HESSE CUBICS AND GIT STABILITY MCGILL UNIV. MONTREAL, 2014 MAY 22

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Abstract. The moduli space of nonsing. curves of genus g is compactified by adding Deligne-Mumford stable curves of genus g. The moduli space of stable curves is a projective variety, known as Deligne-Mumford compactification. We compactify in a similar way the moduli space of abelian varieties as the moduli space of some mildly degenerating limits of abelian varieties.

A typical case is the moduli space of Hesse cubics. Any Hesse cubic is GIT-stable, and any GIT stable planar cubic is one of Hesse cubics. Similarly in arbitrary dimension, the moduli space of abelian varieties is compactified by adding only GIT-stable limits of abelian varieties. Our moduli space is a projective "fine" moduli space of (possibly degenerate) abelian schemes for families over reduced base schemes

with non-classical (non-commutative) level structure

over $\mathbf{Z}[\zeta_N, 1/N]$ for some $N \geq 3$. The objects at the boundary are mild limits of abelian varieties, which we call PSQASes, projectively stable quasi-abelian schemes.

Contents

A reference for this talk is [N04].

1. Introduction

Roughly our problem is the following diagram completion :

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The Deligne-Mumford compactification completes the following diagram

the moduli of smooth curves

= the set of all isom. classes of smooth curves

⊂ the set of all isom. classes of stable curves

= the Deligne-Mumford compactification M_g

Therefore our problem is to complete the following diagram :

the moduli of smooth AVs (= abelian varieties)

= {smooth polarized AVs + extra structure}/ isom.

⊂ {smooth polarized AVs or

singular polarized degenerate AVs + extra structure}/ isom.

= the compactification $SQ_{g,K}$ of the moduli of AVs

The compactification problem of the moduli space of abelian varieties have been discussed by many people

(i) Satake compactification, Igusa monoidal transform of it
(ii) Mumford toroidal compactification (Ash-Mumford-Rapoport-Tai [AMRT75])
(iii) Faltings-Chai arithmetic compactification (arithmetic version of Mumford compactification) [FC90]
(iv) 1975-76 Nakamura, Namikawa,
(v) 1999- Nakamura, Alexeev, Olsson

The compactificatioins (i)–(iii) are not moduli of compact objects,

We wish to construct compactification as a moduli of compact objects, the compactifications in (v) are the moduli of compact objects,

We explain mainly [N99] (1999) of (v). See also [N13]. We construct *a natural compactification, projective, as the fine moduli of compact geometric objects for families over reduced base schemes:* thereby

1. proper = to collect suff. many limits
2. separated = to choose the minimum possible among the above
3. both are necessary for compactification

2. Hesse cubics

2.1. **Hesse cubics.** Let k be a closed field of chara. $\neq 3$. A Hesse cubic curve is defined by

$$C(\mu)\ :\ x_0^3 + x_1^3 + x_2^3 - 3\mu x_0x_1x_2 = 0 \tag{1}$$

for some $\mu \in k$, or $\mu = \infty$ (in which case we understand that $C(\infty)$ is the curve defined by $x_0x_1x_2 = 0$).

1. $C(\mu)$ is nonsingular elliptic for $\mu \neq \infty, 1, \zeta_3, \zeta_3^2$, where ζ_3 is a primitive cube root of unity.
2. $C(\mu)$ is a 3-gon for $\mu = \infty, 1, \zeta_3, \zeta_3^2$
3. any elliptic $C(\mu)$ has 9 inflection points(=flexes), independent of μ,

$$K := 9 \text{ flexes}$$

say, $(0, 1, -\zeta_3^k)$, $(-\zeta_3^k, 0, 1)$, $(1, -\zeta_3^k, 0)$, Note $K \subset C(\mu)$ ($\forall\ \mu$),

4. σ and τ act on $C(\mu)$, where $\sigma(x_k) = \zeta 3^k x_k$ and $\tau(x_k) = x_{k+1}$,
5. over $\mathbf{C}$, any Hesse cubic is the image of $E(\omega) := \mathbf{C}/\mathbf{Z} + \mathbf{Z}\omega$, a complex torus by thetas

$$\begin{aligned} x_k = \theta_k(q, w) &= \sum_{m\in\mathbf{Z}} e^{2\pi i(3m+k)^2\omega/6} e^{2\pi i(3m+k)z} \\ &= \sum_{m\in\mathbf{Z}} q^{(3m+k)^2} w^{3m+k} \end{aligned}$$

where $q = e^{2\pi i\omega/6}$, $w = e^{2\pi iz}$.

Then K is the image of $\ker(3 : E(\omega) \to E(\omega)) = \langle \frac{1}{3}, \frac{\omega}{3} \rangle$,
6. It is known by Hecke that $\mu = \vartheta/\chi$ where

$$\begin{aligned} \vartheta &= \sum_{\ell\in\mathbf{Z}^2} \exp(\pi i A[\ell]\tau), \\ \chi &= \sum_{\ell\in\mathbf{Z}^2} \exp(\pi i A[\ell + (1/3)^t(0, 1)]\tau), \\ A &= \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}, \quad A[\ell] = {}^t\ell A\ell. \end{aligned}$$

2.2. **The moduli space of Hesse cubics — Neolithic level structure.** Consider the moduli space of Hesse cubics.

