The extended Dynkin diagram in McKay Correspondence

Iku Nakamura

2011 January 20

1 Reviews — Dynkin diagrams ADE

The following are related with Dynkin ADE

(1) Finite subgroups of SL(2, C), Regular polyhedra

(2) 2-dim simple sing. (deformation-stable critical pts)

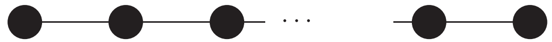
(3) simple Lie algebras ADE, II_1-factors etc.

(4) Partition func. of SL(2, Z)-inv. conformal field th.

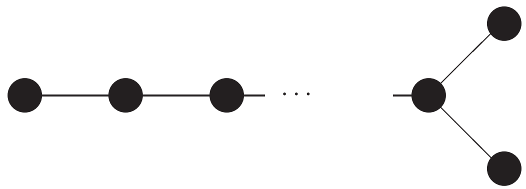
(5) Finite simple groups the derived group of the Fisher F_{24}, the Baby monster B, the Monster M, are related with (E_6, E_7, E_8) (McKay's 3rd observ.)

Dynkin diagrams ADE

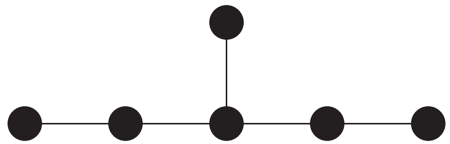
A_n



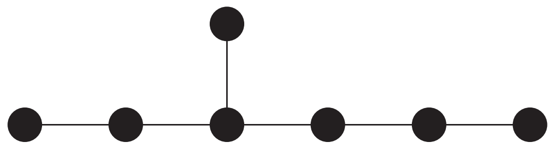
D_n



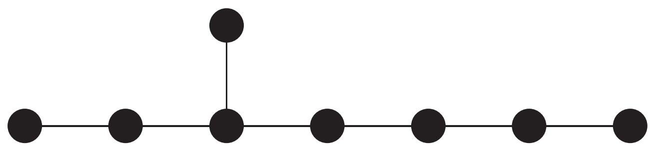
E_6

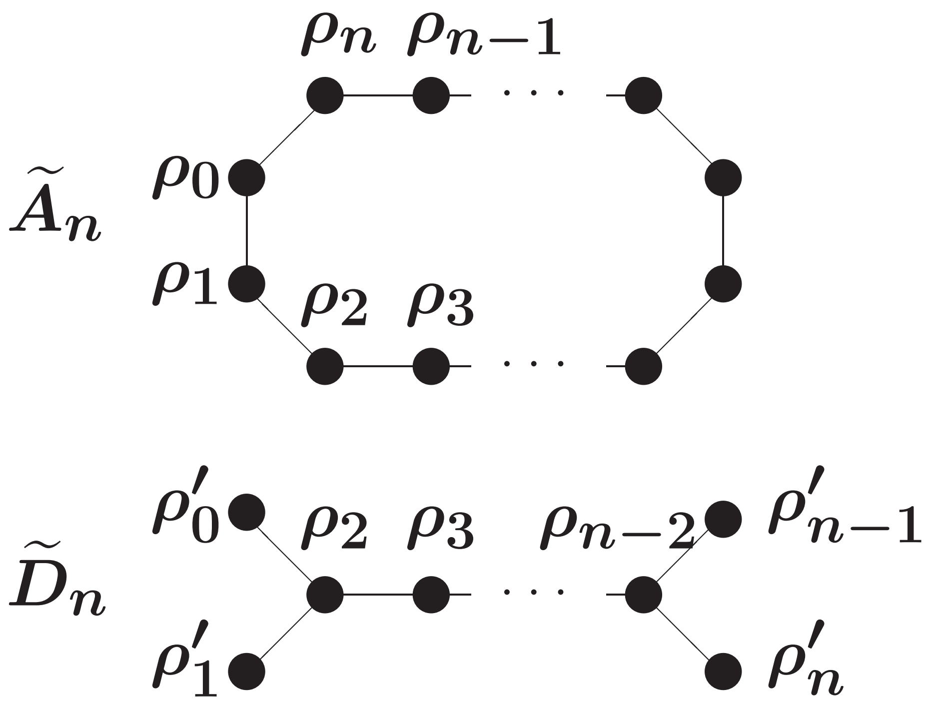


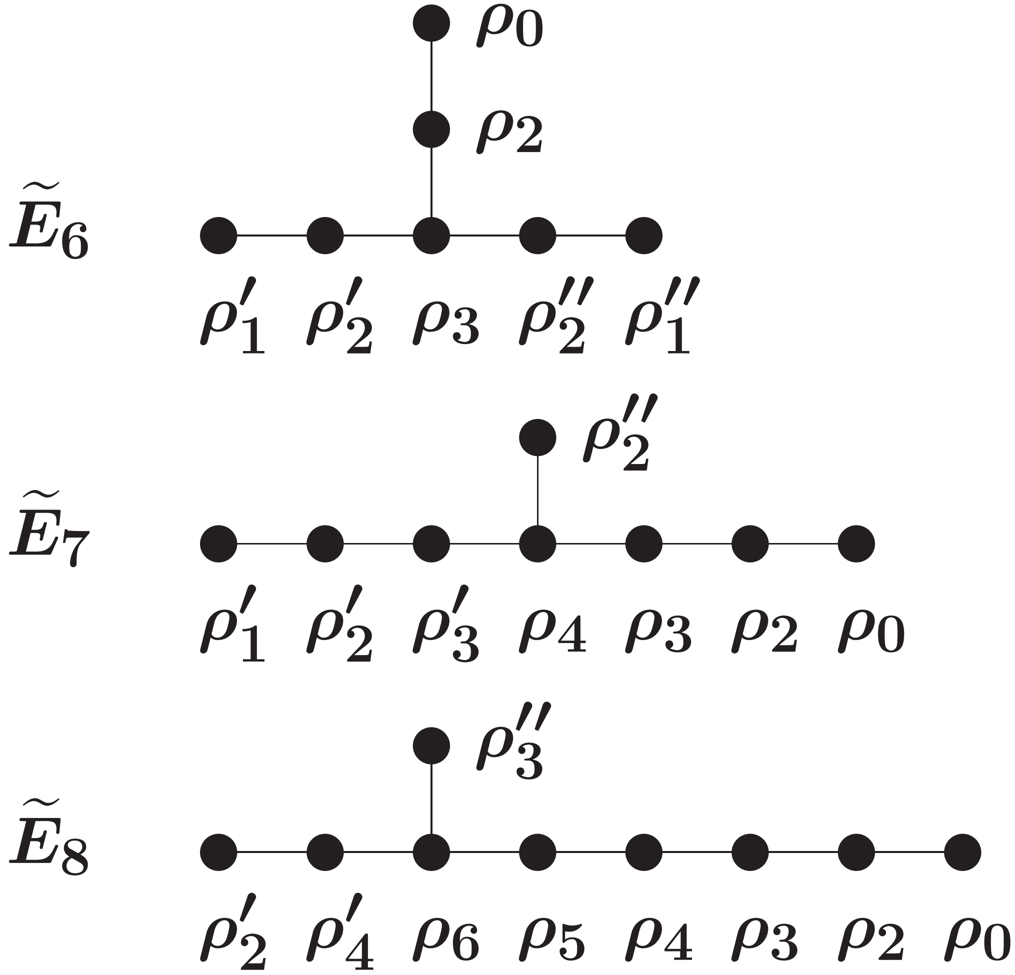
E_7



E_8




$\widetilde{A}_n$
ρ_n
ρ_{n-1}
ρ_0
ρ_1
ρ_2
ρ_3
$\widetilde{D}_n$
ρ'_0
ρ_2
ρ_3
ρ_{n-2}
ρ'_{n-1}
ρ'_1
ρ'_n



$\widetilde{E}_6$
ρ_0
ρ_2
ρ_1' ρ_2' ρ_3 ρ_2'' ρ_1''
$\widetilde{E}_7$
ρ_2''
ρ_1' ρ_2' ρ_3' ρ_4 ρ_3 ρ_2 ρ_0
$\widetilde{E}_8$
ρ_3''
ρ_2' ρ_4' ρ_6 ρ_5 ρ_4 ρ_3 ρ_2 ρ_0

Partition function of SL(2, Z)-inv. CFTs

Definition: $Z = \mathrm{Tr}(q^{L_0+\bar{L}_0})$, where

$q = e^{2\pi\sqrt{-1}\tau}$, $\tau \in$ H : upper half-plane

Assumptions

1. $\exists$ Unique vacuum($\exists$1 state of min. energy)
2. χ_k : a $A_1^{(1)}$-character corresp. to a particle
   (or an operator in a physical theory)
3. Partition func. Z is a sum of $\chi\overline{\chi'}$,
   (χ, χ':$A_1^{(1)}$-characters)
4. Z is SL(2, Z)-inv., invariant under $\tau \mapsto -\tau^{-1}$

Classification of Z

Cappelli, Itzykson-Zuber, A.Kato

Type	$k+2$	Partition function $Z(q,\theta,\bar{q},\bar{\theta})$
A_n	$n+1$	$\sum_{\lambda=1}^{n}\lvert\chi_\lambda\rvert^2$
D_{2r}	$4r-2$	$\sum_{\lambda=1}^{r-1}\lvert\chi_{2\lambda-1}+\chi_{4r+1-2\lambda}\rvert^2+2\lvert\chi_{2r-1}\rvert^2$
D_{2r+1}	$4r$	$\sum_{\lambda=1}^{2r}\lvert\chi_{2\lambda-1}\rvert^2+\sum_{\lambda=1}^{r-1}(\chi_{2\lambda}\bar{\chi}_{4r-2\lambda}+\bar{\chi}_{2\lambda}\chi_{4r-2\lambda})+\lvert\chi_{2r}\rvert^2$
E_6	12	$\lvert\chi_1+\chi_7\rvert^2+\lvert\chi_4+\chi_8\rvert^2+\lvert\chi_5+\chi_{11}\rvert^2$
E_7	18	$\lvert\chi_1+\chi_{17}\rvert^2+\lvert\chi_5+\chi_{13}\rvert^2+\lvert\chi_7+\chi_{11}\rvert^2+\lvert\chi_9\rvert^2$
		$+(\chi_3+\chi_{15})\bar{\chi}_9+\chi_9(\bar{\chi}_3+\bar{\chi}_{15})$
E_8	30	$\lvert\chi_1+\chi_{11}+\chi_{19}+\chi_{29}\rvert^2+\lvert\chi_7+\chi_{13}+\chi_{17}+\chi_{23}\rvert^2$

Fact : **Indices of χ = Coxeter exponents of ADE**

Type	Partition function $\boldsymbol{Z}$ and Coxeter exponents
$\boldsymbol{E_6}$	$\lvert\chi_1+\chi_7\rvert^2+\lvert\chi_4+\chi_8\rvert^2+\lvert\chi_5+\chi_{11}\rvert^2$
	1, 4, 5, 7, 8, 11
$\boldsymbol{E_7}$	$\lvert\chi_1+\chi_{17}\rvert^2+\lvert\chi_5+\chi_{13}\rvert^2+\lvert\chi_7+\chi_{11}\rvert^2+\lvert\chi_9\rvert^2$
	$+(\chi_3+\chi_{15})\bar{\chi}_9+\chi_9(\bar{\chi}_3+\bar{\chi}_{15})$
	1, 5, 7, 9, 11, 13, 17
$\boldsymbol{E_8}$	$\lvert\chi_1+\chi_{11}+\chi_{19}+\chi_{29}\rvert^2+\lvert\chi_7+\chi_{13}+\chi_{17}+\chi_{23}\rvert^2$
	1, 7, 11, 13, 17, 19, 23, 29

(5) McKay's 3rd observation in the Monster case

∃ only 2 conj. classes of involutions of Monster $\mathbb{M}$.

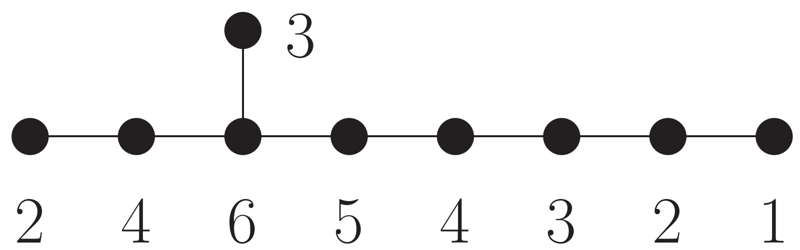
(Fischer) is one of the classes (Fischer involution).

1. **{the conj. class of $a \cdot b; a, b \in$ (Fischer)} = 9 classes**

2. **{order of $a \cdot b; a, b \in$ (Fischer)} = {1, 2, 2, 3, 3, 4, 4, 5, 6},**

   the same as mult. of vertices of $\widetilde{E}_8$, extended E_8.

$\widetilde{E}_8$



Recent progress

Topic(4) : Kawahigashi and other (Ann. Math.)

Topic(5) : Conway, Miyamoto, Lam-Yamada-Yamauchi

For $G \subset \mathrm{SL}(2)$

McKay correspondence (1) $\Longrightarrow$ (0)

[Ito and N. 1999] explains McKay (1) $\Longrightarrow$ (0) + (2)

by Hilbert scheme of G-orbits G-Hilb(C^2)

Missing in [Ito and N. 1999] is the extended Dynkin

Today the extended Dynkin appears ([N, 2007; ϵ])

---

(0) Dynkin diagrams ADE

(1) Finite subgroups of SL(2, C)

(2) 2-dim. Simple sing. and their resol.

