf{P}(V_H)$ so that we can ignore the unit ϕ_h. It is clear that $\tau(\xi(\gamma)(g))\tau(h) = \tau(h)\tau(g)$. It follows that the map $\tau(h) : (P_0, \mathcal{L}_0) \to (P_0, \mathcal{L}_0)$ induces a $\mathcal{G}_H$-isomorphism

$$\sigma = (P_0, \mathcal{L}_0, \phi, \tau) \simeq \sigma(\xi(\gamma)) = (P_0, \mathcal{L}_0, \phi(\xi(\gamma)), \tau(\xi(\gamma))).$$

Next we shall prove (2). Any k-TSQAS $(P_0, \mathcal{L}_0)$ has an automorphism inv_{P_0} which is induced from the algebra endomorphism of $\widetilde{R}$ [N99, p. 670] $\mathrm{inv}_{\widetilde{R}} : a(x)w^x\vartheta \mapsto a(x)w^{-x}\vartheta$, or in other words, induced from $(-\mathrm{id}_Z)$ of an abelian variety $Z := P_\eta$, the generic fibre of P in Definition 2.4 (by choosing an even B, $r = 0$ in Subsec. 3.1 by some base change). Note that $-\mathrm{id}_K \in \mathrm{Aut}(K, e_K)$ lifts to an automorphism $\mathrm{inv}_{\mathcal{G}_H}$ as $\mathrm{inv}_{\mathcal{G}_H}(a, z, \alpha) = (a, -z, -\alpha)$. We denote $\mathrm{inv}_{\mathcal{G}_H}$ by $\xi(-\mathrm{id}_K)$. The automorphism inv_{P_0} gives an isomorphism $(P_0, \phi, \tau) \simeq (P_0, \phi(\xi(-\mathrm{id}_K)), \tau(\xi(-\mathrm{id}_K)))$. This proves (2).

Finally we shall prove (3). If $\sigma \simeq \sigma(\xi)$, then there exists an isomorphism $(f, \delta) : (P_0, \mathcal{L}_0) \simeq (P_0, \mathcal{L}_0)$ such that $(f, \delta) \cdot \tau(g) = \tau(\xi(g)) \cdot (f, \delta)$ for any g. It follows that $f(T_g(x)) = T_{\xi(g)}f(x)$ and $\delta(T_g(x))\phi_g(x) = \phi_{\xi(g)}(f(x))\delta(x)$. Since σ is a general abelian variety over k, $f \in \mathrm{Aut}(P_0)$ is a translation T_h, or the composite of a translation T_h and inv_{P_0} for $h = \omega(t)$ and $t \in K$. If $f = T_h$, then $(f, \delta) = (T_h, \phi_h) = \tau(h)$. This case is reduced to (1). If $f = T_h \cdot (\mathrm{inv}_{P_0})$, then $g := f \cdot (\mathrm{inv}_{P_0})$ is reduced to (1). This completes the proof. □

Corollary 4.6. *The action of* $\mathrm{Aut}_c(\mathcal{G}_H)$ *on* $SQ^{\mathrm{toric}}_{g,K}$ *reduces to* $\overline{\mathrm{Aut}}(K, e_K)$.

Proof. The map $s(\xi) : SQ^{\mathrm{toric}}_{g,K} \to SQ^{\mathrm{toric}}_{g,K}$ sending σ to $\sigma(\xi^{-1})$ is an automorphism of $SQ^{\mathrm{toric}}_{g,K}$. This defines an action of $\mathrm{Aut}_c(\mathcal{G}_H)$ on $SQ^{\mathrm{toric}}_{g,K}$, that is, $s(\xi\xi') = s(\xi)s(\xi')$. By Lemma 4.5 (1), $s(\xi(\gamma))$ $(\gamma \in K^\vee)$ acts on $SQ^{\mathrm{toric}}_{g,K}$ trivially. by Lemma 4.3, the action of $\mathrm{Aut}_c(\mathcal{G}_H)$ reduces to $\overline{\mathrm{Aut}}(K, e_K)$. □

Definition 4.7. Let $\xi \in \mathrm{Aut}_c(\mathcal{G}_H)$, and $G(\xi)$ be the subset of $\mathbf{P}(V_H)$ consisting of all eigenvectors of $A(\xi) \neq \mathrm{id}$. Let $G_{g,K}$ be the union of all $G(\xi)$ for $\xi \in \mathrm{Aut}_c(\mathcal{G}_H)$. $G_{g,K}$ is at most $(N-2)$-dimensional. See Subsec. 5.4.

Lemma 4.8. *Let k be an algebraically closed field over $\mathcal{O}_N$, and $(P_0, \mathcal{L}_0, \phi, \tau)$ be a rigid-$\mathcal{G}_H$ k-TSQAS, and $(P_0, \mathcal{L}_0, \psi, \sigma)$ be another rigid-$\mathcal{G}_H$ k-TSQAS. Then there exists $\xi \in \mathrm{Aut}_c(\mathcal{G}_H)$ such that*

$$(P_0, \mathcal{L}_0, \psi, \sigma) \simeq (P_0, \mathcal{L}_0, \phi(\xi), \tau(\xi)).$$

Proof. We *choose and fix* a rigid-$\mathcal{G}_H$ TSQAS (P_0, ϕ, τ) and take another rigid-$\mathcal{G}_H$ TSQAS (P_0, ψ, σ) above $(P_0, \mathcal{L}_0)$. Let $\Phi := \{(T_g, \phi_g)\}_{g \in \mathcal{G}_H}$ (resp.