(i) the moduli space $SQ^{\mathrm{NL}}_{1,3}$:=the set of isom. classes of $(C(\mu), K)$,
 where the definition of an isom.
 $(C(\mu), K) \simeq (C(\mu'), K)$: isom. iff
 $\exists\ f : C(\mu) \to C(\mu')$: an isom. with $f_{|K} = \mathrm{id}_K$,
 This extra condition $f_{|K} = \mathrm{id}_K$ for isom. is the classical level str.,

(ii) if $(C(\mu), K) \simeq (C(\mu'), K)$, then $\mu = \mu'$,

(iii) $SQ^{\mathrm{NL}}_{1,3} \simeq \mathbf{P}^1$, in fact, $SQ^{\mathrm{NL}}_{1,3} \simeq X(3)$ modular curve over $\mathbf{Z}[\zeta_3, 1/3]$, This compactifies $A^{\mathrm{NL}}_{1,3} := \{(C(\mu), K); C(\mu)\text{smooth}\} = \mathbf{P}^1 \setminus \{4 \text{ points}\}$.

Proof of (i). It suffices to prove (i). Suppose we are given an isomorphism

$$f : (C(\mu), K) \simeq (C(\mu'), K).$$

For simplicity suppose f is given by a 3×3 matrix A.

We shall prove that A is a scalar and $f = \mathrm{id}$. In fact, any line $\ell_{x,y}$ connecting two points $x, y \in K$ is fixed by f. Since the line $x_0 = 0$ connects $[0, 1, -1]$ and $[0, 1, -\zeta_3]$, it is fixed by f. Similarly the lines $x_1 = 0$ and $x_2 = 0$ are fixed by f, whence $f^*(x_i) = a_i x_i$ $(i = 0, 1, 2)$ for some $a_i \neq 0$. Thus A is diagonal. Since $[0, 1, -1]$ and $[-1, 0, 1]$ are fixed, we have $a_0 = a_1 = a_2$, hence A is scalar and $f = \mathrm{id}$, $\mu = \mu'$. □

2.3. **The moduli space of smooth cubics — classical level structure.** Consider the (fine) moduli space of smooth cubics over a closed field $k \ni 1/3$.

Definition 2.3.1. Let $K = (\mathbf{Z}/3\mathbf{Z})^{\oplus 2}$, e_i a standard basis of K. Let $e_K : K \times K \to \mu_3$ be a standard symplectic form of K: in other words, e_K is (multiplicatively) alternating and bilinear such that

$$e_K(e_1, e_2) = e_K(e_2, e_1)^{-1} = \zeta_3,\ e_K(e_i, e_i) = 1.$$

Let C be a smooth cubic with zero O, $C[3] = \ker(3\,\mathrm{id}_C)$ the group of 3-division points and e_C the Weil pairing of C, that is,

$$e_C : C[3] \times C[3] \to \mu_3 \quad \text{alternating nondegenerate bilinear.}$$

There exists a symplectic (group) isomorphism

$$\iota : (C[3], e_C) \to (K, e_K).$$

If $C = \mathbf{C}/\mathbf{Z} + \mathbf{Z}\omega$, then

$$1/3 \mapsto e_1, \omega/3 \mapsto e_2,$$
$$e_C(1/3, \omega/3) = \zeta_3.$$

For instance, in this case, we can identify $C(\mu)[3]$ with K by

$$O = [0, 1, -1],\ e_1 = [0, 1, -\zeta_3],\ e_2 = [1, -1, 0]. \tag{2}$$

Definition 2.3.2. The triple $(C, C[3], \iota) \in SQ^{\mathrm{CL}}_{1,3}$ is called *a cubic with classical level-3 structure.* We define $(C, C[3], \iota) \simeq (C', C'[3], \iota')$ to be isomorphic iff there exists an isom. $f : C \to C'$ such that $f_{|C[3]} : C[3] \to C'[3]$ is a symplectic (group) isom. subject to

$$\iota' \cdot f = \iota.$$

This is ess. the same as isoms of Neolithic level str. which fix K, so

$$SQ^{\mathrm{CL}}_{1,3} = \{(C, C[3], \iota)\}/isom. = \{(C(\mu), K, \mathrm{id}_K)\} = SQ^{\mathrm{NL}}_{1,3}.$$

3. Non-commutative level structure

Remark 3.1. If we stick to the definition of classical level structure

$$K = C[3] \subset C,$$

we will have nonseparated moduli in higher dimension.

Instead we consider the actions of (K and) $\mathcal{G}_H$ on C and L.

3.2. **Non-commutative interpretation of Hesse cubics.** Interpret the theory of Hesse cubics as follows: Fix $O = [0, 1, -1] \in C(\mu)$.

1. $C = C(\mu)$, $L := O_C(1)$ hyperplane bundle,
2. $K := \ker(3\,\mathrm{id}_C) \simeq (\mathbf{Z}/3\mathbf{Z})^{\oplus 2}$ with Weil pairing e_K (alt. nondeg.)
3. any T_x ($x \in K$), translation by $x \in K$, is lifted to $\gamma_x \in \mathcal{G}_H \subset \mathrm{GL}(3)$: a lin. transf. of $\mathbf{P}^2$,
4. translation by $1/3$ is lifted to σ (Recall that x_k is theta) $\theta_k(z + 1/3) = \zeta_3^k \theta_k(z)$
5. translation by $1/3$ is lifted to τ $[\theta_0, \theta_1, \theta_2](z + \omega/3) = [\theta_1, \theta_2, \theta_0](z)$
6. $\sigma(x_k) = \zeta_k x_k$, $\tau(x_k) = x_{k+1}$.
7. $[\sigma, \tau] = \zeta_3$, not commute,
8. $G(3) := \langle \sigma, \tau \rangle$ a finite group of order 27,
9. $H^0(C, L) = \{x_0, x_1, x_2\}$ is an irreducible $G(3)$-module of weight one, "weight one" means that $a \in \mu_3$ (center) acts as $a\,\mathrm{id}_V$,
10. the action of $G(3)$ on $H^0(C, L)$ is a special case of Schrödinger repres.,