History well known and trivial results

1. From finite groups to singularities : (1) $\Longrightarrow$ (2)

2. From singularities to Dynkin diag. : (2) $\Longrightarrow$ (0)

---

(0) Dynkin diagrams ADE

(1) Finite subgroups of $\mathbf{SL(2, C)}$

(2) 2-dim. Simple sing. and their resol.

(3) Simple Lie algebras ADE

History — well known, but nontrivial results

1. From Lie algebra to singularities : (3) $\Rightarrow$ (2)

   Grothendieck, Brieskorn, Slodowy

2. From finite gps to Dynkin of reps : (1) $\Rightarrow$ (0) (McKay)

3. Explanation for McKay : (1) + (2) $\Rightarrow$ (0) + (2)

   by vector bundles and their Chern classes

   (Gonzalez-Sprinberg and Verdier 1984)

4. [Ito and N. 1999] explains McKay (1) $\Rightarrow$ (0) + (2)

   by Hilbert scheme of G-orbits : G-Hilb($\mathbf{C}^2$)

5. **Today we refine the item 4.**

Today A relative form of [Ito and N. 1999] is given.

Theorem Let $X = G$-Hilb, let $\mathfrak{n}_X$: the ideal defining the graph $X \to (\mathrm{A}^2/G) \times X$, Define

$$\mathcal{V} := I_{\mathrm{univ}}/\mathfrak{m}I_{\mathrm{univ}} + \mathfrak{n}_X, \quad \mathcal{V}^\dagger := I_{\mathrm{univ}}/(\mathfrak{m} + \mathfrak{n}_X)I_{\mathrm{univ}},$$

Then we have

$$\mathcal{V} := I_{\mathrm{univ}}/(\mathfrak{m}I_{\mathrm{univ}} + \mathfrak{n}_X) \simeq \bigoplus_{i=1}^r O_{E(\rho_i)}(-1) \otimes \rho_i$$

$$\mathcal{V}^\dagger := I_{\mathrm{univ}}/(\mathfrak{m} + \mathfrak{n}_X)I_{\mathrm{univ}} \simeq \bigoplus_{i=0}^r O_{E(\rho_i)}(-1) \otimes \rho_i$$

2 Review : what is McKay correpondence ?

Ex 1 Let $\zeta_6 = e^{2\pi\sqrt{-1}/6}$.

$G = G(D_5)$: the dihedral group of order 12

G : generated by σ and τ

$$\sigma = \begin{pmatrix} \zeta_6 & 0 \\ 0 & \zeta_6^{-1} \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Repres.	Tr(ρ)	e	$-e$	σ	σ^2	τ	τ^3
ρ_0	χ_0	1	1	1	1	1	1
ρ_1	χ_1	1	1	1	1	-1	-1
ρ_2	χ_2	2	-2	1	-1	0	0
ρ_3	χ_3	2	2	-1	-1	0	0
ρ_4	χ_4	1	-1	-1	1	i	$-i$
ρ_5	χ_5	1	-1	-1	1	$-i$	i

Let $\rho_{\mathrm{nat}} : G \subset \mathrm{SL}(2, \mathrm{C})$ (the natural inclusion)

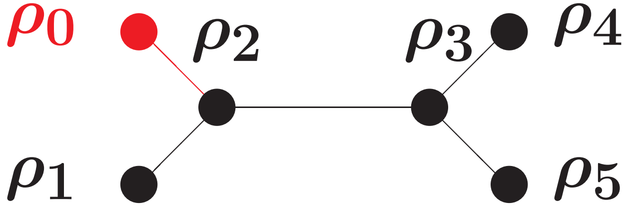
Then repres.ρ_i and their tensor products with ρ_{nat}

$$\rho_2 \otimes \rho_{\mathrm{nat}} = \rho_0 + \rho_1 + \rho_3, \quad \rho_0 \otimes \rho_{\mathrm{nat}} = \rho_1 \otimes \rho_{\mathrm{nat}} = \rho_2,$$

$$\rho_3 \otimes \rho_{\mathrm{nat}} = \rho_2 + \rho_4 + \rho_5, \quad \rho_4 \otimes \rho_{\mathrm{nat}} = \rho_3,$$

$\rho_5 \otimes \rho_{\mathrm{nat}} = \rho_3.$ **Draw the graph Dynkin(Rep(G)):**

$\widetilde{D}_5$



Rule : Connect ρ_i and $\rho_j \iff \rho_i \otimes \rho_{\mathrm{nat}} = \rho_j + \cdots$

Remove ρ_0 (triv. repres.).Then we get Dynkin D_5.

Ex 1 (Continued)

$G = G(D_5)$: a dihedral group of order 12

G : generated by σ and τ

$$\sigma = \begin{pmatrix} \zeta_6 & 0 \\ 0 & \zeta_6^{-1} \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The quotient $\mathbb{C}^2/G$ has a unique sing.

Invariant polynomials of $G(D_5)$ and their relation are

$$F = x^6 + y^6, G = x^2y^2, H = xy(x^6 - y^6)$$

$$G^4 - GF^2 + H^2 = 0$$

Invariant polynomials of $G(D_5)$ and their relation are

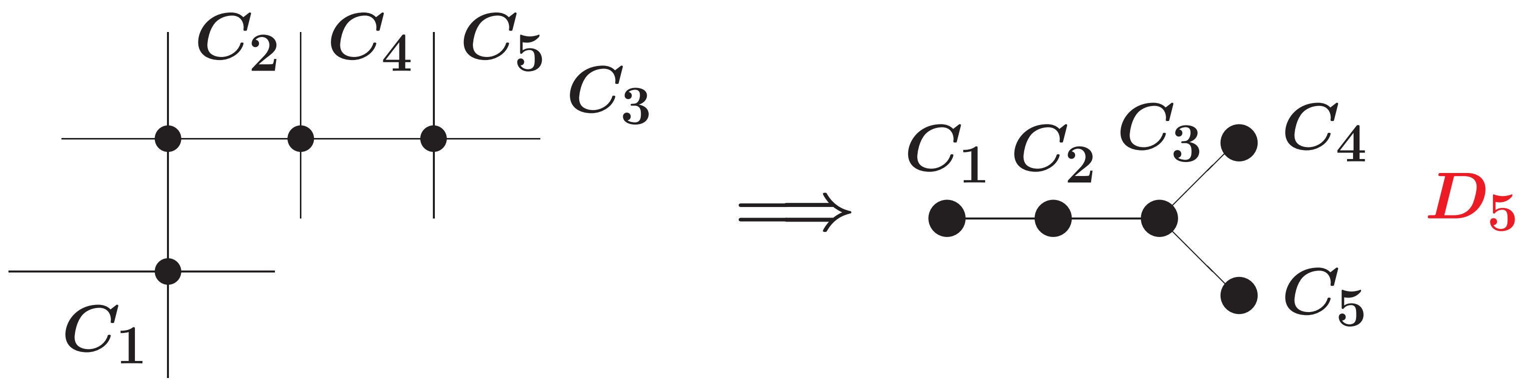
$$G^4 - GF^2 + H^2 = 0$$

The equation of D_5 is usually referred to as

$$X^4 + XY^2 + Z^2 = 0$$

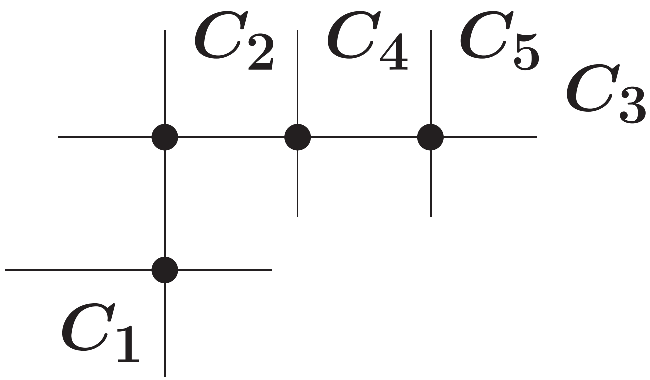
A unique singularity $(X, Y, Z) = (0, 0, 0)$

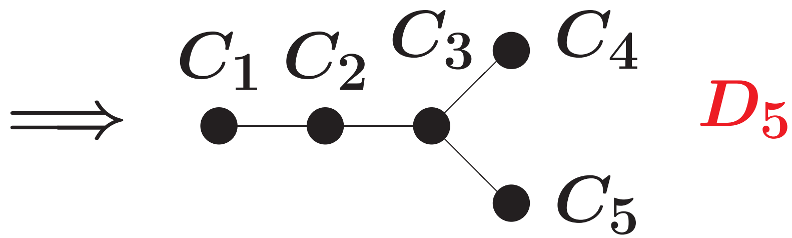
The exceptional set for the sing.$(0, 0, 0)$



C_i is P^1 (a line), intersecting at most transversely

The exceptional set for the sing. $(0,0,0)$





Dynkin$(\mathbf{C}^2/G)$

C_i is $\mathbf{P}^1$ (a line), intersecting at most transv.

The dual graph of it is Dynkin diagram D_5

Rule of dual graph : C_i = a vertex, $C_i \cap C_j$ = an edge

Conclusion : Dynkin(Rep(G))=Dynkin($\mathbf{C}^2/G$)

(McKay correspondence)

McKay corresp. asserts :

$\exists$ a deep relationship between (1) and (2)

(1) Resolution of the sing. of $\mathbf{C}^2/G$

(2) Representation theory of G

Invariant polynom. of $G(D_5)$ and relation :

$$G^4 - GF^2 + H^2 = 0$$

A unique singularity $(F, G, H) = (0, 0, 0)$

When resolve sing, take quotients $H/G = \frac{(x^6 - y^6)}{xy}$ etc.

$x^6 - y^6$, xy are not $G(D_5)$-invariant

But xy, $x^6 - y^6$ belong to the same repres. of $G(D_5)$

Therefore Resolution of $\mathbb{C}^2/G$ and
Repres. of G obviously relate each other.

3 Moduli-theoretic resolution of $\mathbf{C}^2/G$

$G = G(D_5)$, Regard $\mathbf{C}^2/G$ as a moduli space

$\mathbf{C}^2/G$ ={a G-inv subset consisting of 12 points}

: moduli of geometric G-orbits, $|G| = 12$

Resolution of $\mathbf{C}^2/G$ = moduli of ring-theor. G-orbits

G - Hilb($\mathbf{C}^2$) := {an $O_{\mathbf{C}^2}$ - G - module of length 12}

For a module $M \in G$ - Hilb($\mathbf{C}^2$)

$$0 \to I \to O_{\mathbf{C}^2} \to M \to 0 \text{ (exact)}$$

A G-module generating I is almost G-irreducible

Example of generators of I : $F_t = xy - t(x^6 - y^6)$

4 The Hilbert scheme of n points

What is

The Hilbert scheme of n points of the space X ?

Z : n points of X

"n points" Z is a formal sum

$$Z = n_1 P_1 + n_2 P_2 + \cdots + n_r P_r \quad (P_i \neq P_j)$$

$$(\text{where } n = n_1 + \cdots + n_r)$$

Ex 2 Assume $X = \mathbb{C}$.

$$I_Z := \text{the ideal of } Z = f_Z \cdot \mathbb{C}[x]$$

$$n \text{ points } Z = n_1 P_1 + \cdots + n_r P_r,\ P_i : x = \alpha_i$$

$$\overset{\text{equiv.}}{\Longleftrightarrow} \quad \dim_{\mathbb{C}} \mathbb{C}[x]/I_Z = n \overset{\text{equiv.}}{\Longleftrightarrow}$$

$$f_Z(x) = (x - \alpha_1)^{n_1}(x - \alpha_2)^{n_2} \cdots (x - \alpha_r)^{n_r}$$

$$= x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$$

e.g. if $z = n \cdot [0]$, then $I_Z = (x^n)$

$$\mathrm{Hilb}^n(\mathbb{C}) = \{n \text{ points of } X\}$$

$$= \left\{ x^n + \sum_{j=0}^{n-1} a_{n-j}\, x^j ; a_j \in \mathbb{C} \right\} \cong \mathbb{C}^n$$

Ex 3 Assume $X = \mathbb{C}^2$. Then

$$Z = n_1P_1 + \cdots + n_rP_r, \quad \text{(formal sum)},$$

$$P_i \neq P_j, \quad \text{namely},$$

$$Z \in X \underbrace{\times \cdots \times}_{n} X/\text{order forgotten} =: X^{(n)}$$

$$X^{(n)} = X \underbrace{\times \cdots \times}_{n} X/S_n$$

$X^{(n)}$ is very singular, has a lot of sing.

Caution $X^{(n)}$ is different from $\mathrm{Hilb}^n(\mathbb{C}^2)$.

Assume $X = \mathbf{C}^2$. Let $X^{[n]}$ =Hilb$^n(\mathbf{C}^2)$.

$$X^{[n]} = \{\text{an ideal } I \subset \mathbf{C}[x,y]; \dim \mathbf{C}[x,y]/I = n\}$$

$$= \left\{ \begin{array}{c} I \subset \mathbf{C}[x,y];\ I : \text{a vector subsp of } \mathbf{C}[x,y] \\ xI \subset I, yI \subset I, \\ \dim \mathbf{C}[x,y]/I = n \end{array} \right\}$$

Thm 1 (Fogarty 1968) $X^{[n]}$ is a resolution of $X^{(n)}$.

