6/28.

今日の話題 ・偏微分の変数変換

・2変数関数の極大,極小.

命題 $f(x,y)$ を微分可能な2変数関数

$g(r,\theta)$, $h(r,\theta)$

偏微分可能な2変数関数

$$\frac{\partial f(g(r,\theta),h(r,\theta))}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial h}{\partial r}$$

$$\frac{\partial f(g(r,\theta),h(r,\theta))}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial h}{\partial \theta}$$

例 $f(x,y) = \log(x^2+y^2+1)$

$g(r,\theta) = r\cos\theta$ $\quad h(r,\theta) = r\sin\theta$

$$f(g(r,\theta),h(r,\theta)) = \log(r^2+1)$$

$$\frac{\partial f}{\partial r} = \frac{2r}{r^2+1} \qquad \frac{\partial f}{\partial \theta} = 0$$

一方 $\dfrac{\partial f}{\partial x} = \dfrac{2x}{x^2+y^2+1}$ $\quad \dfrac{\partial f}{\partial y} = \dfrac{2y}{x^2+y^2+1}$

$$\frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial r} = \frac{2r\cos\theta}{r^2+1}\cdot\cos\theta$$

$$\frac{\partial f}{\partial y}\cdot\frac{\partial h}{\partial r} = \frac{2r\sin\theta}{r^2+1}\cdot\sin\theta$$

$$\frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial h}{\partial r} = \frac{2r}{r^2+1}$$

記号

$$\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)^n f$$

$$\therefore \sum_{k=0}^{n} {}_nC_k \frac{\partial^n f}{\partial^k x\,\partial^{n-k} y}$$

例 $f(g(t),h(t))$ を t で微分して見る.

$$\frac{df}{dt} = \frac{\partial f}{\partial x}g'(t) + \frac{\partial f}{\partial y}h'(t)$$

$\dfrac{d^2f}{dt^2}$ の計算をするために準備をする.

$$\left(\frac{\partial f}{\partial x}\right)' = \frac{\partial}{\partial x}\cdot\frac{\partial f}{\partial x}g'(t) + \frac{\partial}{\partial y}\frac{\partial f}{\partial x}h'(t)$$

$$\approx \frac{\partial^2 f}{\partial x^2} g'(t) + \frac{\partial^2 f}{\partial x \partial y} h'(t)$$

$$\left(\frac{\partial f}{\partial y}\right)' = \frac{\partial^2 f}{\partial x \partial y} g'(t) + \frac{\partial^2 f}{\partial y^2} h'(t)$$

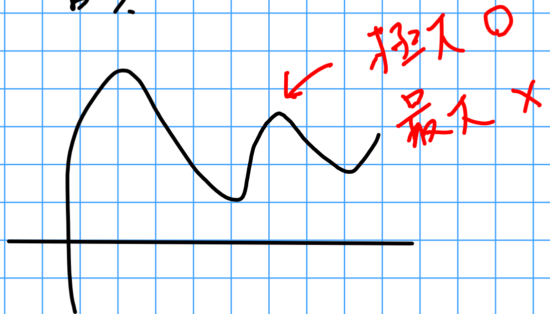
これから

$$\frac{d^2 f}{dt^2} = \frac{\partial^2 f}{\partial x^2}\{g'(t)\}^2 + 2\frac{\partial^2 f}{\partial x \partial y} g'(t) h'(t) + \frac{\partial^2 f}{\partial y^2}\{h'(t)\}^2 + \frac{\partial f}{\partial x} g''(t) + \frac{\partial f}{\partial y} h''(t)$$

2変数関数の極大, 極小

定義 $z = f(x, y)$ が $P(x_0, y_0)$ で極大

とは, P のある近くの領域で P が $f(x, y)$ の最大値を取る点となっているときを言う.



極値をどうやって求めるか?

① $\cancel{f_x =}$ $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ の解を求める.

② ①で得た解を (x_0, y_0) とする.

$$\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$ の値を計算し,

$+$ であれば 極値 $\Rightarrow$ ③へ

$-$ であれば 極値ではない.

0 であれば 判定不能

③ $\frac{\partial^2 f}{\partial x^2} > 0$ … 極小

< 0 … 極大.

例題.

$f(x, y) = x^2 + 3xy + y^2$ の極値を(あれば)求めよ

解). $\cancel{f_x =}$ $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$

$$\Longleftrightarrow \begin{cases} 2x + 3y = 0 \\ 3x + 2y = 0 \end{cases} \Longleftrightarrow x = y = 0.$$

$$\frac{\partial^2 f}{\partial x^2} = 2 \qquad \frac{\partial^2 f}{\partial x \partial y} = 3 \qquad \frac{\partial^2 f}{\partial y^2} = 2$$

よって $\dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial^2 f}{\partial x \partial y}\right)^2 = 4 - 9 = -5 < 0$

これから $f(x, y)$ に極値はない.

例 2. ← 7/5 の講義で再び扱う.

極値が求まるからくり.

まず 1 変数の場合を復習する.

$y = f(x)$ が $x = x_0$ で極大 (or 極小)

をもつ十分条件として $f'(x_0) = 0$ かつ $f''(x_0) < 0$

[> 0]

実際 $f(x)$ の $x = x_0$ のまわりのテイラー展開

を 4 項まで書いてみると,

$$f(x) = f(x_0) + \underbrace{f'(x_0)(x - x_0)}_{0} + \frac{1}{2!} f''(x_0)(x - x_0)^2 + \frac{1}{\cancel{4!}\; 3!} f'''(x_0)(x - x_0)^3 + \cdots$$

だいたい等しい.

$$\fallingdotseq f(x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2$$

↑

頂点 $(x_0, f(x_0))$ の放物線.

後はグラフを考える.

次に 2 変数で考える. $\partial f / \partial x = \partial f / \partial y = 0$ より

$$f(x, y) = f(x_0, y_0) + \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}(x - x_0)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x - x_0)(y - y_0) + \frac{\partial^2 f}{\partial y^2}(y - y_0)^2\right) + \cdots$$

ここで

$$z = ax^2 + 2bxy + cy^2$$

のグラフを考えよう.

$$z = a\left(x + \frac{b}{a}y\right)^2 + \left(c - \frac{b^2}{a}\right)y^2$$

$X = x + \frac{b}{a}y \quad Y = y$ とすれば

$$z = aX^2 + \frac{ac - b^2}{a}Y^2$$

$a,\ ac - b^2/a > 0$ … 極小

$a,\ ac - b^2/a < 0$ … 極大

$$\left.\begin{array}{l} a < 0,\ ac - b^2/a > 0 \\ a > 0,\ ac - b^2/a < 0 \end{array}\right\}$$ 極値ではない.