$\Psi := \{(S_g, \psi_g)\}_{g\in\mathcal{G}_H})$ be a $\mathcal{G}_H$-linearization of $\mathcal{L}_0$ such that $\tau = \tau_\Phi$, $\sigma = \tau_\Psi$. Let $\tau^{ab}(g) = T_g$ and $\sigma^{ab}(g) = S_g$. By Definition 2.4 and by [N10, 2.19] $\tau^{ab}(\mathcal{G}_H) = \sigma^{ab}(\mathcal{G}_H) = K(P_0, \mathcal{L}_0)$. Hence via the isomorphisms $\tau^{ab}(\mathcal{G}_H) \simeq K$ and $= \sigma^{ab}(\mathcal{G}_H) \simeq K$ the identity of $K(P_0, \mathcal{L}_0)$ induces an isomorphism $\eta \in \mathrm{Aut}(K)$ such that $\eta(T_g) = S_g$ for $\forall g \in \mathcal{G}_H$, which keeps e_K invariant because $e_K(S_g, S_h) = [g, h] = e_K(T_g, T_h) \in k$. Hence $\eta \in \mathrm{Aut}(K, e_K)$. By Lemma 4.3 η is lifted to $\xi(\eta) \in \mathrm{Aut}_c(\mathcal{G}_H)$ with $S_g = \eta(T_g) = T_{\xi(\eta)(g)}$.

It follows $\gamma(g) := \psi_g \cdot \phi^{-1}_{\xi(\eta)(g)} \in \mathrm{Aut}_{P_0}(\mathcal{L}_0) = \mathrm{Hom}_{O_{P_0}}(\mathcal{L}_0, \mathcal{L}_0)^\times = k^\times$. Then γ is a character of $\mathcal{G}_H$ because

$$\begin{aligned}\gamma(gh) &= \psi_{gh} \cdot \phi^{-1}_{\xi(\eta)(g)\xi(\eta)(h)} = (S_h^*\psi_g \cdot \psi_h)(T^*_{\xi(\eta)(h)}\phi_{\xi(\eta)(g)} \cdot \phi_{\xi(\eta)(h)})^{-1}\\ &= (S_h^*\psi_g \cdot \psi_h)(S_h^*\phi_{\xi(\eta)(g)} \cdot \phi_{\xi(\eta)(h)})^{-1} = S_h^*\gamma(g)\gamma(h) = \gamma(g)\gamma(h).\end{aligned}$$

Let $\xi(g) := \gamma(g)\xi(\eta)(g) \in \mathcal{G}_H$. Then $\xi \in \mathrm{Aut}_c(\mathcal{G}_H)$. Hence

$$\begin{aligned}\phi_{\xi(g)} &= \phi_{\xi(\eta)(g)\gamma(g)} = T^*_{\gamma(g)}\phi_{\xi(\eta)(g)}\phi_{\gamma(g)} = \phi_{\xi(\eta)(g)}\gamma(g) = \psi_g,\\ \tau(\xi(g)) &= (T_{\xi(g)}, \phi_{\xi(g)}) = (T_{\xi(\eta)(g)}, \psi_g) = (S_g, \psi_g) = \tau_\Psi(g) = \sigma(g).\end{aligned}$$

Hence $\sigma = \tau\xi$. Let $A := (\phi^*)^{-1}(\psi^*) \in \mathrm{GL}(V_H \otimes k)$. Then

$$\begin{aligned}U_H(g) &= \rho(\psi, \sigma)(g) = (\psi^*)^{-1}S^*_{g^{-1}}\psi_g\psi^* = (\psi^*)^{-1}T^*_{\xi(g)^{-1}}\phi_{\xi(g)}\psi^*\\ &= A^{-1}\rho(\phi, \tau)(\xi(g))A = A^{-1}U_H(\xi(g))A\end{aligned}$$

by Definition 2.4 (3). We can identify $A = A(\xi)$ so that $\psi = p(\xi)\phi$, $\sigma = \tau\xi$, hence $(P_0, \psi, \sigma) = (P_0, \phi(\xi), \tau(\xi))$. □

Lemma 4.9. *Let k be a local ring with $N = |H|$ invertible, R a local k-algebra, I an ideal of R with $I^2 = 0$ such that $k = R/I$. Let $\sigma_0 = (P_0, \mathcal{L}_0, \phi_0, \tau_0)$ be a rigid-$\mathcal{G}_H$ k-TSQAS, and $\sigma := (P, \mathcal{L}, \phi, \tau)$ a rigid-$\mathcal{G}_H$ R-TSQAS such that $\sigma \otimes_R (R/I) \simeq \sigma_0$, If $(P, \mathcal{L})$ is the pull back of $(P_0, \mathcal{L}_0)$ to R, then σ is the pull back of σ_0 to R.*

Proof. By the assumption, $(P, \mathcal{L}) \simeq \mathrm{Spec}\ R \times_k (P_0, \mathcal{L}_0)$, and R is a k-algebra with $R = k \oplus I$, and $H^0(P, \mathcal{L}) \simeq H^0(P_0, \mathcal{L}_0) \otimes_k R$ is an R-isomorphism with $\mathcal{G}_H$-action. Hence there exists $B \in I \cdot \mathrm{End}\,(V_H \otimes R)$ such that

$$\phi^* = \phi_0^* + \phi_0^* \cdot B,\ \ \phi_0^* : V_H \otimes k \simeq H^0(P_0, \mathcal{L}_0).$$