A natural map $\pi : X^{[n]} \to X^{(n)}$ is defined,

$\pi : Z \mapsto n_1P_1 + \cdots + n_rP_r$ where

$|Z| = \{P_1, \cdots, P_r\}, n_i =$ multiplicity of P_i in Z.

Thm 1 (Fogarty 1968)(revisited)

The natural morphism $X^{[N]} = \mathrm{Hilb}^N(\mathrm{C}^2) \xrightarrow{\pi} X^{(N)}$ is a resol (minimal) of sing.

The map π sends G-fixed points to G-fixed points.

where $G \subset \mathrm{SL}(2)$, $N = |G|$. Then

$$\pi^{G\text{-inv.}} : (X^{[N]})^{G\text{-inv.}} \to (X^{(N)})^{G\text{-inv.}}$$

By Theorem of Fogarty $(X^{[N]})^{G\text{-inv.}}$ is nonsing.

G -inv. part of Fogarty = The Next Theorem

5 The G-orbit Hilbert scheme of C^2

Lemma 2 $(X^{(N)})^{G\text{-inv.}} = \mathrm{C}^2/G$.

Def 3 $G\text{-}\mathrm{Hilb}(\mathrm{C}^2) := (X^{[N]})^{G\text{-inv.}}$

$G\text{-}\mathrm{Hilb}(\mathrm{C}^2)$ is called the G-orbit Hilbert scheme of C^2.

For $I \in G\text{-}\mathrm{Hilb}(\mathrm{C}^2)$, $\mathrm{C}[x,y]/I \cong \mathrm{C}[G]$: regular repres.

Thm 4 (Ito and N. 1999)

$G\text{-}\mathrm{Hilb}(\mathrm{C}^2)$ is a minimal resol. of C^2/G with enough information about repres. of G.

Thm 4 (Ito and N. 1999)

G - Hilb($\mathbf{C}^2$) is a minimal resol. of $\mathbf{C}^2/G$ with enough information about repres. of G.

This gives a new explanation for McKay corresp.

「Vertices of Dynkin diagram」

$\Updownarrow$ (bijective)

「Irred. components of except. set」

(bijective) $\Updownarrow$ (McKay corresp.)

「Equiv. classes of irred. reps($\neq$ trivial) of G」

6 McKay correspondence

For $I \in E$: (except. set), we define

$$V(I) := I/(\mathfrak{m}I + \mathfrak{n})$$

Then $V(I)$ is either irred. or $\rho \oplus \rho'(\rho, \rho'$ irred.)

For ρ, ρ' irred rep of $G, \rho \neq \rho'$, define subsets of E by

$$E(\rho) := \{I \in E; V(I) \supset \rho\}$$

$$P(\rho, \rho') := \{I \in E; V(I) \supset \rho \oplus \rho'\}$$

Thm 5 (Ito-Nakamura 1999)

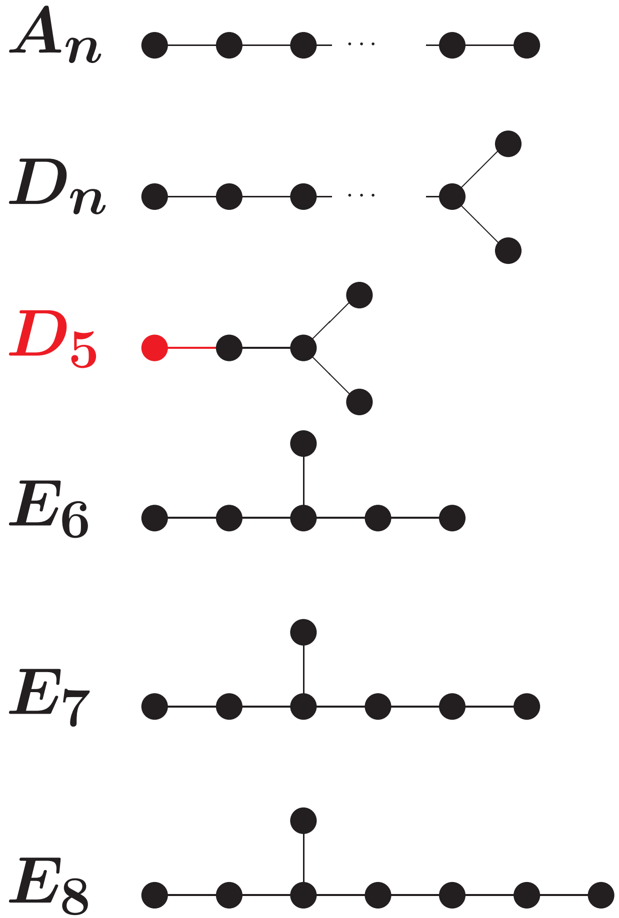
Let G be a finite subgroup of $\mathrm{SL}(2, \mathrm{C})$. Then

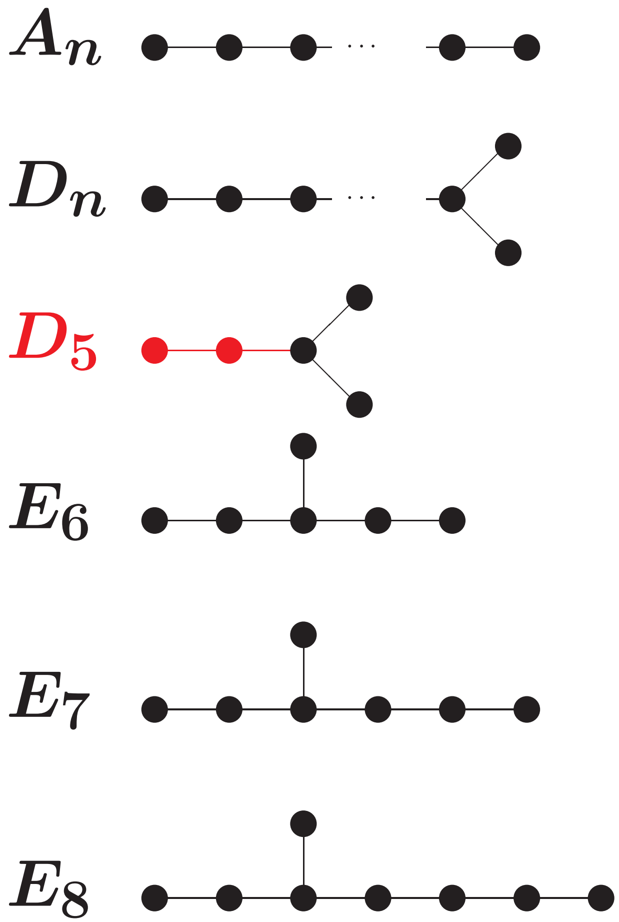
(1) G-Hilb(C^2) is a min. resol. of C^2/G.

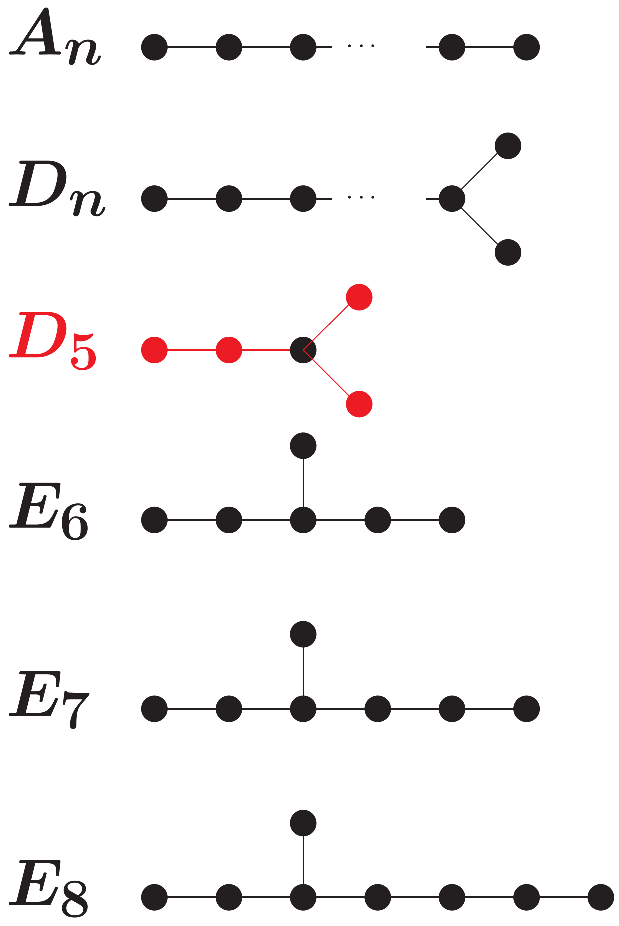
(2) For any irred rep. ρ of G, $\rho \neq$ trivial, $E(\rho) = \mathrm{P}^1$,

The map $\rho \mapsto E(\rho)$ is McKay corresp.

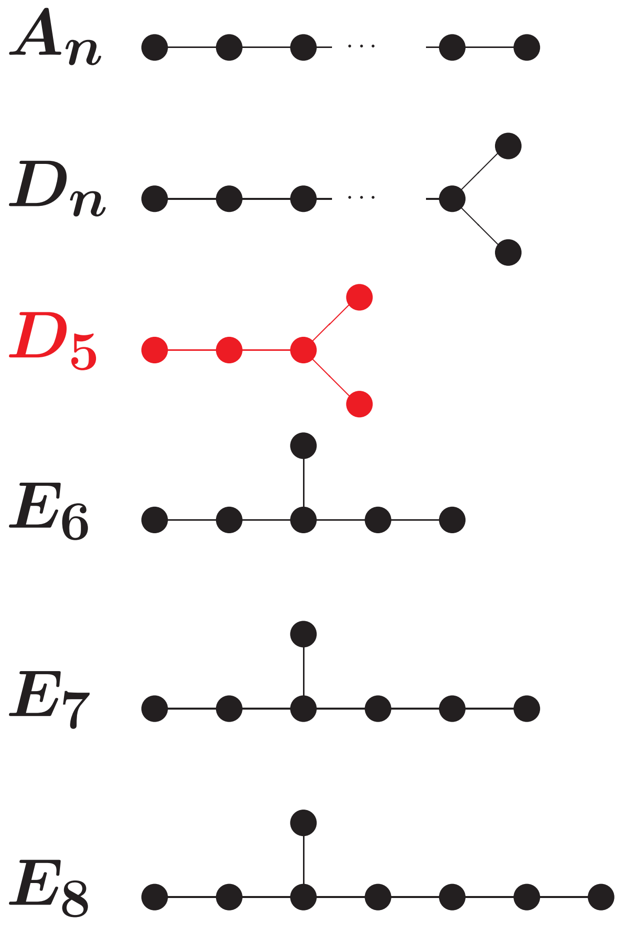
(3) the intersections $P(\rho, \rho') =$ the arrows of Dynkin of rep.

