COMPACTIFICATION OF THE MODULI SPACE OF ABELIAN VARIETIES
KYOTO, 2013 JUNE 11-13

IKU NAKAMURA

Abstract. The moduli space of nonsingular projective curves of genus g is compactified by adding Deligne-Mumford stable curves of genus g, a class of mildly degenerate curves. The moduli space of stable curves is a projective variety, known as Deligne-Mumford compactification. We compactify in a similar way the moduli space of abelian varieties as the moduli space of some mildly degenerating limits of abelian varieties.

A typical case is the moduli space of Hesse cubics. Any Hesse cubic is GIT-stable, and any GIT stable planar cubic is one of Hesse cubics. Similarly in arbitrary dimension, the moduli space of abelian varieties is compactified by adding only GIT-stable limits of abelian varieties. Our moduli space is a projective "fine" moduli space of (possibly degenerate) abelian schemes

with non-classical (non-commutative) level structure

over $\mathbf{Z}[\zeta_N, 1/N]$ for some $N \geq 3$. The objects at the boundary are mild limits of abelian varieties, which we call PSQASes, projectively stable quasi-abelian schemes.

Contents

A reference for this talk is [N04].

Date: June 7, 2013.

1. Introduction

Roughly our problem is the following diagram completion :

The Deligne-Mumford compactification completes the following diagram

the moduli of smooth curves

= the set of all isomorphism classes of smooth curves

$\subset$ the set of all isomorphism classes of stable curves

= the Deligne-Mumford compactification M_g

Therefore our problem is to complete the following diagram :

the moduli of smooth AVs (= abelian varieties)

= {smooth polarized AVs + extra structure}/ isom.

$\subset$ {smooth polarized AVs or

singular polarized degenerate AVs + extra structure}/ isom.

= the new compactification $SQ_{g,K}$ of the moduli of AVs

The compactification problem of the moduli space of abelian varities have been discussed by many people

1. Satake compactification, Igusa monoidal transform of it
2. Mumford toroidal compactification (Ash-Mumford-Rapoport-Tai [AMRT75])
3. Faltings-Chai arithmetic compactification (arithmetic version of Mumford compactification) [FC90]

There are many compactifications, but no canonical choice except Satake. These are compactification as spaces, not as the moduli of compact objects.

We wish to construct *a unique canonical compactification, separated and proper,* of course, more desirably projective, *as the fine/coarse moduli of compact geometric objects :* thereby

1. proper = to collect suff. many limits
2. separated = to choose the minimum possible among the above
3. both are necessary for compactification

2. Hesse cubics

2.1. **Hesse cubics.** Let k be a closed field of chara. $\neq 3$. A Hesse cubic curve is defined by

$$C(\mu)\ :\ x_0^3+x_1^3+x_2^3-3\mu x_0x_1x_2=0 \tag{1}$$

for some $\mu\in k$, or $\mu=\infty$ (in which case we understand that $C(\infty)$ is the curve defined by $x_0x_1x_2=0$).

1. $C(\mu)$ is nonsingular elliptic for $\mu\neq\infty,1,\zeta_3,\zeta_3^2$, where ζ_3 is a primitive cube root of unity.
2. $C(\mu)$ is a 3-gon for $\mu=\infty,1,\zeta_3,\zeta_3^2$
3. any elliptic $C(\mu)$ has 9 inflection points(=flexes), independent of μ,

$$K := 9 \text{ flexes}$$

say, $(0,1,-\zeta_3^k)$, $(-\zeta_3^k,0,1)$, $(1,-\zeta_3^k,0)$, Note $K\subset C(\mu)$ ($\forall\ \mu$),

4. over $\mathbf{C}$, any Hesse cubic is the image of $E(\omega) := \mathbf{C}/\mathbf{Z}+\mathbf{Z}\omega$, a complex torus by thetas

$$x_k = \theta_k(q,w) = \sum_{m\in\mathbf{Z}} e^{2\pi i(3m+k)^2\omega/6}e^{2\pi i(3m+k)z}$$
$$= \sum_{m\in\mathbf{Z}} q^{(3m+k)^2}w^{3m+k}$$

where $q = e^{2\pi i\omega/6}$, $w = e^{2\pi iz}$.

Then K is the image of $\ker(3 : E(\omega) \to E(\omega)) = \langle \frac{1}{3}, \frac{\omega}{3}\rangle$.

2.2. The moduli space of Hesse cubics — the Stone-age (Neolithic) level structure. Consider the moduli space of Hesse cubics.

1. the moduli space $SQ_{1,3}$:=the set of isom. classes of $(C(\mu), K)$,
   where the definition of an isom.
   $(C(\mu), K) \simeq (C(\mu'), K)$: isom. iff
   $\exists\ f : C(\mu) \to C(\mu')$: an isom. with $f_{|K} = \mathrm{id}_K$,
   This extra condition $f_{|K} = \mathrm{id}_K$ for isom. is the classical level str.,
2. if $(C(\mu), K) \simeq (C(\mu'), K)$, then $\mu = \mu'$, because the isom is given by a matrix A, whose eigenvectors are K with $|K| = 9$, hence easy to prove A is scalar.
3. $SQ_{1,3} \simeq \mathbf{P}^1$, in fact, $SQ_{1,3} \simeq X(3)$ modular curve over $\mathbf{Z}[\zeta_3, 1/3]$, This compatifies $A_{1,3} := \{(C(\mu), K); C(\mu)\text{smooth}\} = \mathbf{P}^1 \setminus \{4 \text{ points}\}$.

2.3. The moduli space of smooth cubics — classical level structure. Consider the moduli space of Hesse cubics.

1. the moduli space $A_{1,3}$:=the set of isom. classes of $(C, C[3], \iota)$,
   where C a smooth cubic, $C[3]$ the 3-division points,

$$\iota : (C[3], e_C) \to (K, e_K) \quad \text{a symplectic isom,}$$

   where e_C Weil pairing of C, that is,

$$e_C : C[3] \times C[3] \to \mu_3 \quad \text{alternating nondeg.}$$

   and $K = (\mathbf{Z}/3\mathbf{Z})^{\oplus}$, $e_K(e_1, e_2) = \zeta_3$, e_i stand. basis of K, e_K alt.,
2. the definition of an isom.
   $(C, C[3], \iota) \simeq (C', C'[3], \iota')$: isom. iff
   $\exists\ f : C \to C'$: isom. , $f_{|C[3]} : C[3] \to C'[3]$ isom. $\iota' \cdot f = \iota$,

   This extra condition $f_{|C[3]} : C[3] \to C'[3]$ isom such that $\iota' \cdot f = \iota$ for isom. is the classical level str.,
3. Note that $(C(\mu), C(\mu)[3], \mathrm{id}_K) \in A_{1,3}$ because $C(\mu)[3] = K$,
4. any $(C, C[3], \iota) \simeq (C(\mu), C(\mu)[3], \mathrm{id})$ for some μ,
5. if $(C(\mu), C(\mu)[3], \mathrm{id}) \simeq (C(\mu'), C(\mu')[3], \mathrm{id})$, then $\mu = \mu'$, because f an isom satisfies $\mathrm{id}_K \cdot f = \mathrm{id}_K$, hence $f = \mathrm{id}_K$,Neolithic isom, $\mu = \mu'$,
6. This proves $A_{1,3} := \{(C(\mu), K); C(\mu)\text{smooth}\} = \mathbf{P}^1 \setminus \{4 \text{ points}\}$, hence what to add to $A_{1,3}$ are 3-gons.