11. Matrix forms

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^2 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\sigma\tau = \begin{pmatrix} 0 & 0 & 1 \\ \zeta_3 & 0 & 0 \\ 0 & \zeta_3^2 & 0 \end{pmatrix}, \quad \tau\sigma = \begin{pmatrix} 0 & 0 & \zeta_3^2 \\ 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \end{pmatrix}$$

Definition 3.3. $\mathcal{G}(K) = \mathcal{G}_H$: Heisenberg group;
U_H : Schrödinger representation

$$K = K_H = H \oplus H^\vee, H \text{ finite abelian}, N = |H|$$
$$H = H(e), H(e) = \oplus_{i=1}^g (\mathbf{Z}/e_i\mathbf{Z}), e_i | e_{i+1}, e_{\min}(K) := e_1,$$
$$\mathcal{G}(K) = \mathcal{G}_H = \{(a, z, \alpha); a \in \mathbf{G}_m, z \in H, \alpha \in H^\vee\},$$
$$G(K) = G_H = \{(a, z, \alpha); a \in \mu_N, z \in H, \alpha \in H^\vee\},$$
$$V := V_H = \mathcal{O}[H^\vee] = \bigoplus_{\mu \in H^\vee} \mathcal{O}v(\mu),$$
$$(a, z, \alpha)v(\gamma) = a\gamma(z)v(\alpha + \gamma)$$

where $\mathcal{O} = \mathcal{O}_N = \mathbf{Z}[\zeta_N, 1/N]$.

$$1 \to \mathbf{G}_m \to \mathcal{G}_H \to K_H \to 0 \quad \text{(exact)}$$

The action of $\mathcal{G}_H$ on V is denoted U_H.

In the Hesse cubics case, $\mathcal{O} := \mathbf{Z}[\zeta_3, 1/3]$, $H = H^\vee = \mathbf{Z}/3\mathbf{Z}$, we identify $G(3)$ with G_H:

$$\sigma = (1, 1, 0), \tau = (1, 0, 1) \in \mathcal{G}_H, N = 3.$$
$$V_H = \mathcal{O}[H^\vee] = \mathcal{O} \cdot v(0) \oplus \mathcal{O} \cdot v(1) \oplus \mathcal{O} \cdot v(2)$$

Lemma 3.4. *$\mathcal{G}_H$ (and G_H) has a unique irreducible representation of weight one over $\mathbf{Z}[\zeta_N, 1/N]$.*

3.5. **New formulation of the moduli problem.**

1. classical level 3 str. = to choose a syml. basis of K
2. new level 3 str.= to choose an action of $\mathcal{G}_H$ on $V \simeq H^0(C, L)$

Definition 3.6. For C any cubic with $L = O_C(1)$, (C, ψ, τ) is a level-$\mathcal{G}_H$ structure if

1. τ is a $\mathcal{G}_H$-action on the pair (C, L),
2. $\psi : C \to \mathbf{P}(V_H) = \mathbf{P}^2$ is the inclusion (it is a $\mathcal{G}_H$-equivariant closed immersion by τ, hence $\phi^* O(1)$ very ample)

Any smooth cubic (C, L) with $L = O_C(1)$, always has a $\mathcal{G}_H$-action τ.
Define : $(C, \psi, \tau) \simeq (C', \psi', \tau')$ isom. iff
$\exists\ (f, F) : (C, L) \to (C', L')$ $\mathcal{G}_H$-isom. with $\psi' \cdot f = \psi$
(This is equivalent to $f_{|K} = \mathrm{id}_K$ in the classical case.)

Lemma 3.7. *Any (C, ψ, τ) is isom. to a unique Hesse cubic $(C(\mu), i, U_H)$.*

Proof. Let $\mathbf{P}^2$ be $\mathbf{P}(V_H)$ and $\mathbf{H}$ the hyperplane bundle of $\mathbf{P}^2$. U_H induces an action on $H^0(\mathbf{P}^2, O_{\mathbf{P}^2}(1)) = V_H$.

Any (C, ψ, τ) is isomorphic to some Hesse cubic $(C(\mu), i, U_H)$. Here we prove the uniqueness of it only.

$$H^0(O_{C(\mu)}(1)) \simeq H^0(O_{\mathbf{P}^2}(1))$$
$$H^0(U_H, O_{C(\mu)}(1)) \simeq H^0(U_H, O_{\mathbf{P}^2}(1)) = U_H \quad \text{on } V_H$$

where $H = \mathbf{Z}/3\mathbf{Z}$.