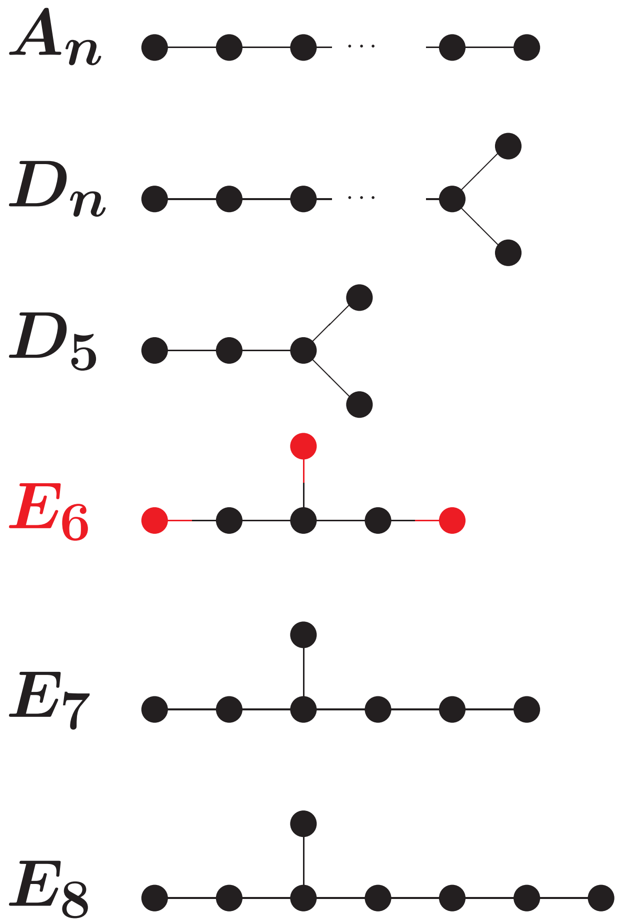

A_n
D_n
D_5
E_6
E_7
E_8

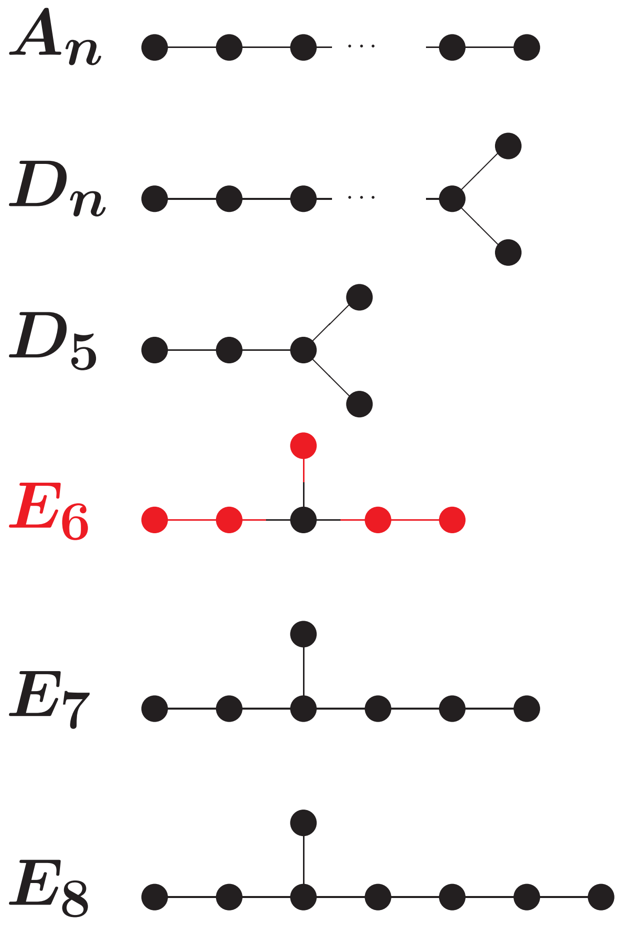

A_n
D_n
D_5
E_6
E_7
E_8

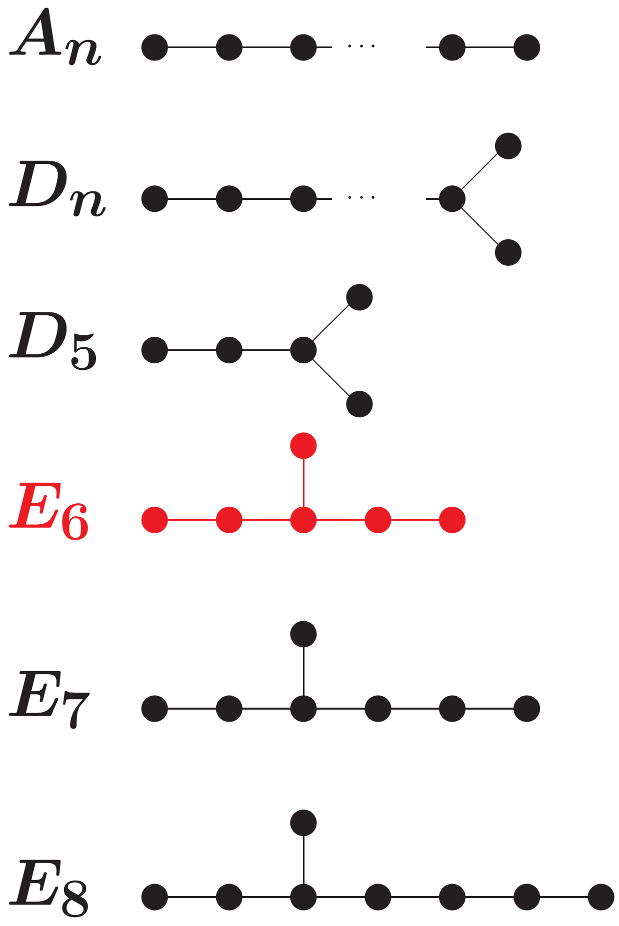


A_n
D_n
D_5
E_6
E_7
E_8


A_n
D_n
D_5
E_6
E_7
E_8


A_n
D_n
D_5
E_6
E_7
E_8


A_n
D_n
D_5
E_6
E_7
E_8


A_n
D_n
D_5
E_6
E_7
E_8

7 The exceptional set — D_5 case

$\pi : G\text{-}\mathrm{Hilb}(\mathbf{C}^2) \to \mathbf{C}^2/G$ the natural map

:isomorphism over $\mathbf{C}^2/G \setminus \{0\}$

Let $E = \pi^{-1}(0)$ be the exceptional set. Then

$$E = \{I \in G\text{-}\mathrm{Hilb}(X); I \subset \mathfrak{m}\}$$

where For $I \in G\text{-}\mathrm{Hilb}$, $\mathbf{C}[x,y]/I = \mathbf{C}[G]$: Regular rep.

$\mathfrak{m} = (x,y)\mathbf{C}[x,y]$: the maximal ideal.

For $I \in E$, we set $V(I) = I/(\mathfrak{m}I + \mathfrak{n})$

where $\mathfrak{n} = (F,G,H)\mathbf{C}[x,y]$. $V(I)$ is a G-module of generators of I other than G-inv.

Define subsets of E by

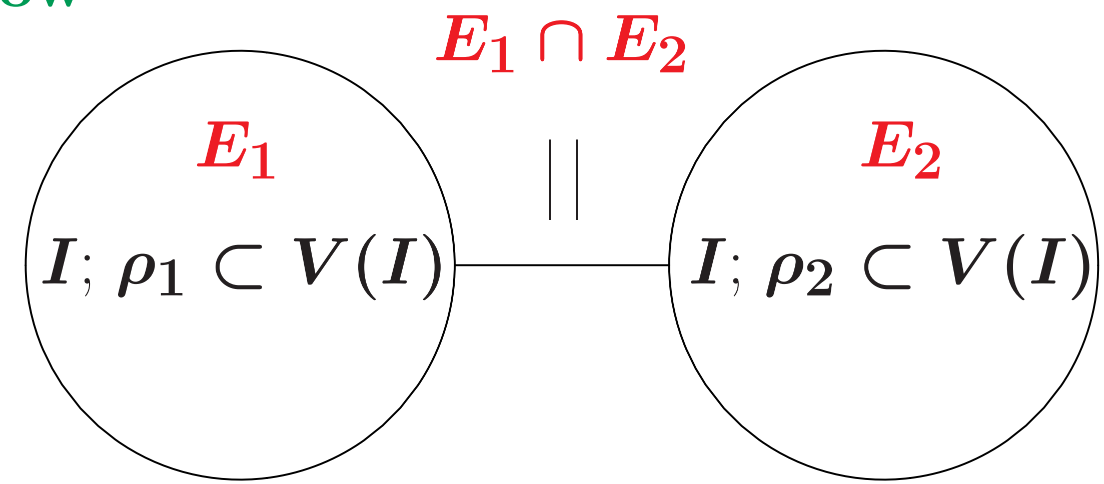
$$E_1 := \{I \in E;\ \rho_1 \subset V(I)\}$$

$$= \{I \in G\text{-Hilb};\ \mathfrak{m} \subset I, \rho_1 \subset V(I)\}$$

$$E_2 := \{I \in E;\quad V(I) \supset \rho_2\}$$

$$E_1 \cap E_2 := \{I \in E;\ \rho_1 \oplus \rho_2 \subset V(I)\}.$$

Want to show



Have defined

$$E_1 := \{I \in E;\ \rho_1 \subset V(I)\}$$

$$= \{I \in G\text{-}\mathrm{Hilb};\ \mathfrak{m} \subset I, \rho_1 \subset V(I)\}$$

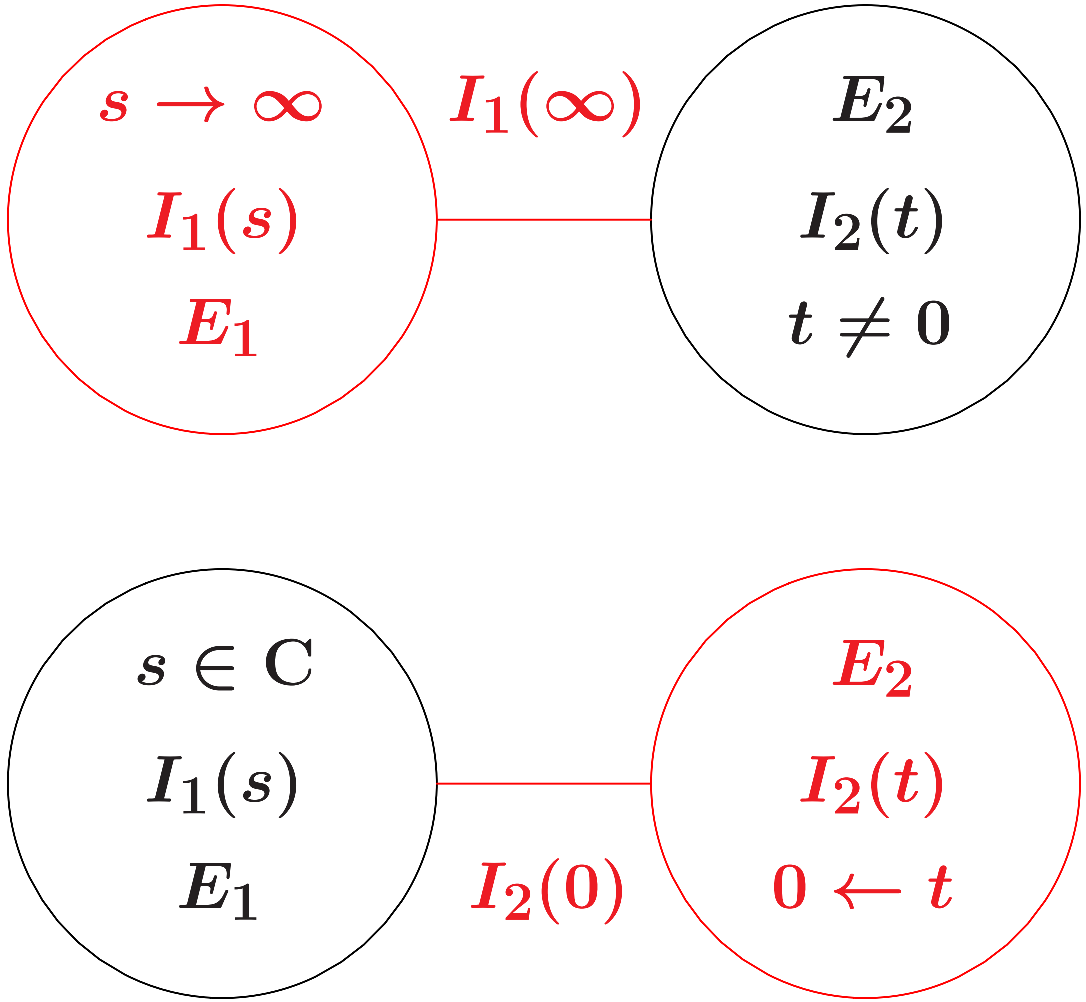
$$E_2 := \{I \in E;\quad V(I) \supset \rho_2\}.$$

We will see for $I_1(s)$, $I_2(s)$ explicit

$$E_1 = \{I_1(s); s \in \mathrm{C}\} \cup I_1(\infty) \simeq \mathrm{P}^1,$$

$$E_2 = \{I_2(s); s \in \mathrm{C}^*\} \cup I_2(0) \cup I_2(\infty) \simeq \mathrm{P}^1,$$

$$E_1 \cap E_2 = I_1(\infty) = I_2(0) \text{ (one point)}$$



We will see

$$I_1(\infty) = I_2(0)$$

The coinv. algebra of G gets involved in computing E_i.

m	$V_m(\rho)$ (: deg m, type ρ)	Eq. class
1	$\{x, y\}_{\rho_2}$	ρ_2
2	$\{xy\}_{\rho_1} \oplus \{x^2, y^2\}_{\rho_3}$	$\rho_1 + \rho_3$
3	$\{x^2y, -xy^2\}_{\rho_2} \oplus \{x^3 \pm iy^3\}$	$\rho_2 + \rho_4^+ + \rho_5^-$
4	$\{y^4, x^4\}_{\rho_3} \oplus \{x^3y, -xy^3\}_{\rho_3}$	$\rho_3^{\oplus 2}$
5	$\{y^5, -x^5\}_{\rho_2} \oplus \{xy(x^3 \pm (-iy^3))\}$	$\rho_2 + \rho_4^+ + \rho_5^-$
6	$\{x^6 - y^6\}_{\rho_1} \oplus \{x^5y, -xy^5\}_{\rho_3}$	$\rho_1 + \rho_3$
7	$\{xy^6, x^6y\}_{\rho_2}$	ρ_2

Coinv. alg. : $\mathbb{C}[x, y]/\mathfrak{n} = \mathbb{C}[x, y]/(F, G, H)$

The exceptional set E,

How to compute E_1, an irred. comp. of E

Consider G-submodules of $\{xy\}_{\rho_2} \oplus \{x^6 - y^6\}_{\rho_2}$

$$I_1(s) := (\{xy + s(x^6 - y^6)\}) + \mathfrak{n} \quad (s \in \mathrm{C})$$

where $\mathfrak{n} := (F, G, H) = (x^6 + y^6, x^2y^2, xy(x^6 - y^6))\mathrm{C}[x, y]$.