3. Non-commutative level structure

Remark 3.1. If we stick to the definition of classical level structure

$$K = C[3] \subset C,$$

we will have nonseparated moduli in higher dimension.

Instead we consider the actions of (K and) $G(K)$ on C and L.

3.2. **Non-commutative interpretation of Hesse cubics.** Interpret the theory of Hesse cubics as follows: Fix $O = [0, 1, -1] \in C(\mu)$.

1. C: any smooth cubic, $L := O_C(1)$ hyperplane bundle,
   Let $\lambda(L) : C \to C^\vee := \mathrm{Pic}\ ^0(C) \simeq C$ be the map $x \to T_x^* L \otimes L^{-1}$,
   Then $K :=$ 9 flexes $= \ker(\lambda(L))$ if $C = C(\mu)$, where $\lambda(L) = 3\,\mathrm{id}_C$,
2. $K := \ker \lambda(L) \simeq (\mathbf{Z}/3\mathbf{Z})^{\oplus 2}$ with Weil pairing e_K (alt. nondeg.)
3. any T_x ($x \in K$), translation by $x \in K$, is lifted to $\gamma_x \in G(K) \subset \mathrm{GL}(3)$: a lin. transf. of $\mathbf{P}^2$,
4. translation by $1/3$ is lifted to σ
   (Recall that x_k is theta)
   $\theta_k(z + 1/3) = \zeta_3^k \theta_k(z)$
5. translation by $1/3$ is lifted to τ
   $[\theta_0, \theta_1, \theta_2](z + \omega/3) = [\theta_1, \theta_2, \theta_0](z)$
6. $\sigma(x_k) = \zeta_k x_k$, $\tau(x_k) = x_{k+1}$.
7. $[\sigma, \tau] = \zeta_3$, not commute,
8. $G(3) := \langle \sigma, \tau \rangle$ a finite group of order 27,
9. $H^0(C, L) = \{x_0, x_1, x_2\}$
   is an irreducible $G(3)$-module of weight one,
   "weight one" means that $a \in \mu_3$ (center) acts as $a\,\mathrm{id}_V$,
10. the action of $G(3)$ on $H^0(C, L)$ is a special case of more general Schrödinger representations,
11. Matrix forms

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^2 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\sigma\tau = \begin{pmatrix} 0 & 0 & 1 \\ \zeta_3 & 0 & 0 \\ 0 & \zeta_3^2 & 0 \end{pmatrix}, \quad \tau\sigma = \begin{pmatrix} 0 & 0 & \zeta_3^2 \\ 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \end{pmatrix}$$

Definition 3.3. $G(K) = G_H$: Heisenberg group;

U_H : Schrödinger representation

$$K = H \oplus H^\vee, H \text{ finite abelian}, N = |H|$$
$$H = H(e), H(e) = \oplus_{i=1}^{g} (\mathbf{Z}/e_i\mathbf{Z}), e_i | e_{i+1}, e_{\min}(K) := e_1,$$
$$G_H = \{(a, z, \alpha); a \in \mu_N, z \in H, \alpha \in H^\vee\},$$
$$(a, z, \alpha) \cdot (b, w, \beta) = (ab\beta(z), z + w, \alpha + \beta),$$
$$V := V_H = \mathcal{O}[H^\vee] = \bigoplus_{\mu \in H^\vee} \mathcal{O} v(\mu),$$
$$(a, z, \alpha) v(\gamma) = a\gamma(z) v(\alpha + \gamma)$$

where $\mathcal{O} = \mathcal{O}_N = \mathbf{Z}[\zeta_N, 1/N]$.

The action of $G(K)$ on V is denoted U_H.

In the Hesse cubics case, $\mathcal{O} := \mathbf{Z}[\zeta_3, 1/3]$, $H = H^\vee = \mathbf{Z}/3\mathbf{Z}$, we identify $G(3)$ with $G(K)$:

$$\sigma = (1,1,0), \tau = (1,0,1) \in G(K), N = 3.$$
$$V_H = \mathcal{O}[H^\vee] = \mathcal{O} \cdot v(0) \oplus \mathcal{O} \cdot v(1) \oplus \mathcal{O} \cdot v(2)$$

3.4. **New formulation of the moduli problem.**

1. classical level 3 str. = Fix the 3-division points K
2. new level 3 str.=Fix the matrix form of $G(K)$ on $V \simeq H^0(C, L)$
3. Let C: any smooth cubic, $L = O_C(1)$, Then the pair (C, L) always has a $G(K)$-action τ

Definition 3.5. For C any cubic with $L = O_C(1)$, (C, ψ, τ) is a level-$G(K)$ structure if

1. τ is a $G(K)$-action on the pair (C, L),
2. $\psi : C \to \mathbf{P}(V_H) = \mathbf{P}^2$ is the inclusion (it is a $G(K)$-equivariant closed immersion by τ)

Define : $(C, \psi, \tau) \simeq (C', \psi', \tau')$ isom. iff
$\exists\ (f, F) : (C, L) \to (C', L')$ $G(K)$-isom. with $\psi' \cdot f = \psi$
(This is equivalent to $f_{|K} = \mathrm{id}_K$ in the classical case.)

Lemma 3.6. *Any (C, ψ, τ) is isom. to a unique Hesse cubic $(C(\mu), i, U_H)$.*

Proof. Suppose $(C(\mu), i, U_H) \simeq (C(\mu'), i, U_H)$. Let $h : C(\mu) \to C(\mu')$ the $G(K)$-isom, hence $h \in \mathrm{GL}(3)$ such that

$$hU_H(g) = U_H(g)h \quad \text{for } \forall g \in G(K).$$

Since U_H is irred, hence h is scalar by Schur's lemma. Hence $h = \mathrm{id} \in \mathrm{PGL}(3)$, $C(\mu) = C(\mu')$, $\mu = \mu'$. □

Proposition 3.7. *Over a closed field of char. $\neq 3$,*

$$\begin{aligned} SQ_{1,3} :&= \{(C, \psi, \tau)\}/isom \\ &= \{(C(\mu), i, U_H)\}/isom = \{\mu \in \mathbf{P}^1\} = \{(C(\mu), K)\} \end{aligned}$$

In other words,

$\{$*cubic with new level 3-structure*$\} = \{$*cubic with classical level 3-structure*$\}$

We call this new level 3-structure *level-$G(K)$ structure*.

This is the noncommutative level structure that we can generalize into higer dimension.