Moreover τ maps $\mathcal{G}_H$ into $\mathrm{Aut}_R(\mathcal{L}/P) \simeq \mathrm{Spec}\ R \times_k \mathrm{Aut}_k(\mathcal{L}_0/P_0)$. Hence $\tau^{ab}(\mathcal{G}_H) \subset \mathrm{Aut}^\dagger(P) = \mathrm{Spec}\ R \times_k \mathrm{Aut}^\dagger(P_0)$. Let $\tau^{ab} = T^0 + T^1$, $T^0 = \tau^{ab} \otimes k$ and $T^1 = \{T^1_g\} \in C^1(\mathcal{G}_H, IH^0(\Theta^\dagger_{P_0}))$ where $\tau^{ab}(g) := T^0_g + T^1_g$, $T^0_g \in \mathrm{Aut}^\dagger(P_0)$, $T^1_g \in IH^0(\Theta^\dagger_{P_0})$. Let $\epsilon_g = T^0_{g^{-1}}T^1_g$. Since τ^{ab} is a group homomorphism, we have $\epsilon_{gh} = \mathrm{Ad}(T^0_{h^{-1}})\epsilon_g + \epsilon_h$. Thus $\epsilon := \{\epsilon_g\}_{g\in\mathcal{G}_H} \in H^1(\mathcal{G}_H, IH^0(\Theta^\dagger_{P_0}))$. Let $W := H^0(P_0, \Theta^\dagger_{P_0})$. Then $W \simeq k^{\oplus g}$ by Lemma 3.5 and Nakayama's lemma. Since T_h ($h \in \mathcal{G}_H$) acts on P_0 as translation by $K(P_0, \mathcal{L}_0) \simeq K$, T^0_h keeps any $\theta \in W$ invariant. Hence $\epsilon_{gh} = \epsilon_g + \epsilon_h$, and $\epsilon \in \mathrm{Hom}(K, IW) = \mathrm{Hom}(K, I^{\oplus g}) = 0$ because N is invertible in R, hence $\epsilon = 0$, $\tau^{ab} = T^0$.

Let $\phi_g = \phi_g^0 + \phi_g^1$ and $\varepsilon_g := (\phi_g^0)^{-1} \cdot \phi_g^1$. Then $\varepsilon_g \in IH^0(O_{P_0})$. In fact, we can write ϕ_g in down-to-earth terms as follows. Since $(P, \mathcal{L}) \simeq (P_0, \mathcal{L}_0)_R$, we can choose, by [N10, p. 94], a $\mathcal{G}_H$-invariant affine open covering U_j of P and a one-cycle $A_{ij}(x)$ of $\mathcal{L}_0$ such that $\mathcal{L}_0$ is trivial over U_j. Then we obtain $\phi_i^\nu(g,x) = \frac{A_{ij}(gx)}{A_{ij}(x)}\phi_j^\nu(g,x)$ $(\nu = 0, 1)$, where $(\phi_g^\nu)_{|U_i} =: \phi_i^\nu(g,x)$. Hence $\phi_i^0(g,x)^{-1}\phi_i^1(g,x) = \phi_j^0(g,x)^{-1}\phi_j^1(g,x)$. This implies $\varepsilon_g \in IH^0(O_{P_0})$.

Since ϕ_g is a $\mathcal{G}_H$-linearization of $\mathcal{L}$,

$$\phi_{gh}^0 = (T_h^0)^*\phi_g^0 \cdot \phi_h^0,\ \phi_{gh}^1 = (T_h^0)^*\phi_g^0 \cdot \phi_h^1 + (T_h^0)^*\phi_g^1 \cdot \phi_h^0,$$

whence $\varepsilon_{gh} = (T_h^0)^*\varepsilon_g + \varepsilon_h = \varepsilon_g + \varepsilon_h$ because $(T_h^0)^*\varepsilon_g = \varepsilon_g \in IH^0(O_{P_0})$. It follows $\varepsilon := \{\varepsilon_g\} \in \mathrm{Hom}(\mathcal{G}_H, IH^0(O_{P_0})) = \mathrm{Hom}(K, IH^0(O_{P_0})) = 0$ because N is invertible in R and $IH^0(O_{P_0}) = I$ by $H^0(O_{P_0}) = k$. Hence $\varepsilon = 0$, $\phi_g = \phi_g^0$ $(\forall g \in \mathcal{G}_H)$, and $\tau = \tau_0$. Hence we see

$$U_H = \rho(\phi, \tau) = \rho(\phi, \tau_0) = \rho(\phi_0, \tau_0) + [\rho(\phi_0, \tau_0), B] = U_H + [U_H, B],$$

whence $[U_H, B] = 0$. Since U_H is an irreducible representation of $\mathcal{G}_H$, B is a scalar. Hence $\sigma \simeq (\sigma_0)_R$. □

Definition 4.10. Let $(P_0, \mathcal{L}_0)$ be a k-TSQAS with $\mathcal{L}_0$ $\mathcal{G}_H$-linearized. Then a maximal isotropic subgroup H of K is said to be *hereditary* for (P_0, ϕ_0, τ_0) if $\tau_0^{ab}(H) \subset G_0 := \mathrm{Aut}_k^{\dagger 0}(P_0)$. Therefore if P_0 is an abelian variety, then any maximal isotropic subgroup is hereditary. If $(P_0, \mathcal{L}_0)$ is totally degenerate, then a maximal isotropic subgroup H (denoted H_{hd}) of K is hereditary iff $H \subset G_0 := \mathrm{Aut}_k^{\dagger 0}(P_0) = \mathrm{Hom}_k(X, \mathbf{G}_m)$.

Definition 4.11. We freely use the notation of [N99, pp.670-671]. Let $(P_0, \mathcal{L}_0)$ be a totally degenerate k-TSQAS with $\mathcal{L}_0$ $\mathcal{G}_H$-linearized, H_{hd} a hereditary maximal isotropic subgroup of K for $(P_0, \mathcal{L}_0)$ with $H_{\mathrm{hd}}^\vee = X/Y$.