We see,

$$\dim \mathrm{C}[x, y]/I_1(s) = 12,$$

$$\therefore \quad I_1(s) \in G\text{-}\mathrm{Hilb}(\mathrm{C}^2) \quad (\forall s \in \mathrm{C})$$

As $s \to \infty$, $\quad E = \pi^{-1}(0)$

If $E \ni I$, we have $I \supset \mathfrak{n} = (F, G, H)$. Hence

$$\mathrm{C}[x, y]/\mathfrak{n} \twoheadrightarrow \mathrm{C}[x, y]/I \text{ (surjective)}$$

$$\therefore E \subset \mathrm{Grass.}(\mathrm{C}[x, y]/\mathfrak{n}, \mathrm{codim}\ 11)$$

Grass is compact, so that the sequence $I_1(s)$ $(s \to \infty)$ converges. Now we define $I_1(\infty)$ by:

$$I_1(\infty)/\mathfrak{n} = \lim_{s \to \infty} I_1(s)/\mathfrak{n}$$

To compute $I_1(\infty) \Longrightarrow$ McKay corresp. appears !

$$I_1(s) = (\{xy + s(x^6 - y^6)\}_{\rho_1}) + \mathfrak{n} \quad (s \neq 0)$$
$$= (\{\tfrac{1}{s}xy + (x^6 - y^6)\}_{\rho_1}) + \mathfrak{n} \quad (s \neq 0)$$

What happens when $s = \infty$? Let $\frac{1}{s} = 0$.

But $I_1(\infty) \neq (\{x^6 - y^6\}_{\rho_1}) + \mathfrak{n}$, A Correct Answer is

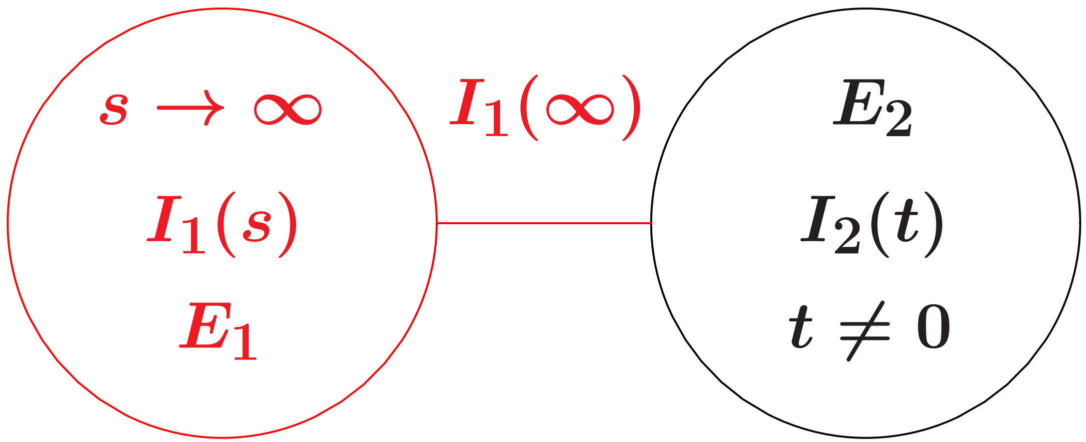
$$I_1(\infty) = (\{x^6 - y^6\}_{\rho_1}) + (\{x^2y, xy^2\}_{\rho_2}) + \mathfrak{n}$$
$$= (\{x^6 - y^6\}_{\rho_1}) + (\{x, y\}_{\rho_2} \cdot \{xy\}_{\rho_1}) + \mathfrak{n}$$

$$V_3(\rho_2) = \{x^2y, xy^2\}_{\rho_2} =$$
$$\{x, y\}_{\rho_2} \cdot \{xy\}_{\rho_1} = \rho_{\mathrm{nat}} \cdot (V_2(\rho_1))$$

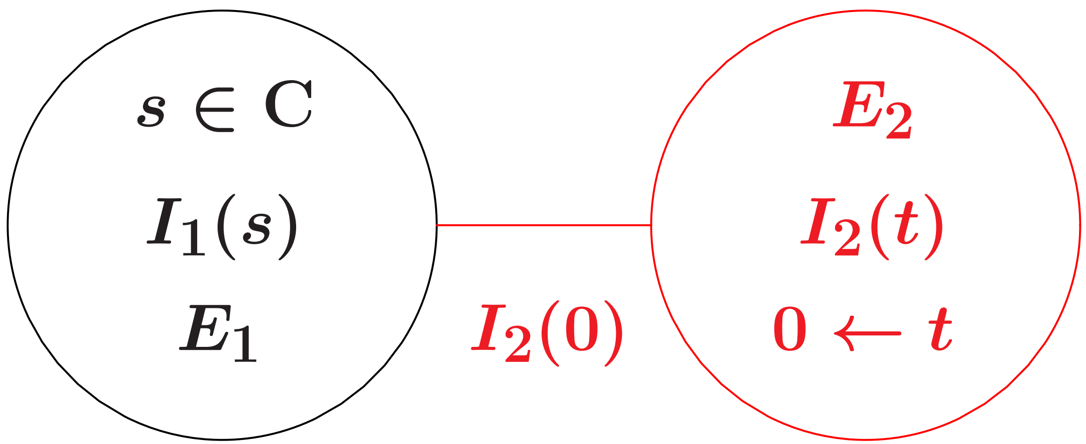
This reminds us of the McKay rule : $\rho_2 = \rho_{\mathrm{nat}} \otimes \rho_1$

$I_1(\infty) = E_1 \cap E_2$ comes out from the McKay rule !

We have just computed



Next we will compute $I_2(0)$ to see $I_1(\infty) = I_2(0)$



Recall $E_i = \{I \in E;\ \rho_i \subset V(I)\} =: E(\rho_i)$.

We saw

$$E(\rho_1) = \{I_1(s); s \in \mathrm{C}\} \cup I_1(\infty)$$

$$= \mathrm{P}(\{xy\}_{\rho_1} \oplus \{x^6 - y^6\}_{\rho_1}) \simeq \mathrm{P}^1$$

(the set of all irred. G**-submod.)**

We will see $I_2(0) = I_1(\infty)$ **and**

$$E(\rho_2) = I_2(0) \cup \{I_2(t); 0 \neq t \in \mathrm{C}\} \cup I_2(\infty)$$

$$= \mathrm{P}(\{x^2y, xy^2\}_{\rho_2} \oplus \{-y^5, x^5\}_{\rho_2}) \simeq \mathrm{P}^1$$

(the set of all irred. G**-submod.)**

$$V_3(\rho_2) = \{x^2y, xy^2\}_{\rho_2},\ \ V_5(\rho_2) = \{-y^5, x^5\}_{\rho_2}$$

Similarly for $t \neq 0$, we define

$$I_2(t) := (x^2y - ty^5, xy^2 + tx^5) + \mathfrak{n}$$

$V(I_2(t)) := I_2(t)/(\mathfrak{m}I_2(t) + \mathfrak{n}) \simeq \rho_2$. Then we see,

$$\begin{aligned} I_2(0) &= (\{x^2y, xy^2\}_{\rho_2}) + (\{x^6 - y^6\}_{\rho_1}) + \mathfrak{n} \\ &= (\{x^2y, xy^2\}_{\rho_2}) + (\{x, y\}_{\rho_2} \cdot \{-y^5, x^5\}_{\rho_2}) + \mathfrak{n} \\ &= I_1(\infty) + \mathfrak{n}, \quad \text{because} \end{aligned}$$

$$I_1(\infty) = (\{x^6 - y^6\}_{\rho_1}) + (\{x^2y, xy^2\}_{\rho_2}) + \mathfrak{n},$$

$$\mathfrak{n} = (F, G, H) = (x^6 + y^6, x^2y^2, xy(x^6 - y^6))$$

$$I_2(0) = (\{x^2y, xy^2\}_{\rho_2}) + (\{x^6 - y^6\}_{\rho_1}) + \mathfrak{n} = I_1(\infty)$$

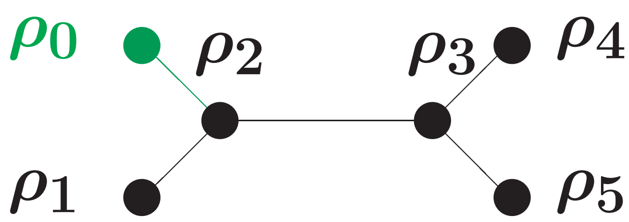
Intersection of E_1 and E_2 !!

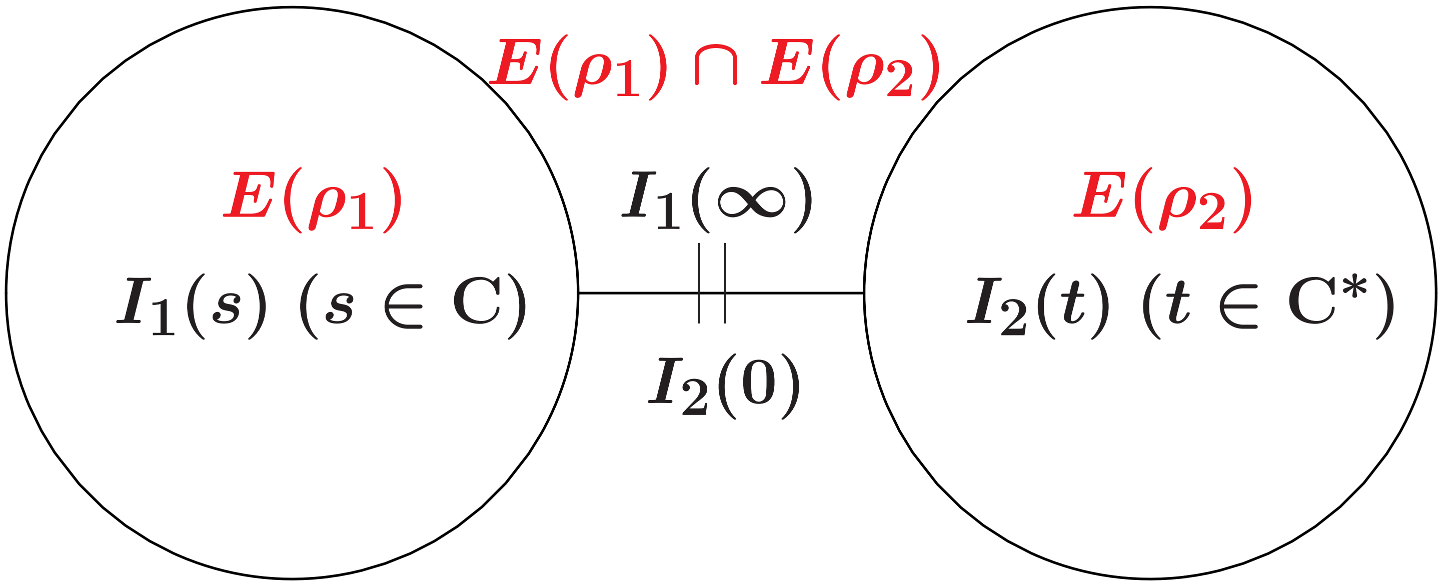
This comes from McKay rule because

$$V_6(\rho_1) = \{x^6 - y^6\}$$
$$= \{x, y\} \cdot \{-y^5, x^5\} \quad \text{mod } \mathfrak{n} + (x^2y, xy^2)$$
$$= \rho_{\text{nat}} \cdot V_5(\rho_2) \quad \text{mod } \mathfrak{n} + (x^2y, xy^2)$$

This reminds us of McKay rule : $\rho_1 + \cdots = \rho_{\text{nat}} \otimes \rho_2$

$\widetilde{D}_5$





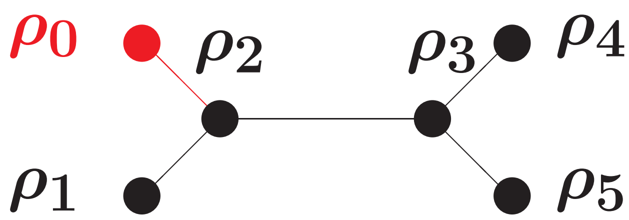
$$V(I_1(s)) \simeq \rho_1,\ V(I_2(t)) \simeq \rho_2,$$

$$V(I_1(\infty)) = V(I_2(0)) \simeq \rho_1 \oplus \rho_2$$

$$s \in \mathrm{C},\ t \in \mathrm{C}\ (s \neq 0)$$

where $V(I) = I/(\mathfrak{m}I + \mathfrak{n})$: generators of I

$\widetilde{D}_5$



$$\rho_2 \otimes \rho_{\mathrm{nat}} = \rho_0 + \rho_1 + \rho_3$$

$$\rho_1 \otimes \rho_{\mathrm{nat}} = \rho_2$$

How does the total Dynkin diag. come out ?

m	$V_m(\rho)$ (: deg m, type ρ)	Eq. class
1	$\{x, y\}_{\rho_2}$	ρ_2
2	$\{xy\}_{\rho_1} \oplus \{x^2, y^2\}_{\rho_3}$	$\rho_1 + \rho_3$
3	$\{x^2y, -xy^2\}_{\rho_2} \oplus \{x^3 \pm iy^3\}$	$\rho_2 + \rho_4^+ + \rho_5^-$
4	$\{y^4, x^4\}_{\rho_3} \oplus \{x^3y, -xy^3\}_{\rho_3}$	$\rho_3^{\oplus 2}$
5	$\{y^5, -x^5\}_{\rho_2} \oplus \{xy(x^3 \pm (-iy^3))\}$	$\rho_2 + \rho_4^+ + \rho_5^-$
6	$\{x^6 - y^6\}_{\rho_1} \oplus \{x^5y, -xy^5\}_{\rho_3}$	$\rho_1 + \rho_3$
7	$\{xy^6, x^6y\}_{\rho_2}$	ρ_2

Decomposition of the coinv. alg. into repres. of $G(D_5)$

Coinv. alg. : $\mathbb{C}[x, y]/\mathfrak{n} = \mathbb{C}[x, y]/(F, G, H)$

Deg. **Rep.**	**1** ρ_2	**2** $\boxed{\rho_1} + \rho_3$	**3** $\boxed{\rho_2} + \rho_4 + \rho_5$	**4** $\rho_3^{\oplus 2}$
Deg. **Rep.**	**7** ρ_2	**6** $\boxed{\rho_1} + \rho_3$	**5** $\boxed{\rho_2} + \rho_4 + \rho_5$	

The Quiver str. of the Red part determ. the Dynkin diag.