Suppose $h : (C(\mu), i, U_H) \simeq (C(\mu'), i, U_H)$ is a $\mathcal{G}(3)$-isomorphism. Since h is linear by $\psi h = \psi'$, so $h^*\psi^*O(1) = (\psi')^*O(1)$, h induces an autom. of $(\mathbf{P}^2, O_{\mathbf{P}^2}(1))$ (also denoted h) so that we have a commutative diagram

$$\begin{array}{ccc}
H^0(O_{\mathbf{P}^2}(1)) = V_H & \xrightarrow{H^0(h^*)} & H^0(O_{\mathbf{P}^2}(1)) = V_H \\
\downarrow \| & & \downarrow \| \\
H^0(O_{C(\mu')}(1)) & \xrightarrow{H^0(h^*)} & H^0(O_{C(\mu)}(1)), \\
\downarrow U_H(g) & & \downarrow U_H(g) \\
H^0(O_{C(\mu')}(1)) & \xrightarrow{H^0(h^*)} & H^0(O_{C(\mu)}(1)),
\end{array}$$

whence we have

$$U_H(g)H^0(h^*) = H^0(h^*)U_H(g) \in \mathrm{End}\,(V_H)$$

for any $g \in \mathcal{G}(3)$, where we also regard $H^0(h^*) \in \mathrm{End}\,(V_H)$. Since U_H is irreducible, $H^0(h^*)$ is a scalar by Schur's lemma. Hence $H^0(h^*) = c\,\mathrm{id}_{V_H} \in \mathrm{PGL}(V_H)$, $h = \mathrm{id}_{\mathbf{P}(V_H)}$, $C(\mu) = C(\mu')$, $\mu = \mu'$. □

Proposition 3.8. *Over a closed field of char.* $\neq 3$,

$$\begin{aligned}
SQ_{1,3} :&= \{(C, \psi, \tau) : \textit{level-}\mathcal{G}(3)\}/\textit{isom} \\
&= \{(C(\mu), i, U_H) : \textit{level-}\mathcal{G}(3)\}/\textit{isom} = \{\mu \in \mathbf{P}^1\} \\
&= \{(C(\mu), K) : \textit{Neolithic level-3}\} = SQ^{\mathrm{NL}}_{1,3}
\end{aligned}$$

In other words,

$$\{\textit{cubic with level-}\mathcal{G}(3)\textit{-str.}\} = \{\textit{cubic with (Neo. or) classical level 3-str.}\}$$

We call this new level 3-structure *level-$\mathcal{G}_H$ structure*. This is the noncommutative level structure that we can generalize into higher dimension.

Summary 3.9. Nonsingular Hesse cubics are $\mathcal{G}(3)$-invariant abelian varieties embedded in the projective space. This suggests that the following will compactify the moduli of abelian varieties:

1. consider all $\mathcal{G}_H$-invariant abelian varieties embedded in $\mathbf{P}(V_H)$,
2. collect all the limits of $\mathcal{G}_H$-invariant abelian varieties,
3. then what are the limits ? The answer is our PSQASes.
4. Caution: an example in dimension two, $H = (\mathbf{Z}/3\mathbf{Z})^2$ shows that it is too hard to see what happens. In fact, abelian varieties embedded in $\mathbf{P}(V_H) = \mathbf{P}^8$ are defined by 12 equations.

4. The space of closed orbits

Let us forget the above Hesse cubic case for a while.

4.1. **Example.** To convince that the compactif. is natural, we recall GIT.

Let us look at the following example. Let $\mathbf{C}^2$ be the complex plane, (x, y) its coordinates. Let us consider the action of $\mathbf{C}^*$ on $\mathbf{C}^2$:

$$(\alpha, x, y) \mapsto (\alpha x, \alpha^{-1} y) \quad (\alpha \in \mathbf{C}^*) \tag{3}$$

What is the quotient space of $\mathbf{C}^2$ by the action of $\mathbf{C}^*$? There are four kinds of orbits:

$$\begin{gathered} O(a,1) = \{(x,y) \in \mathbf{C}^2; xy = a\} \ \ (a \neq 0), \\ O(0,1) = \{(0,y) \in \mathbf{C}^2; y \neq 0\}, \\ O(1,0) = \{(x,0) \in \mathbf{C}^2; x \neq 0\}, \\ O(0,0) = \{(0,0)\} \end{gathered} \tag{4}$$

where there are the closure relations of orbits

$$\overline{O(1,0)} \supset O(0,0), \ \overline{O(0,1)} \supset O(0,0).$$

If we define the quotient to be the orbit space, its natural topology is not Hausdorff, because

$$O(1,0) = \lim_{x \to 0} O(1,x) = \lim_{x \to 0} O(x,1) = O(0,1) \tag{5}$$

because $O(a,1) = O(1,a)$ $(a \neq 0)$.

In order to avoid this, we use the ring of invariants. By (3) we define the quotient space to be

$$\mathbf{C}^2 /\!/ \mathbf{C}^* = \{t; t \in \mathbf{C}\} \simeq \operatorname{Spec} \mathbf{C}[t]. \tag{6}$$

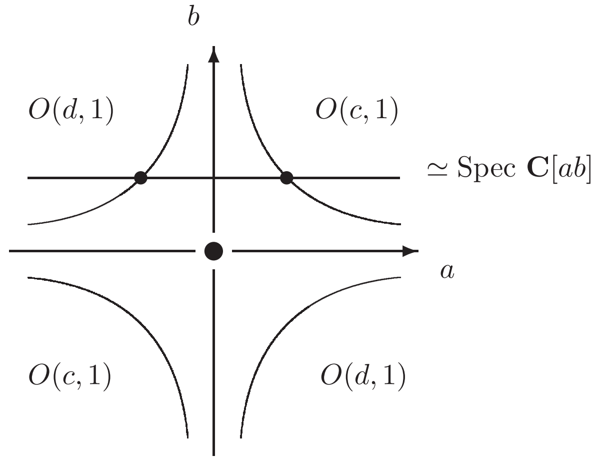
where $t = xy$. Let $\pi : \mathbf{C}^2 \to \mathbf{C}^2 /\!/ \mathbf{C}^*$ be the natural morphism. Hence π sends $(x, y) \mapsto t = xy$, so

$$O(1,0), O(0,1), O(0,0) \mapsto t = 0$$

where $O(0,0)$ is the unique closed orbit.