Let $\phi_{\mathrm{hd}} : P_0 \to \mathbf{P}(V_{H_{\mathrm{hd}}})$ be

$$\phi_{\mathrm{hd}}^*(v_{H_{\mathrm{hd}}}(\alpha)) = \theta(\alpha) := \sum_{y \in Y} a(x+y)w^{x+y},$$

where $x \equiv \alpha \in H_{\mathrm{hd}}^\vee = X/Y$. We define

$$\begin{gathered}\tau_{\mathrm{hd}}(a,z,u)(a(x)w^x\vartheta) := a\alpha(z)a(x+u)w^{x+u}\vartheta,\\ \tau_{\mathrm{hd}}(a,z,\alpha) = \tau_{\mathrm{hd}}^R(a,z,u) \bmod Y,\\ \rho_{\mathrm{hd}}(a,z,\alpha)\theta(\beta) = a\beta(z)\theta(\alpha+\beta),\end{gathered}$$

where $u \in X$, $\alpha \equiv u \in H_{\mathrm{hd}}^\vee$, and $(a,z,\alpha) \in \mathcal{G}_H$. It is clear that

$$\rho(\phi_{\mathrm{hd}}, \tau_{\mathrm{hd}}) = (\phi_{\mathrm{hd}}^*)^{-1}\rho_{\mathrm{hd}}\phi_{\mathrm{hd}}^* = U_{H_{\mathrm{hd}}}.$$

Lemma 4.12. *Let $(P_0, \mathcal{L}_0)$ be a totally degenerate k-TSQAS with $\mathcal{L}_0$ strictly $\mathcal{G}_H$-linearized and $G_0 = \mathrm{Aut}_k^{\dagger 0}(P_0)$. Let $D = (f)$ and $f = \sum_{x \in X/Y} a_x\theta(x)$, $a_x \in k$. Then D contain no G_0-orbits iff $a_x \neq 0$ for any $x \in X/Y$.*

Proof. Let $(Q_0, \mathcal{L}_0)$ be the unique PSQAS associated with $(P_0, \mathcal{L}_0)$ via sq [N10, p.71]. By [NS06, Theorem 2] and [N99, 4.2] $H^0(P_0, \mathcal{L}_0) = H^0(Q_0, \mathcal{L}_0)$ and there is a bijective correspondence between G_0-orbits $O(\sigma)$ of P_0 and $O_Q(\sigma)$ of Q_0. Any zero-dimensional G_0-orbit of Q_0 is $O_Q(c) \in W_0(c)$ ($c \in X$), which is defined by $\xi_{x,c} = 0$ for all $x \neq 0, x \in \mathrm{Star}\,(0)$ by [N99, § 3, § 5]. By [N99, 4.2], $H^0(P_0, \mathcal{L}_0) = H^0(Q_0, \mathcal{L}_0)$ is spanned by

$$\theta(x) := \sum_{y \in Y} a_0(x+y)\xi_{x+y} \quad (x \in X/Y).$$

Suppose that any of g elementary divisor of $X/Y = H_{\mathrm{hd}}^\vee$ is at least 3. Then by [N99, 6.3], The restriction of $\theta(x)/\xi_c$ to $W_0(c)$ is equal to

$$\theta(x)/\xi_c = \begin{cases} a_0(x+y)(\xi_{x+y}/\xi_c) & \text{if } \exists\, y \in Y \text{ with } x+y \in \mathrm{Star}\,(c), \\ 0 & \text{otherwise} \end{cases}$$

where $\xi_c/\xi_c = 1$. Hence $(\theta(x)/\xi_c)_{|W_0(c)}$ is at most a single term, and $\theta(x)$ is zero at $O_Q(c)$ if $x \notin c + Y$. It follows that $\theta(c)$ is the unique element of $H^0(Q_0, \mathcal{L}_0)$ that does not vanish at $O(c)$. Hence $\theta(c)$ is the unique element of $H^0(P_0, \mathcal{L}_0)$ that does not vanish at $O(c)$.

Let $D = (f)$ and $f = \sum_{x \in X/Y} a_x\theta(x)$, $a_x \in k$. Thus we see that the divisor D does not contain $O(c)$ iff $a_x \neq 0$ for $x \equiv c \mod Y$. Hence D contains no $O(c)$ ($c \in X/Y$) iff $a_x \neq 0$ for any $x \in X/Y$. Meanwhile, D contains no G_0-orbits iff D contains no zero-dimensional G_0-orbits iff D contains no $O(c)$ ($c \in X/Y$). This proves the lemma in this case.

In the general case, let $D = (f)$, and $f = \sum_{\alpha \in X/Y} a_\alpha\theta(\alpha) \in H^0(P_0, \mathcal{L}_0)$. There exists an étale $Y/3Y$-covering $\pi : P_0' \to P_0$. Let $\mathcal{L}_0' := \pi^*(\mathcal{L}_0)$. Then $\pi^* f = \sum_{\alpha' \in X/3Y} b_{\alpha'}\vartheta(\alpha') \in H^0(P_0', \mathcal{L}_0')$, where $\vartheta(\alpha') = \sum_{x \in \alpha'} a(x)w^x \in H^0(P_0', \mathcal{L}_0')$, $b_{\alpha'} = a_\alpha$ for $\alpha' \equiv \alpha \mod Y$. Then D contains no $O_{P_0}(c)$ ($c \in X/Y$) iff $\pi^* D$ contains no $O_{P_0'}(c)$ ($c \in X/3Y$) iff $b_{\alpha'} \neq 0$ for any $\alpha' \in X/3Y$ iff $a_\alpha \neq 0$ for any $\alpha \in X/Y$. This proves the lemma. □

Lemma 4.13. *Let H be a maximal isotropic subgroup of (K, e_K), and $v := \sum_{\beta \in H^\vee} a_\beta v_H(\beta) \in V_H$. Then the following are equivalent :*

1. $(\mathrm{Aut}_k^{\dagger 0}(P_0), P_0, \mathcal{L}_0, \mathrm{div}\,\phi^*(v))$ *is a semiabelic pair for any rigid-$\mathcal{G}_H$ k-TSQAS $(P_0, \mathcal{L}_0, \phi, \tau)$,*
2. $\sum_{\alpha \in H^\vee} a_\alpha a_{\alpha,\beta}(\xi) \neq 0$ *for* $\forall \beta \in H^\vee$, $\forall \xi \in \mathrm{Aut}_c(\mathcal{G}_H)$,
3. $\sum_{\alpha \in H^\vee} a_\alpha a_{\alpha,\beta}(\xi(\eta)) \neq 0$ *for* $\forall \beta \in H^\vee$, *and some* $\xi(\eta)$ *with* $\pi(\xi(\eta)) = \eta$ *for* $\forall \eta \in \mathrm{Aut}(K, e_K)$.