Quiver str. of Coinv alg. connects the irred. comp. of E

$$\{x, y\} \cdot \{xy\}_{\rho_1} = \{x^2y, -xy^2\}_{\rho_2} \mod (*),$$

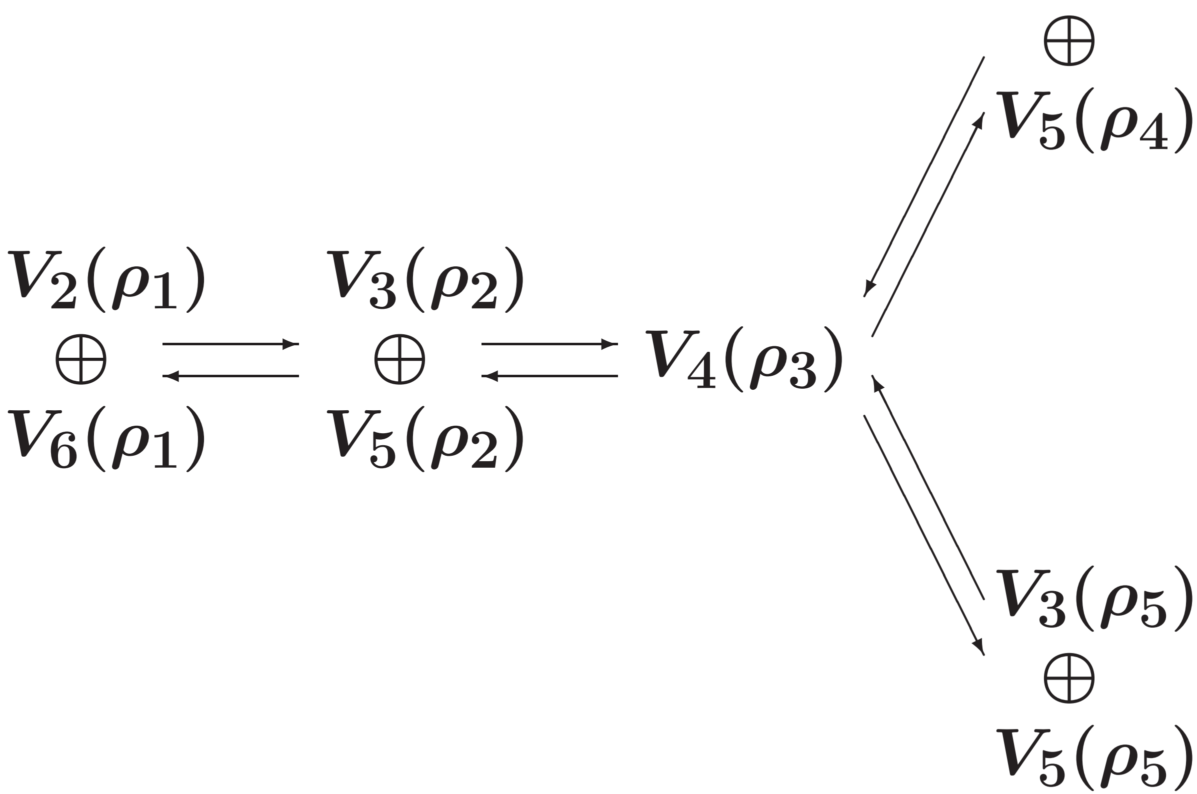
$$\{x, y\} \cdot \{y^5, -x^5\}_{\rho_2} = \{x^6 - y^6\}_{\rho_1} \mod (*)$$

connects $E(\rho_1)$ and $E(\rho_2)$.

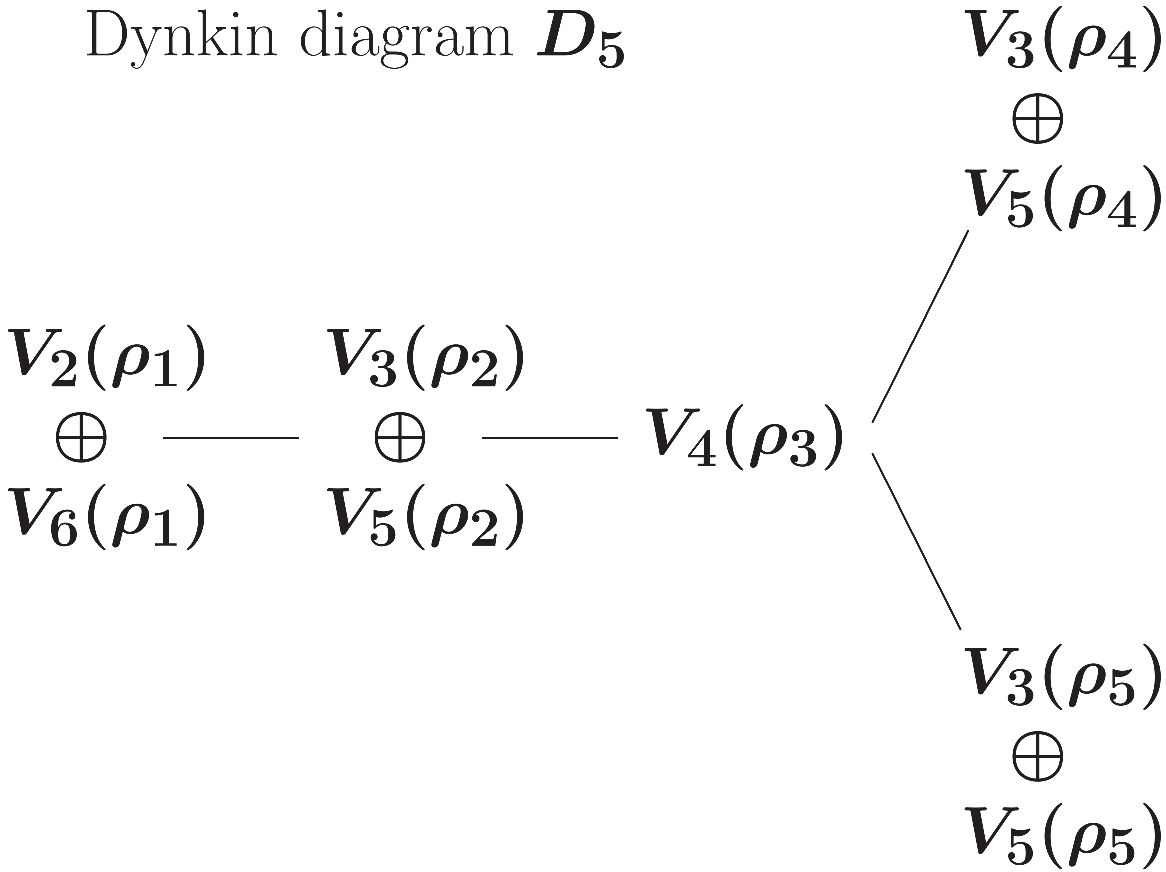
Deg.	1	2	3	4
Rep.		ρ_1	$\rho_2 + \rho_4 + \rho_5$	$\rho_3^{\oplus 2}$
Deg.	7	6	5	
Rep.		ρ_1	$\rho_2 + \rho_4 + \rho_5$	

The McKay Quiver of D_5 in the Coinv alg.

Quiver str. of Red part of $\boldsymbol{D_5}$



Dynkin diagram $\boldsymbol{D_5}$



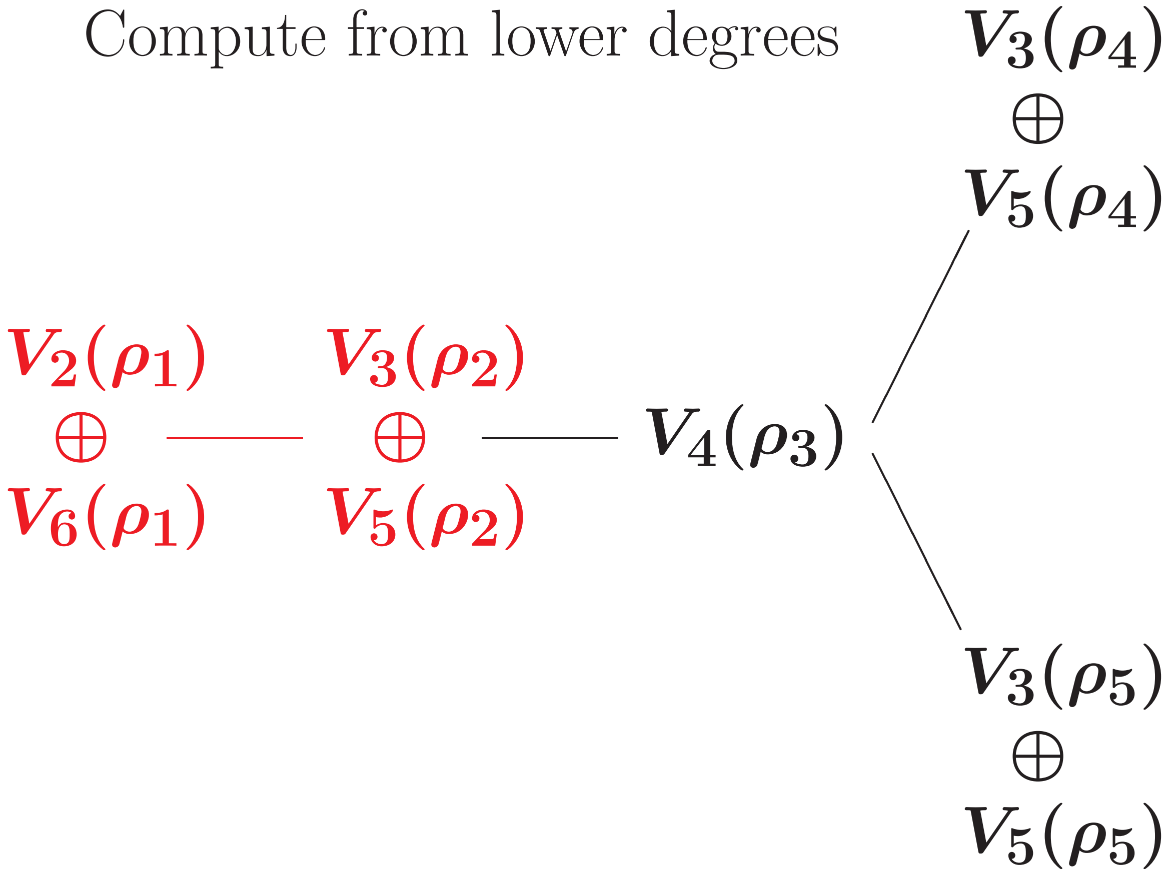
Compute from lower degrees

$$\boldsymbol{V_3(\rho_4) \oplus V_5(\rho_4)}$$

$$\boldsymbol{V_2(\rho_1) \oplus V_6(\rho_1)} \;—\; \boldsymbol{V_3(\rho_2) \oplus V_5(\rho_2)} \;—\; \boldsymbol{V_4(\rho_3)}$$

$$\boldsymbol{V_3(\rho_5) \oplus V_5(\rho_5)}$$

Compute from lower degrees



Compute from lower degrees

$$V_2(\rho_1) \oplus V_6(\rho_1) \;—\; V_3(\rho_2) \oplus V_5(\rho_2) \;—\; V_4(\rho_3)$$

$$V_4(\rho_3) \;—\; V_3(\rho_4) \oplus V_5(\rho_4)$$

$$V_4(\rho_3) \;—\; V_3(\rho_5) \oplus V_5(\rho_5)$$

Compute from lower degrees

$$V_3(\rho_4) \oplus V_5(\rho_4)$$

$$V_2(\rho_1) \oplus V_6(\rho_1) \;—\; V_3(\rho_2) \oplus V_5(\rho_2) \;—\; V_4(\rho_3)$$