4. PSQAS and TSQAS

Our goal of constructing a compactification is achieved by

1. finding limit objects PSQAS and TSQAS (Theorem 4.2)
2. constructing the moduli $SQ_{g,K}$ as a projective scheme (Section 7)
3. proving that any point of $SQ_{g,K}$ is the isom. class of a PSQAS (Q, ϕ, τ) with level-$G(K)$ str. (Theorem 4.3)

4.1. **Limit objects.** First we note

• Any PSQAS is a scheme-theoretic limit of the images of AV by theta functions. It is also a compactification of a generalized Tate curve.

Let R be a CDVR, and $k(\eta)$ the fraction field of R. We start with an abelian scheme $(G_\eta, \mathcal{L}_\eta)$ and a polarization morphism $\lambda(\mathcal{L}_\eta) : G_\eta \to G_\eta^t$. Let $K_\eta = \ker(\mathcal{L}_\eta)$ the finite group scheme, and $\mathcal{G}(K_\eta) := \mathrm{Aut}(\mathcal{L}_\eta/G_\eta)$: the autom. gp of the pair $(G_\eta, \mathcal{L}_\eta)$ linear in the fibers of $\mathcal{L}_\eta$ over G_η.

For simplicity, we assume the characteristic of $k(0) = R/m_R$ is prime to rank K_η. Then there exists a finite symplectic abelian group K such that $K_\eta \simeq K$ and $\mathcal{G}(K_\eta) \simeq \mathcal{G}(K)$ by some base change

$$1 \to \mathbf{G}_m \to \mathcal{G}(K) \to K \to 0 \quad \text{(exact)}$$

Theorem 4.2. (A refined version of Alexeev-Nakamura's stable reduction theorem) ([AN99], [N99]) *For an abelian scheme $(G_\eta, \mathcal{L}_\eta)$ and a polarization morphism $\lambda(\mathcal{L}_\eta) : G_\eta \to G_\eta^t$ over $k(\eta)$, there exist proper flat projective schemes $(Q, \mathcal{L}_Q)$ (PSQAS) and $(P, \mathcal{L}_P)$ (TSQAS) over R, by a finite base change if necessary, such that*

(o) $(Q_\eta, \mathcal{L}_\eta) \simeq (P_\eta, \mathcal{L}_\eta) \simeq (G_\eta, \mathcal{L}_\eta)$,
(i) *$(P, \mathcal{L}_P)$ is the normalization of $(Q, \mathcal{L}_Q)$,*
(ii) *P_0 is reduced,*
(iii) *if $e_{\min}(K) \geq 3$, then $\mathcal{L}_Q$ is very ample, and in general, $(Q, \mathcal{L}_Q)$ is an étale quotient of some PSQAS $(Q^*, \mathcal{L}_{Q^*})$ with $\mathcal{L}_{Q^*}$ very ample,*
(iv) *$\mathcal{G}(K)$ acts on $(Q, \mathcal{L}_Q)$ and $(P, \mathcal{L}_P)$ extending the action of it on $(G_\eta, \mathcal{L}_\eta)$.*

- The above theorem proves that *the moduli is proper*,
- $(Q_0, \mathcal{L}_0)$: PSQAS — projectively stable quasi-abelian scheme,
- $(P_0, \mathcal{L}_0)$: TSQAS — torically stable quasi-abelian scheme (= variety),
- In dim. one, any PSQAS=TSQAS is a smooth elliptic or an N-gon,
- The next theorem proves that *the moduli is separated.*

Theorem 4.3. ([N99],[N10],[N13]) *Suppose $e_{\min}(K) \geq 3$. Then $(Q, \mathcal{L})$ and $(P, \mathcal{L})$ are uniquely determined by $(G_\eta, \mathcal{L}_\eta)$.*

5. PSQAS and TSQAS in low dimension

5.1. **Hesse cubics and thetas.** Now we calculate the limit of $[\theta_0, \theta_1, \theta_2]$ as $q \to 0$.

Let R be a CDVR and $I = qR$. Then the power series θ_k converge I-adically: First we shall show a rather strange computation, which may embarass you.

$$\begin{aligned}
\theta_0(q,w) &= \sum_{m\in\mathbf{Z}} q^{9m^2}w^{3m}\\
&= 1+q^9w^3+q^9w^{-3}+q^{36}w^6+\cdots,\\
\theta_1(q,w) &= \sum_{m\in\mathbf{Z}} q^{(3m+1)^2}w^{3m+1}\\
&= qw+q^4w^{-2}+q^{16}w^4+\cdots,\\
\theta_2(q,w) &= \sum_{m\in\mathbf{Z}} q^{(3m+2)^2}w^{3m+2}\\
&= qw^{-1}+q^4w^2+q^{16}w^{-4}+q^{25}w^5+\cdots.
\end{aligned}$$

Hence in $\mathbf{P}^2$

$$\lim_{q\to 0}\,[\theta_0,\theta_1,\theta_2](q,w)] = [1,0,0]$$

The elliptic curves converge to one point ? This looks strange.

To understand this, we need to understand something much deeper, Néron model. We cannot explain it in detail. Instead we show a necessary modification of the above computation. The reason why we got the above is that we treated w as constant.

Let $w = q^{-1}u$ for $u \in R\setminus I$ and $\overline{u} = u \mod I$. Then the power series θ_k converge I-adically:

$$\begin{aligned}
\theta_0(q,q^{-1}u) &= \sum_{m\in\mathbf{Z}} q^{9m^2-3m}u^{3m}\\
&= 1+q^6u^3+q^{12}u^{-3}+q^{30}u^6+\cdots,\\
\theta_1(q,q^{-1}u) &= \sum_{m\in\mathbf{Z}} q^{(3m+1)^2-3m-1}u^{3m+1}\\
&= u+q^6u^{-2}+q^{12}u^4+\cdots,\\
\theta_2(q,q^{-1}u) &= \sum_{m\in\mathbf{Z}} q^{(3m+2)^2-3m-2}u^{3m+2}\\
&= q^2u^2+q^2u^{-1}+q^{20}u^5+q^{20}u^{-4}+\cdots.
\end{aligned}$$

Hence in $\mathbf{P}^2$

$$\lim_{q\to 0}\,[\theta_0,\theta_1,\theta_2](q,q^{-1}u) = [1,\overline{u},0]$$

Similarly

$$\begin{aligned}
\theta_0(q,q^{-2}u) &= 1+q^3u^3+q^{15}u^{-3}+q^{24}u^6+\cdots,\\
\theta_1(q,q^{-2}u) &= q^{-1}u+q^{12}u^{-2}+q^8u^4+\cdots,\\
\theta_2(q,q^{-2}u) &= u^2+q^3u^{-1}+q^{15}u^5+q^{24}u^{-4}+\cdots,\\
\lim_{q\to 0}\,[\theta_0,\theta_1,\theta_2](q,q^{-2}u) &= \lim_{q\to 0}\,[1,q^{-1}u,u^2] = [0,1,0] \quad \text{in } \mathbf{P}^2.
\end{aligned}$$