We summarize:

Theorem 4.2. *The quotient space $\mathbf{C}^2 /\!/ \mathbf{C}^*$ is set-theoretically the space of closed orbits.*



The same is true in general.

Theorem 4.3. (Seshadri-Mumford) *Let X be a projective variety, G a reductive group acting on X. Let X_{ss} be the open subscheme of X consisting of all semistable points in X. Let R be the graded ring of all G-inv. homog. polynomials on X. Let $Y := X_{ss}//G = \mathrm{Proj}\,(R)$. Then*

$$Y = \textit{the space of orbits closed in } X_{ss}.$$

Moreover let $\pi : X_{ss} \to Y$ be the natural morphism. Then $\pi(a) = \pi(b)$ if and only if $\overline{O(a)} \cap \overline{O(b)} \neq \emptyset$ where $a, b \in X_{ss}$.

A reductive group in Theorem 4.3 is by definition an algebraic group whose maximal solvable normal subgroup is an algebraic torus; for example $\mathrm{SL}(n)$ and $\mathbf{G}_m$ are reductive.

Now we give the definition of the term "semistable" in Theorem 4.3.

Definition 4.4. We keep the same notation as in Theorem 4.3. Let $p \in X$.

(1) p is *semistable* if there exists a G-invariant homogeneous polynomial F on X such that $F(p) \neq 0$, or equivalently,

$$\begin{aligned} X \setminus X_{ss} &= \text{the common zero locus of all } G\text{-invariant} \\ &\qquad \text{homogeneous polynomials on } X \\ &= \text{the subset of } X \text{ where no } G\text{-invariant} \\ &\qquad \text{functions are defined } (0/0\,!). \end{aligned}$$

(2) p is *Kempf-stable or closed orbit* if the orbit $O(p)$ is closed in X_{ss},

(3) p is *properly-stable* if p is Kempf-stable and the stabilizer subgroup of p in G is finite.

We denote by X_{ps} or X_{ss} the set of all properly-stable points or the set of all semistable points respectively. The implications are

$$\text{properly stable} \Longrightarrow \text{Closed orbit} \Longrightarrow \text{Semistable}$$

We note that if $a, b \in X_{ps}$, (hence they have closed orbits)

$$\begin{aligned}\pi(a) = \pi(b) &\Longleftrightarrow \overline{O(a)} \cap \overline{O(b)} \neq \emptyset \Longleftrightarrow O(a) \cap O(b) \neq \emptyset \\ &\Longleftrightarrow O(a) = O(b) \Longleftrightarrow \ a \text{ and } b \text{ are isom.}\end{aligned}$$

Very roughly speaking our moduli of abelian varieties is

$$X_{ps}//G = X_{ps}/G \fallingdotseq \text{the set of abelian varieties}$$

where abelian varieties have closed orbits by [Kempf78].

The compactification of $X_{ps}//G$ is

$$\text{Compactification} = X_{ss}//G = Y = \text{the set of closed orbits.}$$

By slightly modifying the formulation so as to fit in the classical theory, we will see that there exists a projective scheme $SQ_{g,K}$

$$\begin{aligned}SQ_{g,K} = \text{Compactification} &= \text{the set of closed orbits} \\ &= \text{the set of level-}\mathcal{G}_H \text{ PSQASes}\end{aligned}$$

where $\mathcal{G}_H$ is the Heisenberg group. If $H = (\mathbf{Z}/n\mathbf{Z})^g$,

$$SQ_{g,K} \fallingdotseq X_{ss}//G \supset X_{ps}//G \fallingdotseq A_{g,K} = \mathbf{H}_g/\Gamma(n)$$

Thus $SQ_{g,K}$ compactifies $\mathbf{H}_g/\Gamma(n)$.

5. Limits–Stable reduction theorem

Our goal of constructing a compactification is achieved by

1. finding limit objects PSQAS and TSQAS (Theorem 5.2)
2. constructing the moduli $SQ_{g,K}$ as a projective scheme (Section 7) so as to fit in the classical theory, for example, $SQ_{g,K}$ compactifies $A_{g,K} = \mathbf{H}_g/\Gamma(n)$ if $H = (\mathbf{Z}/n\mathbf{Z})^g$,
3. proving that any point of $SQ_{g,K}$ is the isom. class of a PSQAS (Q, ϕ, τ) with level-$\mathcal{G}_H$ str. (Ths. 5.2 (4), 7.2, 8.4)

5.1. **Limit objects.** First we note

• Any PSQAS is a scheme-theoretic limit of the images of AV by theta functions. It is also a compactification of a generalized Tate curve.

Let R be a CDVR, and $k(\eta)$ the fraction field of R. We start with an abelian scheme $(G_\eta, \mathcal{L}_\eta)$ and a polarization morphism $\lambda(\mathcal{L}_\eta) : G_\eta \to G_\eta^t$. Let $K_\eta = \ker(\mathcal{L}_\eta)$ the finite group scheme, and $\mathcal{G}(K_\eta) := \mathrm{Aut}(\mathcal{L}_\eta/G_\eta)$: the autom. gp of the pair $(G_\eta, \mathcal{L}_\eta)$ linear in the fibers of $\mathcal{L}_\eta$ over G_η.