$$V_3(\rho_5) \oplus V_5(\rho_5)$$

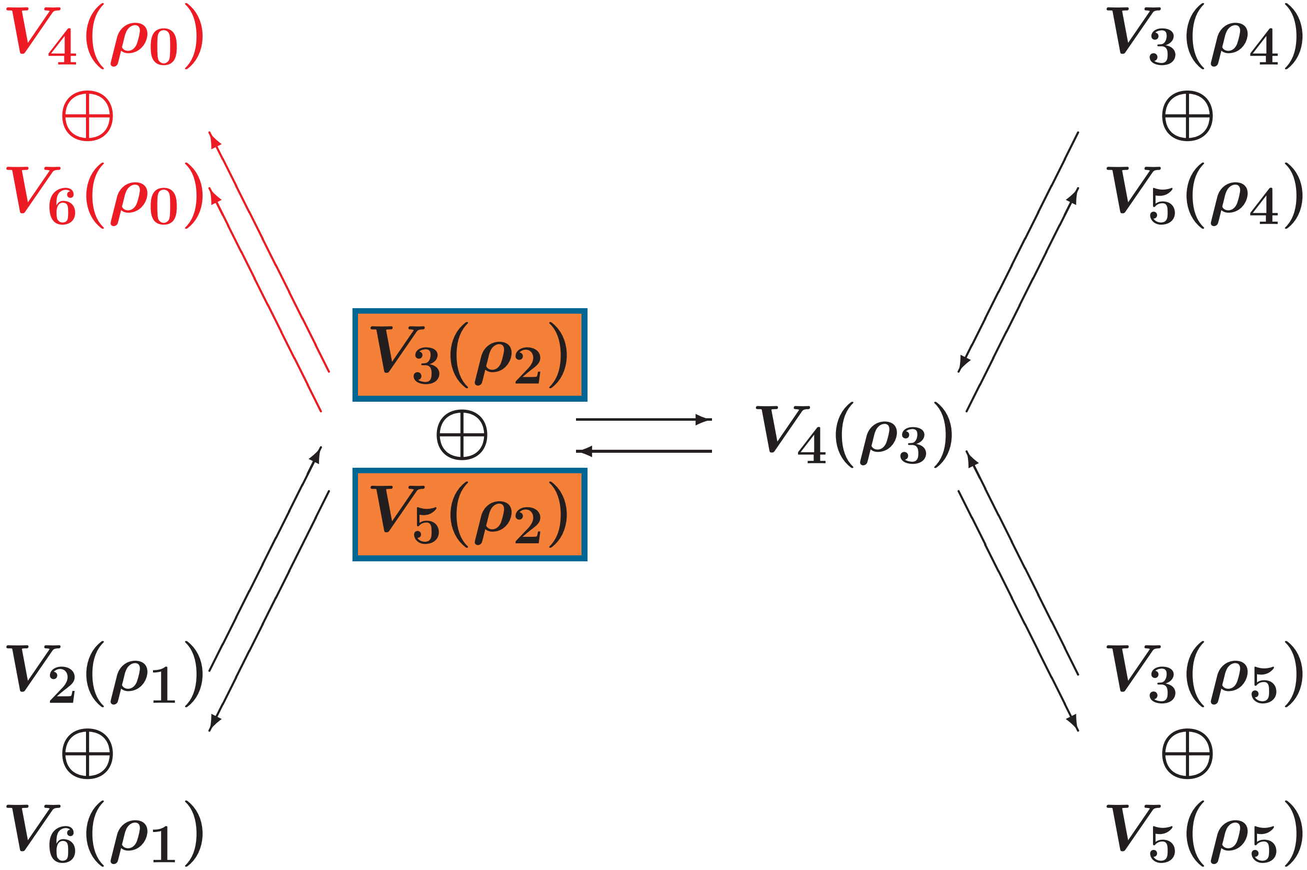
8 The extended Dynkin diagram— D_5-case

$V(I) = I/(\mathfrak{m}I + \mathfrak{n})$ gives the Dynkin diag. D_5.

An extended version of $V(I)$ very roughly

$$V^{\dagger}(I) = I/(\mathfrak{m} + \mathfrak{n})I, \quad (I \in G\text{-}\mathrm{Hilb}).$$

Then $V^{\dagger}(I)$ very roughly gives the extended Dynkin diagram $\widetilde{D}_5$.



$$V_4(\rho_0) = \{x^2y^2\}, \ V_6(\rho_0) = \{x^6 + y^6\}$$

Let X =G-Hilb, minimal resol. of A^2/G. Let $\mathfrak{n}_X$ the ideal defining the graph $X \to (\mathrm{A}^2/G) \times X$.

Theorem [N. 2007+ϵ]

$$\mathcal{V} := I_{\mathrm{univ}}/(\mathfrak{m} I_{\mathrm{univ}} + \mathfrak{n}_X) \simeq \bigoplus_{i=1}^{r} O_{E(\rho_i)}(-1) \otimes \rho_i$$

$$\mathcal{V}^\dagger := I_{\mathrm{univ}}/(\mathfrak{m} + \mathfrak{n}_X) I_{\mathrm{univ}} \simeq \bigoplus_{i=0}^{r} O_{E(\rho_i)}(-1) \otimes \rho_i$$

D_5-case

Let $E(\rho_0) = -E_{\mathrm{fund}}$, $E(\rho_0)$ fits the extended Dynkin.

$L := O_{E(\rho_0)}(-1) := O_{E_{\mathrm{fund}}}(1)$

$L \otimes O_{E(\rho_2)} = O_{E(\rho_2)}(1)$, $L \otimes O_{E(\rho_i)} =$ **trivial** $(i \neq 2)$.

$E(\rho_0) := -E_{\text{fund}}$, $O_{E(\rho_0)}(L) := O_{E_{\text{fund}}}(-L)$

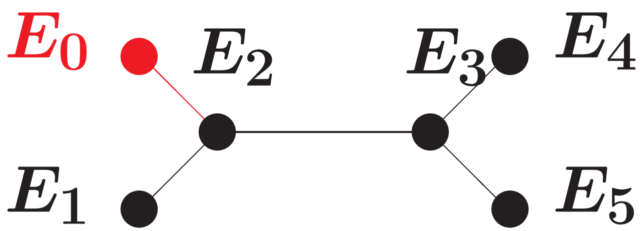
$O_{E(\rho_0)}(-1) := O_{E_{\text{fund}}}(1)$, of degree one on an irred. comp. and trivial elsewhere

D_5-case $\quad E_0 = -E_{\text{fund}}$

$$E_{\text{fund}} = E_1 + 2E_2 + 2E_3 + E_4 + E_5$$
$$= E(\rho_1) + 2E(\rho_2) + 2E(\rho_3) + E(\rho_4) + E(\rho_5)$$

— $\dim \rho_2 = \dim \rho_3 = 2$, $\dim \rho_1 = \dim \rho_4 = \dim \rho_5 = 1$

$\widetilde{D}_5$



9 Exceptional set— E_6-case

$$G(E_6) \subset \mathrm{SL}(2, \mathbb{C}) \text{ (Binary Tetrahedral Group)}$$

$$G(E_6) = \langle \sigma, \tau, \mu \rangle \text{ order } 24$$

$$\sigma = \begin{pmatrix} i, & 0 \\ 0, & -i \end{pmatrix}, \quad \tau = \begin{pmatrix} 0, & 1 \\ -1, & 0 \end{pmatrix}, \quad \mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \epsilon^7, & \epsilon^7 \\ \epsilon^5, & \epsilon \end{pmatrix},$$

where $\epsilon = e^{2\pi i/8}$.

$$G(D_4) := \langle \sigma, \tau \rangle, \text{ normal in } G(E_6)$$

$$1 \to G(D_4) \to G(E_6) \to \mathbb{Z}/3\mathbb{Z} \to 1.$$

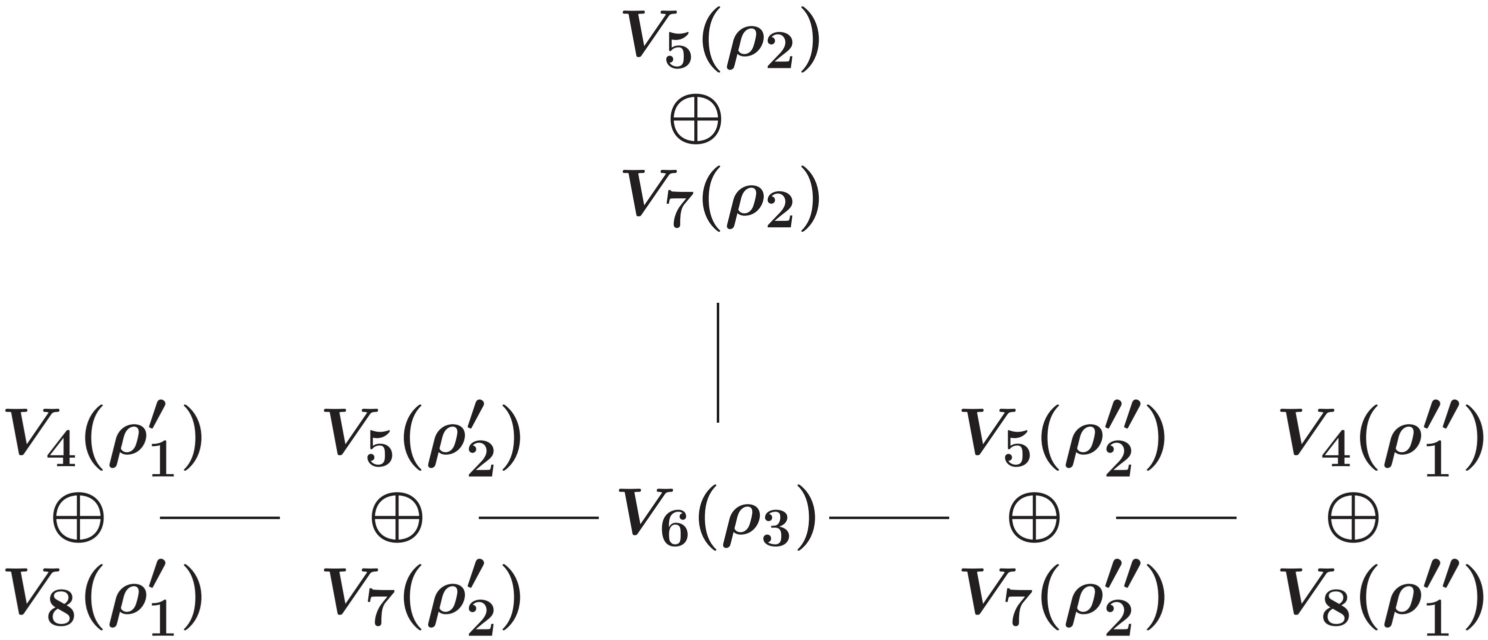
$$\mathbb{C}^2/G(D_4) \xrightarrow{3:1} \mathbb{C}^2/G(E_6)$$

Irred. rep. of E_6 (irred. character)

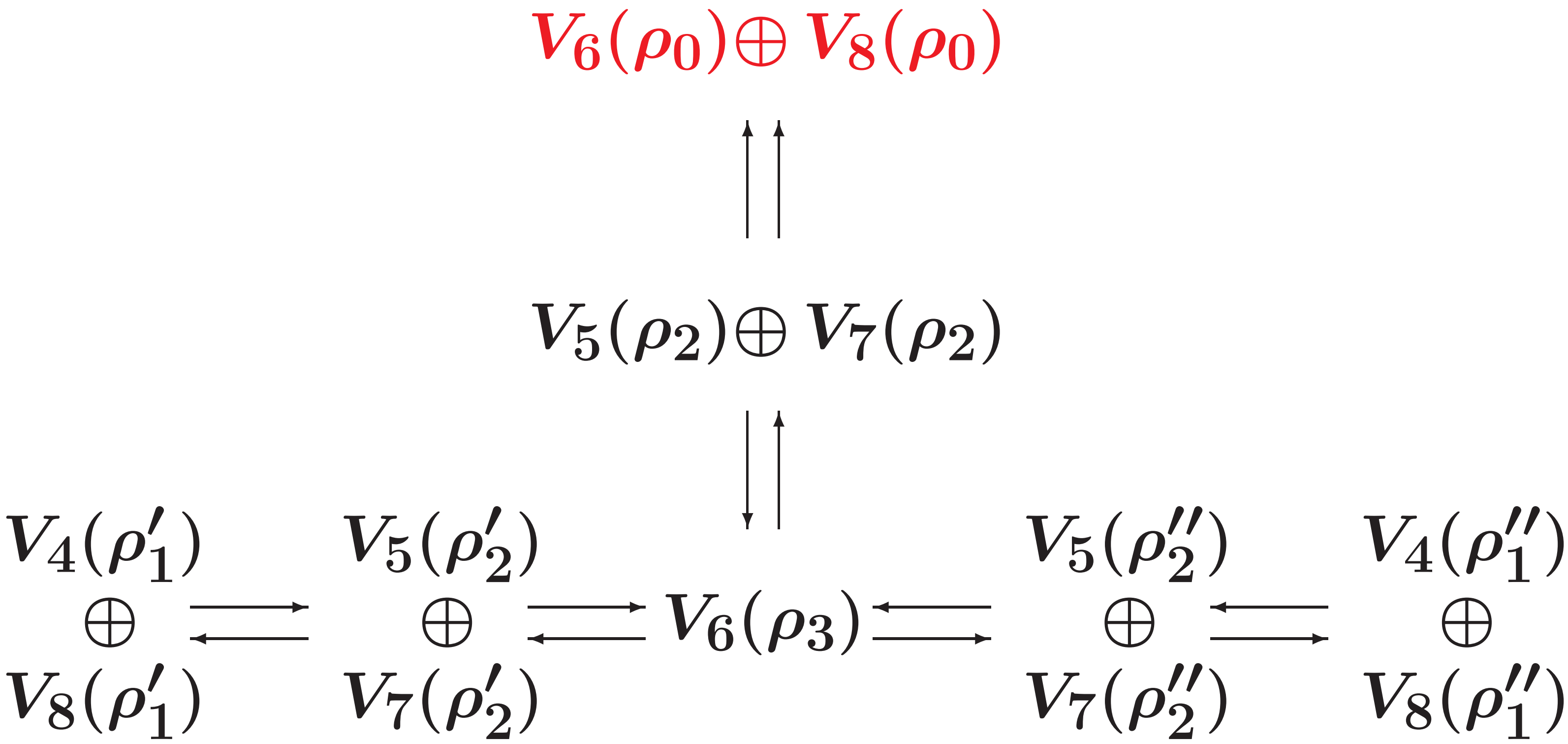
	1	-1	τ	μ	μ^2	μ^4	μ^5
$(\sharp)$	1	1	6	4	4	4	4
ρ_0	1	1	1	1	1	1	1
ρ_2	2	-2	0	1	-1	-1	1
ρ_3	3	3	-1	0	0	0	0
ρ_2'	2	-2	0	ω^2	$-\omega$	$-\omega^2$	ω
ρ_1'	1	1	1	ω^2	ω	ω^2	ω
ρ_2''	2	-2	0	ω	$-\omega^2$	$-\omega$	ω^2
ρ_1''	1	1	1	ω	ω^2	ω	ω^2