Proof. First we assume that P_0 is totally degenerate and then we may assume $H = H_{\mathrm{hd}}$. By Lemma 4.8, $(P, \phi, \tau) \simeq (P, \phi_{\mathrm{hd}}(\xi), \tau_{\mathrm{hd}}(\xi))$ for some $\xi \in \mathrm{Aut}_c(\mathcal{G}_H)$. Since $H_{\mathrm{hd}}^\vee = X/Y$,

$$\phi^*(v) = \phi_{\mathrm{hd}}^* A(\xi)(\sum_{\alpha \in X/Y} a_\alpha v_{H_{\mathrm{hd}}}(\alpha)) = \sum_{\beta \in X/Y}(\sum_{\alpha \in X/Y} a_\alpha a_{\alpha,\beta}(\xi))\theta(\beta)$$

whence (1) and (2) are equivalent by Lemma 4.12.

If P_0 is partially degenerate with A_0 (resp. T_0) its abelian part (resp. torus part), then we choose a hereditary maximal isotropic subgroup H_{hd} of K for $(P_0, \mathcal{L}_0)$ such that X/Y is a direct summand of $H_{\mathrm{hd}}^{\vee}$. See Subsec. 3.1. Assume for simplicity $Y \subset eX$ for some $e \geq 3$. Let $G_0 = \mathrm{Aut}_k^{\dagger 0}(P_0)$ and $F = \phi^*(v) \in H^0(P_0, \mathcal{L}_0)$. Then F is of the form $F = \sum_{\alpha \in H_{\mathrm{hd}}^{\vee}} a_\alpha \theta(\alpha)$, $\theta(\alpha) = \phi^*(v_H(\alpha)) = \sum_{x \equiv \bar{x} \mod Y} \theta_x \zeta_x$ for some $0 \neq \theta_x \in H^0(A_0, \mathcal{M}_x)$, by [N99, 4.10], where $a_\alpha \in k$, $\alpha = (a, \bar{x})$, $\bar{x} \in X/Y$. Since $eX \subset Y$ for some $e \geq 3$, by [N99, 6.3], for $c \in X$, $(\theta(\alpha)/\zeta_c)_{O(c)} = \theta_c \neq 0$ if $c \in \bar{x}$, and $(\theta(\alpha)/\zeta_c)_{O(c)} = 0$ otherwise. Since $O(c)$ is an abelian variety A_0, and since θ_c is not identically zero, $a_\alpha \neq 0$ iff $\mathrm{div}(F)$ does not contains $O(c)$. Hence $a_\alpha \neq 0$ for any α iff $\mathrm{div}(F)$ contains no G_0-orbits. By Lemma 4.8, any (P, ϕ, τ) is isomorphic to $(P, \phi_{\mathrm{hd}}(\xi), \tau_{\mathrm{hd}}(\xi))$ for some $\xi \in \mathrm{Aut}_c(\mathcal{G}_H)$. Hence by the same argument as in the totally degenerate case, (1) and (2) are equivalent. By Lemma 4.5, $(P_0, \phi(\xi \cdot \xi_0), \tau(\xi \cdot \xi_0)) \simeq (P_0, \phi(\xi), \tau(\xi))$ if $\xi_0 = \xi(\gamma)$ or $\xi_0 = \xi(-\mathrm{id}_K)$. Hence $(\mathrm{Aut}_k^{\dagger 0}(P_0), P_0, \mathrm{div}\, \phi(\xi \cdot \xi_0)^*(v))$ is semiabelic if $(\mathrm{Aut}_k^{\dagger 0}(P_0), P_0, \mathrm{div}\, \phi(\xi)^*(v))$ is semiabelic. Hence (2) and (3) are equivalent. □

Theorem 4.14. *Let $K = H \oplus H^\vee$ be a finite symplectic group, H a maximal isotropic subgroup of K, $F_{g,K}$ a hypersurface of $\mathbf{P}((V_H)^\vee)$*

$$F_{g,K} : \prod_{\beta \in H^\vee, \eta \in \overline{\mathrm{Aut}}(K, e_K)} \Big(\sum_{\alpha \in H^\vee} a_\alpha a_{\alpha,\beta}(\xi(\eta)) \Big) = 0, \tag{4}$$

and $D_{g,K} = \mathbf{P}((V_H)^\vee) \setminus (F_{g,K} \cup G_{g,K})$. (See Definition 4.7 for $G_{g,K}$.) We define the map sqap *by*

$$\begin{aligned} \mathrm{sqap} : SQ_{g,K}^{\mathrm{toric}} \times D_{g,K} &\to \overline{AP}_{g,N} \\ (P, \mathcal{L}, \phi, \tau) \times [v] &\mapsto (\mathrm{Aut}^{\dagger 0}(P), P, \mathcal{L}, \mathrm{div}\, \phi^*(v)). \end{aligned}$$

Then the following are true :

