微分積分学 II Jan. 9. 小テスト

問題 1. 次の積分を計算せよ.

$$\int_0^1 \left(\int_0^1 \frac{y^2 - x^2}{(x^2 + y^2)^2} dy \right) dx$$

$$\frac{\partial}{\partial x \partial y} \tan \frac{y}{x} = 中の関数$$

$$\frac{\partial}{\partial x} \tan \frac{y}{x} = -\frac{y}{x^2+y^2}$$

$$= \lim_{\varepsilon \to 0} \int_\varepsilon^1 \left\{ \lim_{\delta \to 0} \int_\delta^1 \frac{y^2 - x^2}{(x^2+y^2)^2} dy \right\} dx$$

$$= \lim_{\varepsilon \to 0} \int_\varepsilon^1 \lim_{\delta \to 0} \left[-\frac{y}{x^2+y^2} \right]_\delta^1 dx$$

$$= \lim_{\varepsilon \to 0} \int_\varepsilon^1 \lim_{\delta \to 0} \left\{ -\frac{1}{x^2+1} + \frac{\delta}{x^2+\delta^2} \right\} dx$$

$$(\mathrm{Arctan}\, x)' = \frac{1}{1+x^2}$$

$$= \lim_{\varepsilon \to 0} \int_\varepsilon^1 -\frac{1}{x^2+1} dx = \left[-\mathrm{ArcTan}\, x \right]_\varepsilon^1$$

$$= \lim_{\varepsilon \to 0} \left\{ -\mathrm{Arctan}\, 1 + \mathrm{Arctan}\, \varepsilon \right\} = -\frac{\pi}{4}$$

問題 2. 半径 a の半球の表面積を求めよ.

公式より 半球の表面積は

$$S = \iint_K \sqrt{1 + f_x^2 + f_y^2}\, dxdy \qquad f(x,y) = \sqrt{a^2 - x^2 - y^2}$$

$$K: \ x^2 + y^2 \leqq a^2$$

$$f_x = \frac{-x}{\sqrt{a^2-x^2-y^2}} \qquad f_y = \frac{-y}{\sqrt{a^2-x^2-y^2}}$$

よって $\sqrt{1+f_x^2+f_y^2} = \sqrt{1 + \frac{x^2}{a^2-x^2-y^2} + \frac{y^2}{a^2-x^2-y^2}}$

$$= \sqrt{\frac{a^2}{a^2-x^2-y^2}}$$

$$= \frac{a}{\sqrt{a^2-x^2-y^2}}$$

以上から $S = \iint_K \frac{a}{\sqrt{a^2-x^2-y^2}}\, dxdy$

ここで $x = r\cos\theta$ $y = r\sin\theta$ とおきかえると,

$$(x, y) \in K \iff \begin{cases} 0 \leqq r \leqq a \\ 0 \leqq \theta \leqq 2\pi \end{cases}$$

又 $|\det J| = r$ となるので

$$S = \iint_{K'} \frac{a}{\sqrt{a^2 - r^2}} \cdot r \cdot drd\theta \qquad K' = [0, a] \times [0, 2\pi]$$

$$= \int_0^a \left\{ \int_0^{2\pi} \frac{a}{\sqrt{a^2-r^2}} r\, d\theta \right\} dr = \int_0^a \left[\frac{a}{\sqrt{a^2-r^2}} r\theta \right]_0^{2\pi} dr$$

$$= \int_0^a \frac{2\pi a}{\sqrt{a^2-r^2}} \cdot r\, dr = \left[2\pi a \left(-\sqrt{a^2-r^2}\right) \right]_0^r = 2\pi a^2_{\,/\!/}$$