For simplicity, we assume the characteristic of $k(0) = R/m_R$ is prime to rank K_η. Then there exists a finite symplectic abelian group K such that $K_\eta \simeq K$ and $\mathcal{G}(K_\eta) \simeq \mathcal{G}_H$ by some base change

$$1 \to \mathbf{G}_m \to \mathcal{G}_H \to K \to 0 \quad \text{(exact)}$$

Theorem 5.2. (A refined version of Alexeev-Nakamura's stable reduction theorem) ([AN99], [N99]) *For an abelian scheme $(G_\eta, \mathcal{L}_\eta)$ and a polarization morphism $\lambda(\mathcal{L}_\eta) : G_\eta \to G_\eta^t$ over $k(\eta)$, there exist proper flat projective schemes $(Q, \mathcal{L}_Q)$ (PSQAS) over R, by a finite base change if necessary, such that*

(1) $(Q_\eta, \mathcal{L}_\eta) \simeq (G_\eta, \mathcal{L}_\eta)$,

(2) *if* $e_{\min}(K) \geq 3$, *then* $\mathcal{L}_Q$ *is very ample, and in general,* $(Q, \mathcal{L}_Q)$ *is an étale quotient of some PSQAS* $(Q^*, \mathcal{L}_{Q^*})$ *with* $\mathcal{L}_{Q^*}$ *very ample,*
(3) $\mathcal{G}_H$ *acts on* $(Q, \mathcal{L}_Q)$ *extending the action of it on* $(G_\eta, \mathcal{L}_\eta)$,
(4) $(Q, \mathcal{L})$ *is uniquely determined by* $(G_\eta, \mathcal{L}_\eta)$ *if it satisfies (1)-(3).*

- (1) [AN99]; (2)-(4) [N99]
- (1) proves that *the moduli is proper*,
- (4) shows that *the moduli is separated.*
- The construction of Q is explicit.
- Summary. $\mathcal{L}$ very ample if $e_{\min}(K) \geq 3$; and $\mathcal{G}_H$ acts on $(Q, \mathcal{L})$.

6. Example

We show an example in dimension one to illustrate Th. 5.2.

We know that Hesse cubics are the images of $E(\omega)$ by theta functions. Nonsingular Hesse cubics have limits 3gons. Thus the next Summary follows.

Remark 6.1. Limits of abelian varieties would be obtained from limit of (normalized, that is, properly ordered by $\mathcal{G}_H$) theta functions.

6.2. **The complex case.** Come back to Hesse cubics, θ_k. Let $X = \mathbf{Z}$.

1. θ_k is Y-inv. where $Y = 3\mathbf{Z}$,
2. we wish to think

$$\begin{aligned} E(\omega) &\simeq \text{Proj } \mathbf{C}[\theta_k \vartheta, k = 0, 1, 2] \\ &=^* \text{Proj}\, (\mathbf{C}[[a(x)w^x\vartheta, x \in X]])^{Y-\text{inv}} \\ &\simeq^* \text{Proj } \mathbf{C}[a(x)w^x\vartheta, x \in X])/Y \end{aligned}$$

3. because $U = \text{Spec } A$ is affine, G a finite group acting on U, then

$$U/G = \text{Spec } A^{G\text{-inv}}.$$

4. Over $\mathbf{C}$, $a(x) \in \mathbf{C}^\times$, and

$$\mathbf{G}_m = \text{Proj } \mathbf{C}[a(x)w^x\vartheta, x \in X] = \bigcup_{k \in \mathbf{Z}} U_k,$$

because

$$U_k = \text{Spec } \mathbf{C}[a(x)w^x\vartheta/a(k)w^k\vartheta; x \in X] = \text{Spec } \mathbf{C}[w, w^{-1}] = \mathbf{G}_m,$$

5. Hence over $\mathbf{C}$ we may think so: if $0 < |q| < 1$, then

$$\begin{aligned} E(\omega) &\simeq \mathbf{G}_m/w \mapsto q^6 w \\ &\simeq \mathbf{G}_m/\{w \mapsto q^{2y}w; y \in 3\mathbf{Z}\} \\ &\simeq (\text{Proj } \mathbf{C}[a(x)w^x\vartheta, x \in X])/Y, \\ E(\omega) &\simeq \text{Proj } \mathbf{C}[\theta_k\vartheta, k = 0, 1, 2] \end{aligned}$$

where θ_k is something like the average by Y, because $\theta_k = \sum_{y \in Y} a(y+k)w^{y+k}$ converges.

6.3. **The scheme-theoretic limit.** What happens over a CDVR R? We can do the same because we have R-adic convergence.

Let $a(x) = q^{x^2}$ for $x \in X$, $X = \mathbf{Z}$, $Y = 3\mathbf{Z}$.

1. let

$$\widetilde{R} := R[a(x)w^x\vartheta, x \in X],$$
$$\mathcal{X} = \text{Proj } \widetilde{R}, \quad Z = \text{Proj } \widetilde{R}/Y.$$

2. define S_y action of Y on $\widetilde{R}$

$$S_y(a(x)w^x\vartheta) = a(x+y)w^{x+y}\vartheta$$

by imitating the summation in θ_k.