1. $\mathrm{sqap} \otimes \mathcal{O}_{N^3}$ *is an étale Galois covering with* $\mathrm{Gal}(\mathrm{sqap}) \simeq \mathrm{Aut}_c(\mathcal{G}_H)$*,*
2. $\mathrm{sqap}_v := \mathrm{sqap}_{|SQ_{g,K}^{\mathrm{toric}} \times [v]}$ *is a closed immersion for any fixed* $[v] \in D_{g,K}(k)$*, where k is any field over $\mathcal{O}_N$.*

Proof. First we prove that sqap is well-defined. Since any k-TSQAS is seminormal by [N10, 3.3, 3.8] for any algebraically closed field k over $\mathcal{O}_N$, we have $\mathrm{sqap}(\sigma \times v) \in \overline{AP}_{g,N}(T)$ by Lemma 4.13. Let T be any $\mathcal{O}_N$-scheme and $v \in D_{g,K}(T)$. If $\sigma := (P, \mathcal{L}, \phi, \tau) \simeq (P', \mathcal{L}', \phi, \tau')$ in $SQ_{g,K}^{\mathrm{toric}}(T)$, then there exists an isomorphism $(f, \delta) : \sigma \to \sigma'$ such that $\phi' \cdot f = \phi$ and $(f, \delta)\tau(g) = \tau'(g)(f, \delta)$ $(g \in \mathcal{G}_H)$. Hence $\phi^* v = f^*(\phi')^* v$ for any $v \in V_H$, hence $(f^*)^{-1} \mathrm{div}(\phi^* v) = \mathrm{div}((\phi')^* v)$. Hence the map $(\mathrm{Ad}(f), f, \delta, (f^*)^{-1})$ is an isomorphism from $\mathrm{sqap}(\sigma, [v])$ to $\mathrm{sqap}(\sigma', [v])$ where $\mathrm{Ad}(f)(g) = fgf^{-1}$ for $g \in \mathrm{Aut}^{\dagger 0}(P)$. Thus sqap is a well-defined $\mathcal{O}_N$-morphism.

For $\xi \in \mathrm{Aut}_c(\mathcal{G}_H)$, $\sigma := (P, \mathcal{L}, \phi, \tau) \in SQ^{\mathrm{toric}}_{g,K}(T)$ and $[v] \in D_{g,K}$, let $\sigma(\xi) := (P, \mathcal{L}, \phi(\xi), \tau(\xi))$. We define an action of ξ by

$$\xi \cdot (\sigma, [v]) := (\sigma(\xi^{-1}), [A(\xi)v]).$$

This is also well-defined. We see

(i) $(\xi\xi') \cdot (\sigma, [v]) = \xi \cdot (\xi' \cdot (\sigma, [v]))$ for $\forall \xi, \xi' \in \mathrm{Aut}_c(\mathcal{G}_H)$,
(ii) $\mathrm{sqap}(\xi \cdot (\sigma, [v])) = \mathrm{sqap}(\sigma, [v])$ for $\forall \xi \in \mathrm{Aut}_c(\mathcal{G}_H)$.

Next we prove

$$\mathrm{sqap}^{-1}(\mathrm{sqap}(\sigma, [v])) = \mathrm{Aut}_c(\mathcal{G}_H) \cdot (\sigma, [v]) \tag{5}$$

for any $v \in D_{g,K}(k)$ and any field k over $\mathcal{O}_{N^3}$. The inclusion LHS $\supset$ RHS is clear. Conversely by Lemma 4.8, LHS $\subset$ RHS. By Lemma 4.9, ϕ and τ are rigid for a fixed $(P, \mathcal{L})$ over a local ring k, while $\mathrm{Aut}^{\dagger 0}(P, \mathcal{L})$ is uniquely determined by $(P, \mathcal{L})$. Hence the tangent space of $SQ^{\mathrm{toric}}_{g,K} \times D_{g,K}$ at $(\sigma, [v])$ is isomorphic to the tangent space of $\overline{AP}_{g,N}$ at $\mathrm{sqap}(\sigma, [v])$. Hence sqap is étale. Le k be any field over $\mathcal{O}_{N^3}$ and $(\sigma, [v]) \in SQ_{g,K}(k) \times D_{g,K}(k)$. $A(\xi)[v]$ are all distinct because $[v] \in G^c_{g,K}$, hence $\xi \cdot (\sigma, [v])$ are all distinct for $\xi \in \mathrm{Aut}_c(\mathcal{G}_H)$. This proves (1) by Equality (5).

Next we prove (2). Let k be any field over $\mathcal{O}_N$ and we prove $\mathrm{sqap}_v(k)$ is injective. Suppose $\mathrm{sqap}(\sigma \times [v]) = \mathrm{sqap}(\sigma' \times [v])$ for some $\sigma = (P, \phi, \tau)$, $\sigma' = (P, \phi', \tau') \in SQ^{\mathrm{toric}}_{g,K}(k)$ and $[v] \in D_{g,K}(k)$. By Lemma 4.8, there exists $\xi \in \mathrm{Aut}_c(\mathcal{G}_H)$ such that $(\phi', \tau') \simeq (\phi(\xi), \tau(\xi))$ and $p(\xi)^* = A(\xi)$. It follows that $[\phi^* p(\xi)^*(v)] = [\phi^* v]$, hence $[A(\xi)(v)] = [v]$ because ϕ^* is injective. Hence v is an eigenvector of $A(\xi)$. Since $v \in D_{g,K} \subset G^c_{g,K}$, we have $A(\xi) = \mathrm{id}_{V_H}$. It follows that $\mathrm{sqap}_v(k)$ is injective.