Similarly

$$\begin{aligned}
\theta_0(q,q^{-3}u) &= 1+u^3+q^{18}u^{-3}+q^{18}u^6+\cdots,\\
\theta_1(q,q^{-3}u) &= q^{-2}u+q^{10}u^{-2}+q^4u^4+\cdots,\\
\theta_2(q,q^{-3}u) &= q^{-2}u^2+q^4u^{-1}+q^{10}u^5+q^{28}u^{-4}+\cdots,\\
\lim_{q\to 0}[\theta_0,\theta_1,\theta_2](q,q^{-3}u) &= \lim_{q\to 0}[1,q^{-2}u,u^2]=[0,1,\overline{u}] \quad \text{in } \mathbf{P}^2.
\end{aligned}$$

Let $w=q^{-2\lambda}u$ (a section over a finite extension of $k(\eta)$) and $u\in R\setminus I$.

$$\lim_{q\to 0}[\theta_0,\theta_1,\theta_2](q,q^{-2\lambda}u)=\begin{cases}[1,0,0] & (\text{if } -1/2<\lambda<1/2),\\ [1,\overline{u},0] & (\text{if } \lambda=1/2),\\ [0,1,0] & (\text{if } 1/2<\lambda<3/2),\\ [0,1,\overline{u}] & (\text{if } \lambda=3/2),\\ [0,0,1] & (\text{if } 3/2<\lambda<5/2).\\ [\overline{u},0,1] & (\text{if } \lambda=5/2),\end{cases} \tag{2}$$

When λ ranges in $\mathbf{R}$, the same calculation shows that the same limits repeat mod $Y=3\mathbf{Z}$ because

$$\lim_{q\to 0}[\theta_0,\theta_1,\theta_2](q,q^{6n-a}u)=\lim_{q\to 0}[\theta_0,\theta_1,\theta_2](q,q^{-a}u).$$

Thus we see that $\lim_{\tau\to\infty}C(\mu(\tau))$ is the 3-gon $x_0x_1x_2=0$.

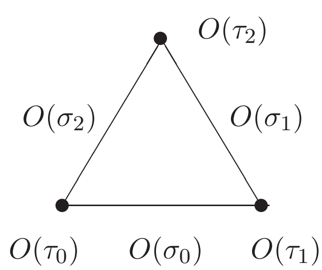

Figure 1

Definition 5.2. For $\lambda\in X\otimes_{\mathbf{Z}}\mathbf{R}$ fixed, let

$$F_\lambda := a^2-2\lambda a \quad (a\in X=\mathbf{Z}).$$

We define a Delaunay cell

$$D(\lambda) := \begin{matrix}\text{the convex closure of all } a\in X\\ \text{that attain the minimum of } F_\lambda\end{matrix}$$

By computations we see

$$D(j+\frac{1}{2}) = [j, j+1] := \{x \in \mathbf{R};\ j \le x \le j+1\},$$
$$D(\lambda) = \{j\} \quad (\text{if } j - \frac{1}{2} < \lambda < j + \frac{1}{2}),$$
$$[\bar{\theta}_k]_{k=0,1,2} := \lim_{q\to 0}[\theta_k(q, q^{-2\lambda}u))]_{k=0,1,2}$$
$$\bar{\theta}_k = \begin{cases} \bar{u}^j & (\text{if } j \in D(\lambda)\cap(k+3\mathbf{Z})) \\ 0 & (\text{if } D(\lambda)\cap(k+3\mathbf{Z}) = \emptyset). \end{cases}$$

For instance $D(\frac{1}{2}) \cap (0+3\mathbf{Z}) = \{0\}$, $D(\frac{1}{2}) \cap (1+3\mathbf{Z}) = \{1\}$ and

$$\lim_{q\to 0}[\theta_k(q, q^{-1}u))] = [\bar{\theta}_0, \bar{\theta}_1, \bar{\theta}_2] = [\bar{u}^0, \bar{u}, 0] = [1, \bar{u}, 0].$$

Similarly for any $\lambda = j + (1/2)$, we have an algebraic torus as a limit

$$\{[\bar{u}^j, \bar{u}^{j+1}] \in \mathbf{P}^1; \bar{u} \in \mathbf{G}_m\} \simeq \mathbf{G}_m \ (= \mathbf{C}^*).$$

Let $\lambda \in X \otimes \mathbf{R}$, and $\sigma = D(\lambda)$ be a Delaunay cell, and $O(\sigma)$ the stratum of $C(\infty)$ consisting of limits of $(q, q^{-2\lambda}u)$. If σ is one-dimensional, then $O(\sigma) = \mathbf{C}^*$, while $O(\sigma)$ is one point if σ is zero-dimensional. Thus we see that $C(\mu(\infty))$ is a disjoint union of $O(\sigma)$, σ being Delaunay cells mod Y, in other words, it is stratified in terms of the Dalaunay decomposition mod Y.

Let $\sigma_j = [j, j+1]$ and $\tau_j = \{j\}$. Then the Dalaunay decomposition in this case and the stratification of $C(\infty)$ are given in Figure 2.

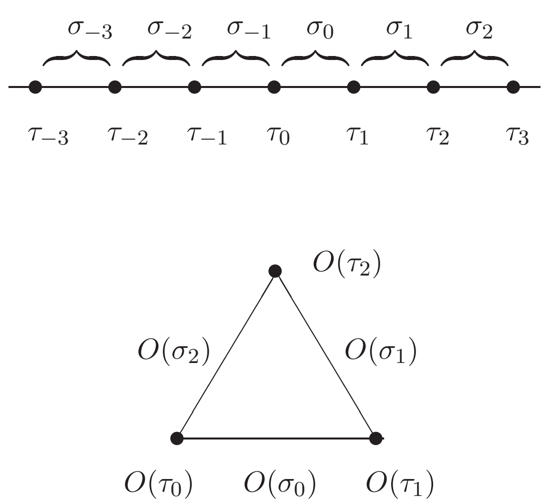

Figure 2

5.3. **The complex case.** The previous computation determines the set-theoretic limit of the image of elliptic curves $E(\omega)$ embedded by thetas. To apply this comutation to the moduli problem, we need to know the set-theoretic limit of the image of $E(\omega)$. Now we explain it.

Now let us write

$$\begin{aligned}\theta_k(q,w) &= \sum_{m\in\mathbf{Z}} q^{(3m+k)^2} w^{3m+k}\\ &= \sum_{m\in\mathbf{Z}} a(3m+k)w^{3m+k}\\ &= \sum_{y\in Y} a(y+k)w^{y+k}\end{aligned}$$

where $Y = 3\mathbf{Z}$ and $a(x) = q^{x^2}$ for $x \in X := \mathbf{Z}$. Thus θ_k is the sum of $a(y+k)w^{y+k}$ over Y, which is Y-invariant.

Since the curve $E(\tau)$ is embedded into $\mathbf{P}^2_{\mathbf{C}}$ by θ_k, we see

$$\begin{aligned}E(\omega) &= \text{Proj } \mathbf{C}[x_k, k=0,1,2]/(x_0^3+x_1^3+x_2^3-3\mu(\omega)x_0x_1x_2)\\ &\simeq \text{Proj } \mathbf{C}[\theta_k\vartheta, k=0,1,2]\\ &= \text{Proj}\,(\mathbf{C}[[a(x)w^x\vartheta, x\in X]])^{Y-\text{inv}}\end{aligned} \tag{3}$$

where $\deg(x_k)=1$, $\deg(\vartheta)=1$ and $\deg(\theta_k)=0$ and $\deg(a(x)w^x)=0$.