3.

$$\begin{aligned}
\mathcal{X} &= \text{Proj } R[a(x)w^x\vartheta, x \in X] \bigcup_{n\in\mathbf{Z}} U_n,\\
U_n &= \text{Spec } R[a(x)w^x/a(n)w^n, x \in X]\\
&= \text{Spec } R[(a(n+1)/a(n))w, (a(n-1)/a(n))w^{-1}]\\
&= \text{Spec } R[q^{2n+1}w, q^{-2n+1}w^{-1}]\\
&\simeq \text{Spec } R[x_n, y_n]/(x_ny_n - q^2),\\
\mathcal{X}_0 \cap U_n &= \text{Spec } k[x_n, y_n]/(x_ny_n).
\end{aligned}$$

4. $\mathcal{X}_0 := \mathcal{X} \otimes_R (R/qR) = \cup_{n\in\mathbf{Z}}\mathcal{X}_0 \cap U_n$ is an infinite chain of $\mathbf{P}^1$,
5. $\mathcal{X}_0/Y$: 3-gon, which recovers a singular Hesse cubic.

Theorem 5.2 generalizes this construction.

7. Stability of PSQASes

Theorem 7.1. ([Gieseker82], [Mumford77]) *For a connected curve C of genus greater than one, the following are equivalent:*

(1) *C is a stable curve,* (moduli-stable)
(2) *Any Hilbert point of C embedded by $|mK_C|$ is* GIT-stable,
(3) *Any Chow point of C embedded by $|mK_C|$ is* GIT-stable.

Theorem 7.2. *Let $K = H \oplus H^\vee$, $N = |H|$, k a closed field,* char $.k \neq N$.

Suppose $e_{\min}(K) \geq 3$, and $(Z, L) \subset (\mathbf{P}(V_H), O_{\mathbf{P}(V_H)}(1))$.

Suppose that (Z, L) is smoothable into an abelian variety whose Heisenberg group is isomorphic to $\mathcal{G}_H$. Then the following are equivalent:

(1) *(Z, L) is a PSQAS,* (moduli-stable)
(2) *the Hilbert points of (Z, L) have closed orbits, that is,* $\text{SL}(V_H)$*-orbit of Hilbert points of (Z, L) is closed,* (GIT-stable)
(3) *(Z, L) is stable under a conjugate of $\mathcal{G}_H$, that is, (Z, L) is a subscheme of $\mathbf{P}(V_H)$ invariant under the action of $\mathcal{G}_H$ on $\mathbf{P}(V_H)$,* ($\mathcal{G}_H$-stable).

Proof. $G = \text{SL}(V_H)$ for all. (1)→(3) Easy (Th. 5.2 (3)).

(3)→(2) by ([Kempf78]+L very ample).

We prove (2)→(1) : PSQAS has a closed orbit. Assume (2) for (Z, L).

- By assumption $\exists$ $(Q, \mathcal{L})$ over a CDVR R such that
  $(Q_\eta, \mathcal{L}_\eta)$ a level-$\mathcal{G}_H$ AV and $(Q_0, \mathcal{L}_0) = (Z, L) =: a$.
  Caution:since (Z, L) may have no $\mathcal{G}_H$-action, cannot apply Th. 5.2 (4), so $(Q, \mathcal{L})$ is a flat family which may not be the family in Th. 5.2,
- $O(a)$: closed by assuming (2).
- by base change may assume
  $\exists$ a level-$\mathcal{G}_H$ PSQAS $(Q', \mathcal{L}')$ s.t. $(Q'_\eta, \mathcal{L}'_\eta) = (Q_\eta, \mathcal{L}_\eta)$.
- Let $(Q'_0, \mathcal{L}'_0) =: b$. Then $\pi(a) = \pi(b)$. $\pi : X_{ss} \to X_{ss}/\!/ \mathrm{SL}$.
- Hence by Seshadri-Mumford, $\overline{O(a)} \cap \overline{O(b)} \neq \emptyset$.
- both are closed orbits. $O(a) \cap O(b) \neq \emptyset$.
- Hence $O(a) = O(b)$. This shows $(Z, L) \simeq (Q'_0, \mathcal{L}'_0)$ PSQAS.

$\square$

Now we are in position to compactify the moduli using Thm 7.2 (3):

$$\begin{aligned} SQ_{g,K} = \text{ Compactification} &= \text{the set of closed orbits} \\ &= \text{the set of all level-}\mathcal{G}_H \text{ PSQASes.} \end{aligned}$$

This will be made more precise in Sec. 8.

7.3. **Stability of planar cubics.** For planar cubics, any GIT-stable curve is either a smooth elliptic curve or a 3-gon by the following table, hence it is isomorphic to one of Hesse cubics. It follows from it that

$$\begin{aligned} C \text{ is GIT-stable} &\Leftrightarrow C \text{ is elliptic or a 3-gon} \\ &\Leftrightarrow C \text{ is isom. to a Hesse cubic} \\ &\Leftrightarrow C \text{ is isom. to a } G(3)\text{-stable cubic.} \end{aligned}$$

where GIT-stable := closed SL(3)-orbit

This is a special case of Theorem 7.2.

Table 1. Stability of cubic curves

curves (sing.)	stability	stab. gr.
smooth elliptic	GIT-stable	finite
3 lines, no triple point	GIT-stable	2 dim
a line+a conic, not tangent	semistable, not GIT-stable	1 dim
irreducible, a node	semistable, not GIT-stable	$\mathbf{Z}/2\mathbf{Z}$
3 lines, a triple point	not semistable	1 dim
a line+a conic, tangent	not semistable	1 dim
irreducible, a cusp	not semistable	1 dim

8. The moduli space $SQ_{g,K}$

By Theorem 5.2, any level $\mathcal{G}_H$ PSQAS $(Q_0, \mathcal{L}_0)$ is embedded into $\mathbf{P}(V)$ if $e_{\min}(K) \geq 3$ where $V = V_H := \mathcal{O}_N[v(\mu); \mu \in H^\vee]$.