In order to prove that sqap_v is a closed immersion, it suffices to prove

$$\mathrm{sqap}_v(R) : SQ^{\mathrm{toric}}_{g,K}(R) \times \{v\} \to \overline{AP}_{g,N}(R)$$

is injective for R an Artin local k-ring, I the maximal ideal of R with $I^2 = 0$, $R/I = k$. Since the set of all R-deformations of a given $\sigma \in SQ^{\mathrm{toric}}_{g,K}(k)$ (resp. $\mathrm{sqap}_v(\sigma) \in \overline{AP}_{g,N}(k)$) with $R/I = k$ admits a k-vector space structure, it suffices to prove that if $\sigma \in SQ^{\mathrm{toric}}_{g,K}(R)$ and if $\mathrm{sqap}_v(\sigma)$ is trivial in $\overline{AP}_{g,N}(R)$, then σ is trivial. Let $\sigma = (P, \mathcal{L}, \phi, \tau) \in SQ^{\mathrm{toric}}_{g,K}(R)$. Suppose $\mathrm{sqap}_v(\sigma)$ is trivial in $\overline{AP}_{g,N}(R)$. Then $(P, \mathcal{L}) = (P_0, \mathcal{L}_0) \times \mathrm{Spec}\, R$. By Lemma 4.9, σ is trivial. This proves the injectivity of $\mathrm{sqap}_v(R)$, hence sqap_v is a closed immersion. □

Corollary 4.15. $SQ^{\mathrm{toric}}_{g,1} \simeq \overline{AP}_{g,1}$.

Proof. We note that $SQ^{\mathrm{toric}}_{g,1}$ is the reduced-coarse-moduli of $(P, \mathcal{L}, \phi, \tau)$ with ϕ and τ trivial. By Lemma 4.13 (2), $(\mathrm{Aut}^{\dagger 0}_k(P_0), P_0, \mathcal{L}_0, \mathrm{div}\, \phi^*(v_0))$ is semi-abelic if $(P_0, \mathcal{L}_0)$ is any $k(0)$-TSQAS with $K = \{1\}$ and v_0 the generator of $V_H = V_{\{1\}} \simeq k(0)$. Hence $\mathrm{sqap} : SQ^{\mathrm{toric}}_{g,1} \to \overline{AP}_{g,1}$ is a birational morphism defined everywhere. Let T be any scheme and $(P, \mathcal{L}) \in SQ^{\mathrm{toric}}_{g,1}(T)$

any T-TSQAS. Hence $h^0(P_s, \mathcal{L}_s) = 1$ for any geometric point $s \in T$. Therefore $\mathrm{sqap}(P, \mathcal{L}) = (\mathrm{Aut}^{\dagger 0}(P), P, \mathcal{L}, \Theta)$ is a semiabelic T-pair where Θ is the divisor defined by a unique generator of the invertible sheaf $\pi_*(\mathcal{L})$. Since $\mathrm{sqap} : A_{g,1} \to AP_{g,1}$ is an isomorphism and $SQ_{g,1}^{\mathrm{toric}}$ is proper, sqap is surjective. Hence if $(G, P, \mathcal{L}, \Theta)$ is a semi-abelic T-pair, then $(P, \mathcal{L})$ is a T-TSQAS. Hence the forgetful map $(G, P, \mathcal{L}, \Theta) \mapsto (P, \mathcal{L})$ is the inverse of sqap. Since $\overline{AP}_{g,1}$ is the closure of a reduced scheme $AP_{g,1}$, it is reduced. $SQ_{g,1}^{\mathrm{toric}}$ is also reduced by the same reason. This proves $SQ_{g,1}^{\mathrm{toric}} \simeq \overline{AP}_{g,1}$. □

5. The one-dimensional case

We use the notation in Subsec. 2.1 and 4.4. Let $H = \mu_3 \simeq \mathbf{Z}/3\mathbf{Z}$, $H^\vee = \mathbf{Z}/3\mathbf{Z}$, $K := K(H) = H \oplus H^\vee$ and $\mathcal{O} := \mathbf{Z}[\zeta_3, 1/3]$. Let $e_0 \in H$, $f_0 \in H^\vee$ be a standard basis of K_H with $e_K(e_0, f_0) = \zeta_3$. Let $C(\mu)$ be a Hesse cubic

$$x_0^3 + x_1^3 + x_2^3 - 3\mu x_0 x_1 x_2 = 0.$$

Let $\phi : C(\mu) \to \mathbf{P}(V_H)$ be $\phi^*(v_H(\beta f_0)) = x_\beta$ and $\tau = U_H$. Then $\sigma := (C(\mu), \phi, \tau)$ is a rigid-$\mathcal{G}_H$ TSQAS of dimension one and conversely. *By abuse of notation* we use the same symbol ϕ ad τ for any $C(\mu)$.

Let $\xi \in \mathrm{Aut}_c(\mathcal{G}_H)$. Then $\sigma(\xi) := (C(\mu), \phi(\xi), \tau(\xi))$ is another Hesse cubic $(C(\mu'), \phi, \tau)$, and the action of $H^\vee$ on σ is transformed into the action of $\xi(H^\vee)$ on $\sigma(\xi)$, which is just the action of $H^\vee$ on $(C(\mu'), \phi, \tau)$ by Subsec. 4.4 Eq.(3).

5.1. The case $\eta_1(e_0) = -f_0$ and $\eta_1(f_0) = e_0$.

Let $\xi_1 \in \mathrm{Aut}_c(\mathcal{G}_H)$ be

$$\xi_1(\omega(e_0)) := \omega(-f_0),\ \xi_1(\omega(f_0)) := \omega(e_0).$$

Let $A(\xi_1) = (a_{\beta,\gamma})$ and $w(\beta) = v_H(\xi_1(\beta f_0))$. Then since $\omega(-f_0) \cdot w(\beta) = \zeta_3^\beta w(\beta)$, $\omega(e_0) \cdot w(\beta) = w(\beta + 1)$ by Subsec. 4.4 Eq.(3), we see $A(\xi_1) = a_{0,0}(\zeta_3^{\beta\gamma})$. Let $P = C(\mu)$ and let $(P, \phi, \tau) := (C(\mu), \phi, \tau)$. Let $y_\beta := \phi(\xi_1)^*(v_H(\beta f_0)) = \sum_\gamma a_{\beta,\gamma} x_\gamma$. Then $(P, \phi(\xi_1), \tau(\xi_1))$ is a Hesse cubic