Recall that if $U = \text{Spec } A$ is affine, G a finite group acting on U, then

$$U/G = \text{Spec } A^{G\text{-inv}}.$$

So we wish to think

$$E(\omega) = (\text{Proj}\,(\mathbf{C}[[a(x)w^x\vartheta, x\in X]]))/Y.$$

Is this realy true ? Over $\mathbf{C}$, $a(x) \in \mathbf{C}^{\times}$, and

$$\mathbf{G}_m = \text{Proj } \mathbf{C}[a(x)w^x\vartheta, x\in X],$$

In fact, the rhs is covered with infinitely many affine U_k

$$U_k = \text{Spec } \mathbf{C}[a(x)w^x\vartheta/a(k)w^k\vartheta; x\in X] = \text{Spec } \mathbf{C}[w,w^{-1}] = \mathbf{G}_m,$$

which is independent of k.

Hence over $\mathbf{C}$

$$\begin{aligned}E(\omega) &\simeq \mathbf{G}_m/w\mapsto q^6 w\\ &\simeq \mathbf{G}_m/\{w\mapsto q^{2y}w; y\in 3\mathbf{Z}\}\\ &\simeq (\text{Proj } \mathbf{C}[a(x)w^x\vartheta, x\in X])/Y.\end{aligned} \tag{4}$$

Thus we see combining (3) and (5)

$$\begin{aligned}E(\omega) &\simeq \text{Proj}\,(\mathbf{C}[[a(x)w^x\vartheta, x\in X]])^{Y-\text{inv}}\\ &\simeq (\text{Proj } \mathbf{C}[a(x)w^x\vartheta, x\in X])/Y,\end{aligned} \tag{5}$$

though we should make the covergence of infinite sum precise. In fact, this is easy in any charactristic when R is a CDVR.

Though it contains a vague point about convergence, the expression (3) of $E(\tau)$ is quite suggestive for the compactification problem.

5.4. **The scheme-theoretic limit.** What happes over a CDVR R ? Let $a(x) = q^{x^2}$ for $x \in X$, $X = \mathbf{Z}$, $Y = 3\mathbf{Z}$ and ϑ is an indeterminate of degree one, where q is the uniformizer of R. We define the action of Y by via the ring homomorphism

$$S_y^*(a(x)w^x\vartheta) = a(x+y)w^{x+y}\vartheta. \tag{6}$$

Then what does Z look like ?

$$Z = \text{Proj } R[a(x)w^x\vartheta, x \in X]/Y.$$

Let $\mathcal{X}$ and U_n be

$$\begin{aligned} \mathcal{X} &= \text{Proj } R[a(x)w^x\vartheta, x \in X], \\ U_n &= \text{Spec } R[a(x)w^x/a(n)w^n, x \in X] \\ &= \text{Spec } R[(a(n+1)/a(n))w, (a(n-1)/a(n))w^{-1}] \\ &= \text{Spec } R[q^{2n+1}w, q^{-2n+1}w^{-1}] \\ &\simeq \text{Spec } R[x_n, y_n]/(x_n y_n - q^2). \end{aligned}$$

Thus

$$U_n = R[x_n, y_n]/(x_n y_n - q^2) = \lim_{\infty \leftarrow n} (R/q^n)[x_n, y_n]/(x_n y_n - q^2)$$

where U_n and U_{n+1} is glued together by

$$x_{n+1} = x_n^2 y_n, \ y_{n+1} = x_n^{-1}.$$

Let $\mathcal{X}_0 := \mathcal{X} \otimes_R (R/qR)$ and $V_n = \mathcal{X}_0 \cap U_n$. Then $\mathcal{X}_0$ is an infinite chain of $\mathbf{P}^1$:

The action of the sublattice $Y = 3\mathbf{Z}$ on $\mathcal{X}_0$ is transfer by 3 components. In fact, S_{-3} sends

$$V_n \overset{S_{-3}}{\to} V_{n+3} \overset{S_{-3}}{\to} V_{n+6} \to \cdots,$$

$$(x_n, y_n) \overset{S_{-3}}{\mapsto} (x_{n+3}, y_{n+3}) = (x_n, y_n)$$

so that we have a cycle of 3 rational curves as the quotient $\mathcal{X}_0/Y$. Thus we have the same consequence as the above theta computation.

6. PSQASes in the general case

6.1. **The degeneraion data of Faltings-Chai.** Now we consider the general case. In the general case let R be a CDVR,$k(\eta)$ te fraction feild of R. Then we can construct similar degenerations of abelian varieties if we are given a lattice X, a sublattice Y of X of finite index and

$$a(x) \in K^\times, \quad (x \in X)$$

such that the following conditions are satisfied

(i) $a(0) = 1$,
(ii) $b(x, y) := a(x+y)a(x)^{-1}a(y)^{-1}$ is a symmetric blinear form on $X \times X$,

(iii) $B(x,y) := \mathrm{val}_q\, b(x,y)$ is positive definite,
(iv)* B is even and $\mathrm{val}_q\, a(x) = B(x,x)/2$.

We assume here a stronger condition $(4)^*$ for simplicity.

These data do exist in general. This is proved by Faltings-Chai [FC90].

Suppose that we are given an abelian scheme $(G_\eta, \mathcal{L}_\eta)$ and a polarization morphism $\lambda(\mathcal{L}_\eta) : G_\eta \to G_\eta^t$. Then there exists the connected Neron model of G_η, which we denote by G. Then by finite base change if necessary we may assume G is semi-abelian.

Now we assume G_0 is a torus over R/qR. This case is called a totally degenerate case, that is, the case when $\mathrm{rank}_{\mathbf{Z}} X = \dim G_\eta$, on which we mainly discuss here. If it is not a torus, then there exists a bit more complicated degeneration data, which enables us to construct a degenerating family of abelian varieties. This is called a partially degenerate case.

When G_0 is a torus over R/qR, the formal competion of G along G_0 is a formal torus over R;

$$G_{\mathrm{for}} \simeq \mathbf{G}^g_{m,R,\mathrm{for}} = \mathrm{Spf}\ R[[w^x; x \in X]]^{I\text{-adic}}$$

where X is a lattice of rank g. We note any line bundle on $\mathbf{G}_{m,R,\mathrm{for}}$ hence $\mathbf{G}_{m,R,\mathrm{for}}$ is trivial. Hence any global section $\theta \in \Gamma(G, \mathcal{L})$ is a formal power series of w^x: we write

$$\theta = \sum_{x \in X} \sigma_x(\theta) w^x.$$

Theorem 6.2. *If G is totally degenerate, then there exists a data $\{a(x); x \in X\}$ satisfying the conditions (i)-(iv)* together with*

(v) *$\Gamma(G_\eta, \mathcal{L}_\eta)$ is the $k(\eta)$ vector subspace of formal Fourier series θ such that*

$$\sigma_{x+y}(\theta) = a(y)b(y,x)\sigma_x(\theta)$$

and $\sigma_x(\theta) \in k(\eta)$ $(\forall x \in X, \forall y \in Y)$.