Coinv. alg. of E_6

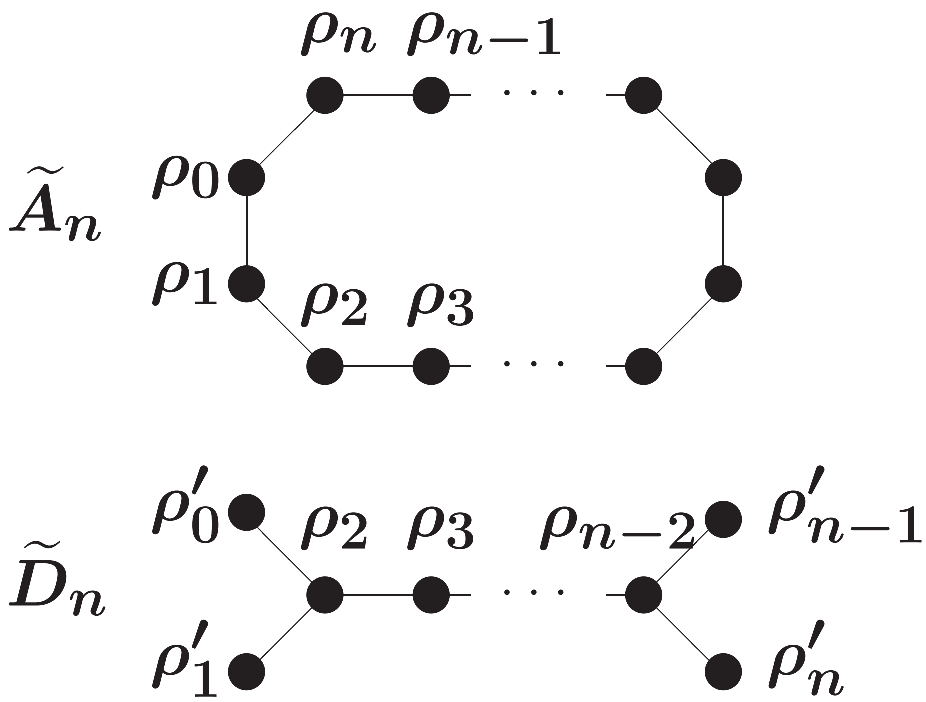
m	$\bar{V}_m$
1	ρ_2
2	ρ_3
3	$\rho_2' + \rho_2''$
4	$\rho_1' + \rho_1'' + \rho_3$
5	$\rho_2 + \rho_2' + \rho_2''$
6	$2\rho_3$
7	$\rho_2 + \rho_2' + \rho_2''$
8	$\rho_1' + \rho_1'' + \rho_3$
9	$\rho_2' + \rho_2''$
10	ρ_3
11	ρ_2

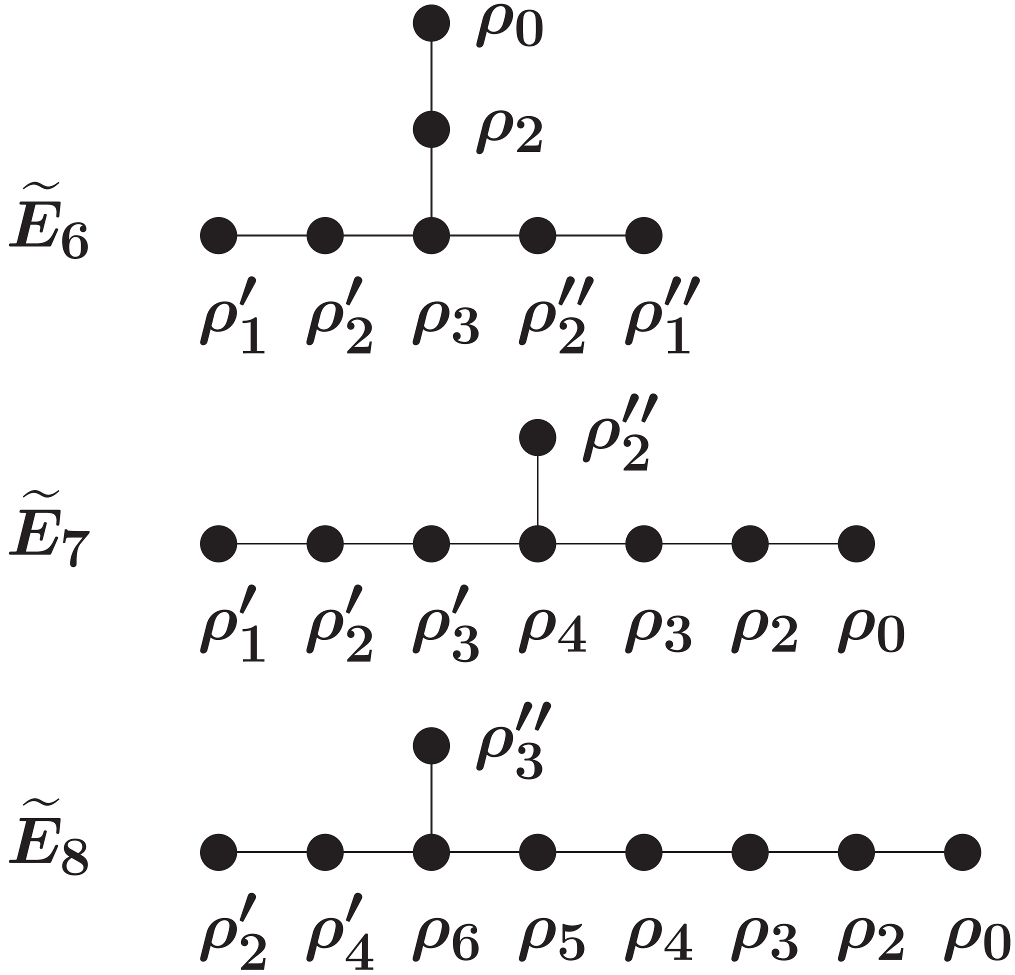


Dynkin diagram $\boldsymbol{E_6}$



The extended McKay quiver of $\boldsymbol{E_6}$


$\widetilde{A}_n$
ρ_n
ρ_{n-1}
ρ_0
ρ_1
ρ_2
ρ_3
$\widetilde{D}_n$
ρ'_0
ρ_2
ρ_3
ρ_{n-2}
ρ'_{n-1}
ρ'_1
ρ'_n


ρ_0
ρ_2
$\widetilde{E}_6$
ρ_1' ρ_2' ρ_3 ρ_2'' ρ_1''
ρ_2''
$\widetilde{E}_7$
ρ_1' ρ_2' ρ_3' ρ_4 ρ_3 ρ_2 ρ_0
ρ_3''
$\widetilde{E}_8$
ρ_2' ρ_4' ρ_6 ρ_5 ρ_4 ρ_3 ρ_2 ρ_0

10 Summary

For $I \in E$: (except. set), we define

$$V(I) := I/(\mathfrak{m}I + \mathfrak{n})$$

Then $V(I)$ is either irred. or $\rho \oplus \rho'(\rho, \rho'$ irred.)

For ρan irred. rep. of G, define

$$E(\rho) := \{I \in E; V(I) \supset \rho\}$$

Thm 5 (Ito-Nakamura 1999)

Let G be a finite subgroup of $\mathrm{SL}(2, \mathrm{C})$. Then

(1) G-Hilb(C^2) is a min. resol. of C^2/G.

(2) For any irred rep. ρ of G, $\rho \neq$ trivial,

$E(\rho) = \mathrm{P}(\rho \oplus \rho) = \mathrm{P}^1$, $\rho \mapsto E(\rho)$ is McKay corresp.

(3) Quiver str. of Coninv. alg.

=Decomp. rule of $\rho_{\mathrm{nat}} \otimes$(rep.) into irred. rep.

=Dynkin diagram

Thm 6 (a refined version of Th. 5, [N.2007+ϵ])

(4) Let $I_{\rm univ}$ be the ideal of the universal G-orbit scheme $Z_{\rm univ}$ over $X := G\text{-}\mathrm{Hilb}(\mathrm{A}^2)$. Let $\mathfrak{m}$ be the maximal ideal of $O \in \mathrm{A}^2$, $\mathfrak{n}_X$ is the ideal of the graph of $X \to (\mathrm{A}^2/G) \times X$. Then

(i) $\mathcal{V} := I_{\rm univ}/(\mathfrak{m} I_{\rm univ} + \mathfrak{n}_X) \simeq \bigoplus_{i=1}^{r} O_{E_i}(-1) \otimes \rho_i$

(ii) $\mathcal{V}^{\dagger} := I_{\rm univ}/(\mathfrak{m} + \mathfrak{n}_X) I_{\rm univ} \simeq \bigoplus_{i=0}^{r} O_{E_i}(-1) \otimes \rho_i$

(iii) The corresp. $E_i \leftrightarrow \rho_i$ is McKay

E_0 is minus the fundamental cycle, ρ_0 is triv.rep.

Thm 6 [N. 2007+ϵ]

$$\mathcal{V} := I_{\mathrm{univ}}/(\mathfrak{m}I_{\mathrm{univ}} + \mathfrak{n}_X) \simeq \bigoplus_{i=1}^{r} O_{E(\rho_i)}(-1) \otimes \rho_i$$

$$\mathcal{V}^{\dagger} := I_{\mathrm{univ}}/(\mathfrak{m} + \mathfrak{n}_X)I_{\mathrm{univ}} \simeq \bigoplus_{i=0}^{r} O_{E(\rho_i)}(-1) \otimes \rho_i$$

D_5-case

Let $E(\rho_0) = -E_{\mathrm{fund}}$, Then $E(\rho_0)$ fits to the extended Dynkin diagram.

$$L := O_{E(\rho_0)}(-1) := O_{E_{\mathrm{fund}}}(1)$$

$L \otimes O_E(\rho_2) = O_{E(\rho_2)}(1)$, $L \otimes O_{E(\rho_i)} =$ trivial $(i \neq 2)$.

End

Thank you for your attention.

Why ?? $\mathcal{V}^{\dagger} \simeq \bigoplus_{i=0}^{r} O_{E(\rho_i)}(-1) \otimes \rho_i$

Let $\phi_3 = \frac{s_3^2 t_3}{1+t_3^2}$. **A universal (local) deformation** $\mathcal{I}_{\mathrm{univ}}$ **is the ideal generated by**

$$D_1 := y^5 + s_3 x^2 y + \phi_3 x, \; D_2 := -x^5 - s_3 x y^2 + \phi_3 y,$$

$$E_1 := x^3 y + t_3 y^4 + s_3 t_3 x^2, \; E_2 := -x y^3 + t_3 x^4 + s_3 t_3 y^2,$$

$$A_6 + 2 s_3 A_4, \; A_8 - 2\phi_3 A_4, \; A_4 - t_3 \phi_3.$$

In $\mathcal{V}^\dagger$, we have

$$t_3D_1 = yE_1 - x(A_4 - t_3\phi_3) = 0, \ t_3D_2 = \tau(t_3D_1) = 0,$$

$$s_3E_1 = xD_1 + y^2E_2 - t_3x^2(A_4 - t_3\phi_3) = 0, \ s_3E_2 = \tau(s_3E_1)$$

$$t_3\phi_3(A_4 - t_3\phi_3) = A_4(A_4 - t_3\phi_3) = 0,$$

Hence

$$\mathcal{V}^\dagger \simeq O_{E(\rho_2)}^{\oplus 2} \oplus O_{E(\rho_3)}^{\oplus 2} \oplus O_{2E(\rho_2)+2E(\rho_3)}(L),$$

where L is a line bundle with $L_0E(\rho_0) = -1$, $E(\rho_0) = -E(\rho_1) - 2E(\rho_2) - 2E(\rho_3) - E(\rho_4) - E(\rho_5)$. The divisor $(t_3\phi_3)$ is equal to $2E(\rho_2) + 2E(\rho_3) = -E(\rho_0)$ near here.