$$(\mu - 1)(y_0^3 + y_1^3 + y_2^3) - 3(\mu + 2) y_0 y_1 y_2 = 0.$$

5.2. The case $\eta_2(e_0) = e_0$ and $\eta_2(f_0) = e_0 + f_0$.

Let $\xi_2 \in \mathrm{Aut}_c(\mathcal{G}_H)$ be

$$\xi_2(\omega(e_0)) = \omega(e_0),\ \xi_2(\omega(f_0)) = \zeta_3 \omega(e_0)\omega(f_0) = \zeta_3^2 \omega(e_0 + f_0).$$

Since $\xi_2(\omega(e_0)) \cdot w(\beta) = \zeta_3^\beta w(\beta)$, $\xi_2(\omega(f_0)) \cdot w(\beta) = w(\beta + 1)$, we see $A(\xi_2) = a_{11}\,\mathrm{diag}(\zeta_3, 1, 1)$. Let $(P, \phi, \tau) := (C(\mu), \phi, \tau)$ as before, and $z_\beta := \phi(\xi_2)^*(v_H(\beta f_0))$. Then $(P, \phi(\xi_2), \tau(\xi_2))$ is a Hesse cubic

$$(z_0^3 + z_1^3 + z_2^3) - 3\zeta_3 \mu z_0 z_1 z_2 = 0.$$

5.3. **The group $\overline{\mathrm{Aut}}(K, e_K)$.** Let $SQ_{1,3} := SQ_{1,K} \simeq SQ_{1,K}^{\mathrm{toric}}$. $SQ_{1,3}$ is the reduced-fine-moduli scheme over $\mathcal{O}$ of Hesse cubics $(C(\mu), \phi, \tau)$.

Let $b_0 = [0, 1, -1]$, $b_1 = [0, 1, -\zeta_3]$, $b_2 = [-1, 0, 1]$. Hence $-b_2 = [1, -1, 0]$. We define $g_i \in \mathrm{PGL}(3, \mathcal{O}_3)$ by

$$g_1 := A(\xi_1) : (x_0, x_1, x_2) \mapsto (y_0, y_1, y_2),$$
$$g_2 := A(\xi_2) : (y_0, y_1, y_2) \mapsto (z_0, z_1, z_2),$$

where

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta_3 & \zeta_3^2 \\ 1 & \zeta_3^2 & \zeta_3 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} z_0 \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \zeta_3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}.$$

Each g_i induces a transformation on the group of 3-torsions $C(\mu)[3]$:

$$\begin{cases} g_1(b_0) = b_0, \\ g_1(b_1) = -b_2, \\ g_1(b_2) = b_1 \end{cases} \qquad \begin{cases} g_2(b_0) = b_0, \\ g_2(b_1) = b_1, \\ g_2(b_2) = b_1 + b_2. \end{cases}$$

We note that $g_1^2 = \mathrm{inv}_{C(\mu)} = A(\xi(-\mathrm{id}_K))$ is 3 times the permutation of x_1 and x_2. $\overline{\mathrm{Aut}}(K, e_K)$ is generated by g_1 and g_2 with g_1^2 regaded as trivial, whence $\overline{\mathrm{Aut}}(K, e_K) \simeq \mathrm{PSL}(2, \mathbf{F}_3) \simeq A_4$. Let $\mathbf{P}^1 = SQ_{1,1}$:=the coarse moduli of one-pointed smooth cubics and a one-pointed nodal cubic. Then $\overline{\mathrm{Aut}}(K, e_K)$ is the Galois group of $SQ_{1,3}$ over $\mathbf{P}^1 = SQ_{1,1}$ under the map $(C(\mu), \phi, \tau) \mapsto (C(\mu), b_0)$.

5.4. **The subset $G_{1,K}$.** Let $K = (\mathbf{Z}/3\mathbf{Z})^{\oplus 2}$. Let $v_i = v_H(if_0)$. Let $G_{1,K}$ be the union of all eigenvectors of nontrivial $A(\xi) \in \mathrm{PGL}(V_H)$ for $\xi \in \mathrm{Aut}_c(\mathcal{G}_H)$ and $F_{1,K}$ the hypersurface of $\mathbf{P}(V_H^\vee)$ of degree 12

$$F_{1,K} : a_0 a_1 a_2 \prod_{j,k \in \mathbf{Z}/3\mathbf{Z}} (a_0 + \zeta_3^j a_1 + \zeta_3^k a_2) = 0.$$

The above g_2 has eigenvectors $a_1 v_1 + a_2 v_2$ with a_i arbitrary. This implies that $G_{1,K}$ contains the hypersurface $a_0 = 0$. Since $G_{1,K}$ is $\mathrm{Aut}_c(\mathcal{G}_H)$-invariant, $G_{1,K}$ contains $F_{1,K} = \mathrm{Aut}_c(\mathcal{G}_H) \cdot \{a_0 = 0\}$. The eigenvectors of g_1 are $w_0 := v_1 - v_2$ and $w_\pm := (1 \pm \sqrt{3}) v_0 + v_1 + v_2$, where $w_0 \in F_{1,K}$. Let $H_{1,K} = G_{1,K} \setminus F_{1,K} = \mathrm{Aut}_c(\mathcal{G}_H)\{w_\pm\}$. Hence

$$H_{1,K} = \{[(1 \pm \sqrt{3}) v_i + \zeta_3^j v_{i+1} + \zeta_3^k v_{i+2}]; i, j, k \in \mathbf{Z}/3\mathbf{Z}\}.$$

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