The condition (v) enables us to prove especially the part (o) of Theorem 4.2.

6.3. **Construction.** So we may assume we are given the data $a(x)$ as above. Then we define Let $\mathcal{X}$ and U_n $(n \in X)$ be

$$\begin{aligned}
\mathcal{X} &= \mathrm{Proj}\ R[a(x)w^x\vartheta, x \in X],\\
U_n &= \mathrm{Spec}\ R[a(x)w^x/a(n)w^n, x \in X]\\
&= \mathrm{Spec}\ R[(a(x)/a(n))w^{x-n}].,\\
Q := \mathcal{X}/Y &= (\mathrm{Proj}\ R[a(x)w^x\vartheta, x \in X])_{\mathrm{for}}/Y,
\end{aligned}$$

where for denote the formal completion along the special fiber. This is what we sought for, $(Q, \mathcal{L}_Q)$ in Theorem 4.2, where $\mathcal{L}$ is given by the homogeneous ideal generatd by the degree one generator ϑ.

The theta function are recovered as

$$\theta_{\bar{x}} := \sum_{y \in Y} a(x+y) w^{x+y}$$

where $\bar{x} \in X/Y$ the class of x mod Y. This is compatible with Theorem 6.2 (v) because

$$\sigma_{x+y}(\theta_{\bar{x}}) = a(x+y) = a(y)b(y,x)a(x) = a(y)b(y,x)\sigma_x(\theta_{\bar{x}}).$$

6.4. **Delaunay decomposition.** Our PSQAS is a scheme-theoretic limit of the image of abelian varieties embedded by (naturally ordered) theta functions. Our computaion in § 5 computes, roughly speaking, a set-theoretic limit of the same.

Let X be an integral lattice of rank g and B a positive symmetric integral bilinear form on X associated with the degeneration data for $(\mathcal{Z}, \mathcal{L})$.

For $\lambda \in X \otimes_{\mathbf{Z}} \mathbf{R}$ fixed, we define:

Definition 6.5. A Delaunay cell σ is a convex hull spanned by the integral vectors (which we call Delaunay vectors) attaining the minimum of the function

$$B(x,x) - 2B(\lambda, x) \quad (x \in X).$$

When λ ranges in $X \otimes_{\mathbf{Z}} \mathbf{R}$, we will have various Delaunay cells. All of them constitute a locally finite polyhedral decomposition of $X \otimes_{\mathbf{Z}} \mathbf{R}$, invariant under the translation by X. We call this the Delaunay decomposition of $X \otimes_{\mathbf{Z}} \mathbf{R}$, which we denote by Del_B.

There are two types of Delaunay decompositions inequivalent under the action of $\mathrm{SL}(2, \mathbf{Z})$. See Figure 3.

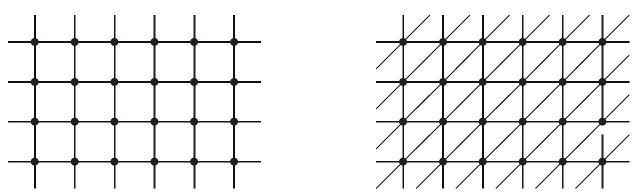

FIGURE 3

The Delaunay decomposition describes a PSQAS as follows:

Theorem 6.6. *Let Z be a PSQAS, X the integral lattice, Y the sublattice of X of finite index and B the positive integral bilinear form on X all of which were defined in §6. Let σ,τ be Delaunay cells in* Del_B.

(1) *For each σ there exists a subscheme $O(\sigma)$ of Z, invariant under the torus-action, which is a torus of dimension* $\dim \sigma$ *over k such that Z is the disjoint union of $O(\sigma)$. Let $Z(\sigma)$ be the closure of $O(\sigma)$.*
(2) *$\sigma \subset \tau$ iff $O(\sigma) \subset O(\tau)$.*
(3) *$\sigma \subset \tau$ iff $Z(\sigma) \subset Z(\tau)$.*
(4) *$Z = \bigcup_{\sigma \in \mathrm{Del} \bmod Y} Z(\sigma)$.*
(5) *The local scheme structure of Z is completely described by B.*

7. THE MODULI SPACE $SQ_{g,K}$

By Theorem 4.2, any level $G(K)$ PSQAS $(Q_0, \mathcal{L}_0)$ is embedded into $\mathbf{P}(V)$ if $e_{\min}(K) \geq 3$ where $V = V_H := \mathcal{O}_N[v(\mu); \mu \in H^\vee]$.

7.1. **The $G(K)$-action and the $G(K)$-linearization.** The $G(K)$-action τ on (Z, L) is translated into $G(K)$-linearization

$$\{\phi_g; g \in G(K)\}$$

(i) $\phi_g : \mathcal{L} \to T_g^*(\mathcal{L})$ is a bundle isomorphism,
(ii) $\phi_{gh} = T_h^*\phi_g \circ \phi_h$ for any $g, h \in G(K)(T)$.

Then the action τ on (Z, L) is recovered from it as follows : By the isomorphisms

$$L \xrightarrow{\phi_h} T_h^*(L) \xrightarrow{T_h^*\phi_g} T_h^*(T_g^*(L)) = T_{gh}^*(L),$$

for $x \in Z$, $\xi \in L_x$,

$$\tau(h) \cdot (z, \xi) = (T_h(z), \phi_h(z) \cdot \xi).$$

We check it is the action:

$$\begin{aligned}
\tau(g) \cdot (\tau(h) \cdot (z, \xi)) &= \tau(g) \cdot (T_h(z), \phi_h(z) \cdot \xi) \\
&= (T_g(T_h(z)), \phi_g(T_h(z))\phi_h(z) \cdot \xi) \\
&= (T_{gh}(z), (T_h^*\phi_g \cdot \phi_h)(z) \cdot \xi) \\
&= (T_{gh}(z), \phi_{gh}(z) \cdot \xi) = \tau(gh) \cdot (z, \xi).
\end{aligned}$$

Then we define the action of $G(K)$ on $H^0(Z, L)$

$$\rho_L(g)(\theta) := T_{g^{-1}}^*(\phi_g(\theta))$$

We check that it is a homom.

$$\begin{aligned}
\rho_L(gh)(\theta) &= T_{h^{-1}g^{-1}}^*(\phi_{gh}\theta) \\
&= T_{g^{-1}}^*\{T_{h^{-1}}^*(T_h^*\phi_g \cdot \phi_h\theta)\} \\
&= T_{g^{-1}}^*\{T_{h^{-1}}^*(T_h^*\phi_g) \cdot (T_{h^{-1}}^*\phi_h\theta)\} \\
&= T_{g^{-1}}^*\{\phi_g \cdot (T_{h^{-1}}^*\phi_h\theta)\} \\
&= \rho_L(g)\rho_L(h)(\theta).
\end{aligned}$$

Thus $H^0(Z, L)$ is a $G(K)$-module of weight one, that is, any $a \in G(K)$ acts as $a \cdot \mathrm{id}_{V_H}$.