Lemma 8.1. *Assume $e_{\min}(K) \geq 3$. For a level-$\mathcal{G}_H$ PSQAS (Q_0, ϕ_0, τ_0), there exists a unique level-$\mathcal{G}_H$ PSQAS (Q'_0, i, U_H) isom. to (Q_0, ϕ_0, τ_0) such that $i : Q'_0 \subset \mathbf{P}(V_H)$, where U_H is the Schrödinger repres. of $\mathcal{G}_H$.*

Let $\mathrm{Hilb}^{\chi(n)}$ be the Hilbert scheme parameterizing all the closed subscheme (Z,L) of $\mathbf{P}(V_H)$ with $\chi(Z,L^n)=n^g\sqrt{|K|}=:\chi(n)$, and $(\mathrm{Hilb}^{\chi(n)})^{\mathcal{G}_H\text{-inv}}$ the $\mathcal{G}_H$-inv. part of it. The following is a closed immersion of $SQ_{g,K}$ (resp. an immersion of $A_{g,K}$) into $(\mathrm{Hilb}^{\chi(n)})^{\mathcal{G}_H\text{-inv}}$:

$$A_{g,K}\ni(A_0,\phi_0,\tau_0)\mapsto(A'_0,i,U_H)\in(\mathrm{Hilb}^{\chi(n)})^{\mathcal{G}_H\text{-inv}}\quad(\mathrm{AV})$$
$$SQ_{g,K}\ni(Q_0,\phi_0,\tau_0)\mapsto(Q'_0,i,U_H)\in(\mathrm{Hilb}^{\chi(n)})^{\mathcal{G}_H\text{-inv}}\quad(\mathrm{PSQAS})$$

Hence

$$SQ_{g,K}=\overline{A_{g,K}}\subset(\mathrm{Hilb}^{\chi(n)})^{\mathcal{G}_H\text{-inv}}$$

Example 8.2. Let $g=1$ and $H=\mathbf{Z}/3\mathbf{Z}$, $\chi(n)=3n$. Hence $\mathrm{Hilb}^{\chi(n)}$ is the space of all cubics in $\mathbf{P}(V_H)=\mathbf{P}^2$. hence the space of ternary cubics. Then $(\mathrm{Hilb}^{\chi(n)})^{\mathcal{G}_H\text{-inv}}$ is $G(3)$-invariant cubics, hence they are Hesse cubics or one of eight cubics defined by $G(3)$-semi-inv cubic poly.

$$x_0^3+\zeta_3x_1^3+\zeta_3^2x_2^3,\quad x_0^2x_1+\zeta x_1^2x_2+\zeta^2x_2^2x_0$$

where $\zeta_3^2+\zeta_3+1=0$, $\zeta^3=1$, so

$$\begin{aligned}(\mathrm{Hilb}^{3n}_{\mathbf{P}^2})^{\mathcal{G}_H\text{-inv}}&=\mathbf{P}^1\bigcup(8\text{ points}),\\(\mathrm{Hilb}^{4n}_{\mathbf{P}^3})^{\mathcal{G}_H\text{-inv}}&=\mathbf{P}^1\bigcup(3\text{ points}),\\(\mathrm{Hilb}^{5n}_{\mathbf{P}^4})^{\mathcal{G}_H\text{-inv}}&=\mathbf{P}^1,\\SQ_{1,K_H}&=\mathbf{P}^1,(H=\mathbf{Z}/n\mathbf{Z},n=3,4,5).\end{aligned}$$

Definition 8.3. The triple (X,ϕ,τ) or (X,L,ϕ,τ) is
a PSQAS with level-$\mathcal{G}_H$ str. if

1. $\phi:(X,L)\to(\mathbf{P}(V),O(1))$ a closed immersion
 such that $\phi^*:V\simeq H^0(X,L)$, $L=\phi^*O_{\mathbf{P}(V)}(1)$,
2. τ is a $\mathcal{G}_H$-action on the pair (X,L) so that ϕ is a $\mathcal{G}_H$-morphism.

Define : $(X,\phi,\tau)\simeq(X',\phi',\tau')$ isom. iff
$\exists\,(f,F):(X,L)\to(X',L')$ $\mathcal{G}_H$-isom. such that $\phi=\phi'\cdot f$.

Theorem 8.4. *Suppose $e_{\min}(K)\geq 3$. Let $N:=\sqrt{|K|}$. The functor $\mathcal{SQ}_{g,K}$ of level-$\mathcal{G}_H$ PSQASes (Q,ϕ,τ) over reduced base schemes is represented by the projective $\mathbf{Z}[\zeta_N,1/N]$-scheme $SQ_{g,K}$:*

$$SQ_{g,K}(T)=\{(Q,\phi,\tau);\textit{PSQAS with level-}\mathcal{G}_H\textit{ str. over }T\}.$$

for T reduced.

It follows

$$\begin{aligned}SQ_{g,K}(k)&=\{(Q,\phi,\tau);\text{PSQAS with level-}\mathcal{G}_H\text{ str. over }k\}\\&=\text{the set of the orbits of level-}\mathcal{G}_H\text{ PSQASes}\\&=\text{the set of closed orbits}\end{aligned}$$

where k is a closed field of chara. prime to N.

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