We recall

Lemma 7.2. *V_H is a unique irreducible representation of weight one of $G(K)$ over $\mathbf{Z}[\zeta_N, 1/N]$.*

Lemma 7.3. *Assume $e_{\min}(K) \geq 3$. Then*

for a level-$G(K)$ PSQAS (Q_0, ϕ_0, τ_0), there exists a unique level-$G(K)$ PSQAS (Q_0', i, U_H) isom. to (Q_0, ϕ_0, τ_0) such that $i : Q_0' \subset \mathbf{P}(V_H)$, where U_H is the Schrödinger repres. of $G(K)$.

Proof of Lemma 7.3. Let $(Q_0, \mathcal{L}_0)$ be (Z, L), and $(Q_0, \phi_0, \tau_0) = (Z, \phi, \tau)$.

For a closed immersion $\phi : Z \to \mathbf{P}(V_H)$, we have an isomorphism

$$\phi^* : V_H \simeq H^0(Z, L),$$

whence

$$\rho(\phi, \tau)(g)(\theta) := (\phi^*)^{-1}\rho_L(g)(\theta)\phi^*. \tag{7}$$

Thus $\rho(\phi, \tau) \in \mathrm{End}\,(V_H)$.

By Schur's lemma, $\exists\ A \in \mathrm{GL}(V_H)$ s.t.

$$U_H = A^{-1}\rho(\phi, \tau)A = (\phi^* A)^{-1}\rho_L(g)(\theta)(\phi^* A).$$

Hence it suffices to choose a closed immersion ψ by $\psi^* = \phi^* A$. Then

$$U_H = \rho(\psi, \tau). \tag{8}$$

This proves Lemma by taking $Z' = \psi(Z)$, i the inclusion of Z'. □

Remark 7.4. Suppose L to be very ample. Then the $G(K)$-linearization on (Z, L) is equiv. to the $G(K)$-equivariant closed immersion of the pair (Z, L) into $(\mathbf{P}(V_H), O_{\mathbf{P}}(1))$. Namely, Z is a $G(K)$-invariant subscheme of $\mathbf{P}(V_H)$ with $L = O_Z(1)$.

Let $\mathrm{Hilb}^{\chi(n)}$ be the Hilbert scheme parametrizing all the closed subscheme (Z, L) of $\mathbf{P}(V_H)$ with $\chi(Z, L^n) = n^g\sqrt{|K|} =: \chi(n)$, and $(\mathrm{Hilb}^{\chi(n)})^{G(K)\text{-inv}}$ the $G(K)$-inv. part of it.

The following is an immersion of $A_{g,K}$ into $(\mathrm{Hilb}^{\chi(n)})^{G(K)\text{-inv}}$:

$$A_{g,K} \ni (A_0, \phi_0, \tau_0) \mapsto (A'_0, i, U_H) \in (\mathrm{Hilb}^{\chi(n)})^{G(K)\text{-inv}} \quad (\mathrm{AV}).$$

Then we define

$$SQ_{g,K} = \overline{A_{g,K}} \subset (\mathrm{Hilb}^{\chi(n)})^{G(K)\text{-inv}}$$

Theorem 7.5. *Suppose $H = \oplus_{i=1}^g (\mathbf{Z}/e_i\mathbf{Z})$. For any closed field k of characteristic prime to $|H| = \prod_{i=1}^g e_i$,*

$$SQ_{g,K}(k) = \{(Q_0, i, U_H); PSQAS, i : Q_0 \subset \mathbf{P}(V_H)\}$$

8. Representability

Definition 8.1. The triple (X, ϕ, τ) or (X, L, ϕ, τ) is
a PSQAS with level-$G(K)$ str. if

1. $\phi : (X, L) \to (\mathbf{P}(V), O(1))$ a closed immersion
   such that $\phi^* : V \simeq H^0(X, L)$, $L = \phi^* O_{\mathbf{P}(V)}(1)$,
2. τ is a $G(K)$-action on the pair (X, L) so that ϕ is a $G(K)$-morphism.

Define : $(X, \phi, \tau) \simeq (X', \phi', \tau')$ isom. iff
$\exists\ (f, F) : (X, L) \to (X', L')$ $G(K)$-isom. such that $\phi = \phi' \cdot f$.

Theorem 8.2. *Suppose $e_{\min}(K) \geq 3$. Let $N := \sqrt{|K|}$. The functor $\mathcal{SQ}_{g,K}$ of level-$G(K)$ PSQASes (Q, ϕ, τ) over reduced base schemes is represented by the projective $\mathbf{Z}[\zeta_N, 1/N]$-scheme $SQ_{g,K}$.*

$$SQ_{g,K}(T) = \{(Q, \phi, \tau); \text{PSQAS with level-}G(K)\text{ str. over } T\}$$

For TSQASes we prove

Theorem 8.3. ([N10],[N13]) *Let $N := \sqrt{|K|}$. No restriction on $e_{\min}(K)$. The functor $\mathcal{SQ}^{\mathrm{toric}}_{g,K}$ of level-$G(K)$ TSQASes (P, ϕ, τ) over reduced base schemes is coarsely represented by the projective $\mathbf{Z}[\zeta_N, 1/N]$-scheme $SQ^{\mathrm{toric}}_{g,K}$.*

Theorem 8.4. *Suppose* $e_{\min}(K) \geq 3$. *Let* $A_{g,K} :=$ *the moduli space of AVs with level-*$G(K)$ *str. Then*

1. *both* $SQ_{g,K} \supset A_{g,K}$ *and* $SQ_{g,K} \supset A_{g,K}$ *(Zariski open),*
2. $\dim SQ_{g,K} = \dim SQ^{\text{toric}}_{g,K} = g(g+1)/2$,
3. $\exists$ *a bijective* $\mathcal{O}_N$*-morphism*
$$\mathrm{sq} : SQ^{\text{toric}}_{g,K} \to SQ_{g,K}$$
extending the identity of $A_{g,K}$.
4. $\mathrm{sq}^{\text{norm}} : (SQ^{\text{toric}}_{g,K})^{\text{norm}} \simeq (SQ_{g,K})^{\text{norm}}$ [N10].

9. Stability of PSQASes

Theorem 9.1. ([Gieseker82], [Mumford77]) *For a connected curve* C *of genus greater than one, the following are equivalent:*

(1) C *is a stable curve, (moduli-stable)*
(2) *Any Hilbert point of* C *embedded by* $|mK_C|$ *is GIT-stable,*
(3) *Any Chow point of* C *embedded by* $|mK_C|$ *is GIT-stable.*

Theorem 9.2. *Let* $K = H \oplus H^{\vee}$, $N = |H|$, k *a closed field,* char $.k \neq N$.

Suppose $e_{\min}(K) \geq 3$, *and* $(Z, L) \subset (\mathbf{P}(V), O_{\mathbf{P}(V_H)}(1))$.

Suppose that (Z, L) *is smoothable into an abelian variety whose Heisenberg group is isomorphic to* $G(K)$.

Then the following are equivalent:

(1) (Z, L) *is a PSQAS, (moduli-stable)*
(2) *any Hilbert point of* (Z, L) *are GIT-stable,*
(3) (Z, L) *is stable under (a conjugate of)* $G(K)$.

9.3. **Stability of planar cubics.** By the following table, a planar cubic is GIT-stable, with respect to $SL(3)$-action on the Hi9lbert schem of cubics, if and only if it is either a smooth elliptic curve or a 3-gon, hence it is isomorphic to one of Hesse cubics. It follows from it that

$$\begin{aligned} C \text{ is GIT-stable} &\Leftrightarrow C \text{ is elliptic or a 3-gon} \\ &\Leftrightarrow C \text{ is isom. to a Hesse cubic} \\ &\Leftrightarrow C \text{ is isom. to a } G(3)\text{-stable cubic.} \end{aligned}$$

This is a special case of Theorem 9.2 because any cubic is a degenerate abelian variety

Table 1. Stability of cubic curves

curves (sing.)	stability	stab. gr.
smooth elliptic	GIT-stable	finite
3 lines, no triple point	GIT-stable	2 dim
a line+a conic, not tangent	semistable, not GIT-stable	1 dim
irreducible, a node	semistable, not GIT-stable	$\mathbf{Z}/2\mathbf{Z}$
3 lines, a triple point	not semistable	1 dim
a line+a conic, tangent	not semistable	1 dim
irreducible, a cusp	not semistable	1 dim

where GIT-stable := closed SL(3)-orbit

10. The other complete moduli space

10.1. **Alexeev's complete moduli space.** [Alexeev02] constructs a complete moduli $\overline{AP}_{g,d}$ of *seminormal degenerate abelian varieties*, each coupled with semiabelian group action and an ample divisor. It is the compactification of the coarse moduli $AP_{g,d}$ of pairs (A, D) with A a g-dimensional abelian variety, D an ample divisor with $h^0(A, D) = d$. $\overline{AP}_{g,d}$ is a proper separated coarse moduli algebraic space over $\mathbf{Z}$ [Alexeev02].

$$\overline{AP}_{g,d} = \left\{ \begin{array}{lr} (G, P, D); & G\text{:semi-abelian, } P\text{:seminormal,} \\ & D \text{ ample div. of } P,\ h^0(P, D) = d, \\ & G \text{ acts on } P + \text{stability cond.} \end{array} \right\}$$

$$\supset AP_{g,d} = \{(G, G, D); G\text{: AV}\}$$

$$\dim \overline{AP}_{g,d} = g(g+1)/2 + d - 1.$$

Theorem 10.2. ([N13]) *Let* $N = \sqrt{|K|}$.

1. $\exists\, U$ *(Zariski open of* $\mathbf{P}(V_H) = \mathbf{P}^{N-1}$*) such that*
$$\mathrm{sqap} : SQ^{\mathrm{toric}}_{g,K} \times U \to \overline{AP}_{g,N}$$
is finite Galois (not surjective) with Galois gp. known,
2. $\mathrm{sqap} : SQ^{\mathrm{toric}}_{g,K} \times \{u\} \to \overline{AP}_{g,N}$ *is a closed immersion for any* $u \in U$*,*
3. $SQ^{\mathrm{toric}}_{g,1} \simeq \overline{AP}_{g,1}$.

References

[Alexeev02] V. Alexeev, Complete moduli in the presence of semiabelian group action, Ann. of Math. **155** (2002), 611–708.

[AN99] V. Alexeev and I. Nakamura, On Mumford's construction of degenerating abelian varieties, Tôhoku Math. J. **51** (1999) 399–420.

[AMRT75] A. Ash, D. Mumford, M. Rapoport and Y. Tai, Smooth compactification of locally symmetric varieties, Math Sci Press, Massachusetts, USA, 1975.

[DM69] P. Deligne and D. Mumford, The irreducibility of the space of curves of given genus, Publ. Math. IHES **36** (1969) 75–110.

[FC90] G. Faltings and C.-L. Chai, Degenerations of abelian varieties, vol. 22, Ergebnisse der Mathematik und ihrer Grenzgebiete, no. 3, Springer-Verlag, 1990.

[Gieseker82] D. Gieseker, Lectures on moduli of curves, Tata Institute of Fundamental Research, Bombay 1982.

[Kempf78] G. Kempf, Instability in invariant theory, Ann. Math. **339** (1978), 299–316.

[Mumford72] D. Mumford, An analytic construction of degenerating abelian varieties over complete rings, Compositio Math. **24** (1972) 239–272.

[Mumford77] D. Mumford, Stability of projective varieties, L'Enseignment Mathematique **23** (1977) 39–110.

[Mumford12] D. Mumford, Abelian varieties, Tata Institute of Fundamental Research, Hindustan Book Agency, 2012.

[N75] I. Nakamura, On moduli of stable quasi abelian varieties, Nagoya Math. J. **58** (1975), 149–214.

[N77] I. Nakamura, Relative compactification of the Néron model and its applications, Complex Analysis and Algebraic Geometry (W.L. Baily, Jr. and T. Shioda, eds.), Cambridge Univ. Press, 1977, pp. 207–225.

[N98] I. Nakamura, Compactification of the moduli of abelian varieties over $\mathbf{Z}[\zeta_N, 1/N]$, C. R. Acad. Sci. Paris, **327** (1998) 875-880.

[N99] I. Nakamura, Stability of degenerate abelian varieties, Invent. Math. **136** (1999), 659–715.

[N04] I. Nakamura, Planar cubic curves, from Hesse to Mumford, Sugaku Expositions **17** (2004), 73-101.

[N10] I. Nakamura, Another canonical compactification of the moduli space of abelian varieties, Algebraic and arithemtic structures of moduli spaces (Sapporo, 2007), Advanced Studies in Pure Math., **58** (2010), 69-135. (arXivmath 0107158)

[N13] I. Nakamura, The complete moduli spaces of degenerate abelian varieties, preprint (to be submitted to arXivmath.)

[NS06] I. Nakamura and K. Sugawara, The cohomology groups of stable quasi-abelian schemes and degenerations associated with the E_8 lattice, Moduli Spaces and Arithmetic Geometry (Kyoto, 2004), Advanced Studies in Pure Math., **45** (2006) 223–281.

[NT01] I. Nakamura and T. Terasoma, Moduli space of elliptic curves with Heisenberg level structure, Moduli of Abelian Varieties, Proceedings of Texel conference 1999, Progress in Math. **195**, Birkhäuser (2001) 299-324.

[Rim72] D.S. Rim, Formal deformaion theory, in Groupes de Monodromie en Géométrie Algébrique (SGA 7 I), Lecture Notes in Math. **288**, Springer-Verlag, Berlin Heidelberg New York, 1972, exp. VI, 32-132.

[SGA1] A. Grothendieck, Revêtements Etales et Groupe Fondamentale (SGA 1), Lecture Notes in Math. **224**, Springer-Verlag, Berlin Heidelberg New York, 1971.

Department of Mathematics, Hokkaido University,Sapporo, 060-0810
E-mail address: `nakamura@math.sci.hokudai.ac.jp`