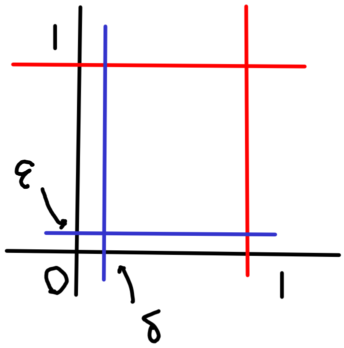
微分積分学 II Dec. 28. 小テスト

先々週の問題 2.　　次の積分を計算せよ.



$$\int_0^1 \left(\int_0^1 \frac{y^2 - x^2}{(x^2 + y^2)^2} dy \right) dx$$

$$= \lim_{\delta \to 0} \int_\delta^1 \left\{ \lim_{\varepsilon \to 0} \int_\varepsilon^1 \frac{y^2 - x^2}{(y^2 + x^2)^2} dy \right\} dx$$

$$= \lim_{\delta \to 0} \int_\delta^1 \left\{ \lim_{\varepsilon \to 0} \left[\frac{-\cancel{x}\,y}{y^2 + x^2} \right]_\varepsilon^1 \right\} dx$$

$$= \lim_{\delta \to 0} \int_\delta^1 \frac{-\cancel{x}\,1}{1 + x^2} dx$$

$$= \lim_{\delta \to 0} \Big[-\mathrm{Arctan}\, x \Big]_\delta^1$$

$$= -\frac{\pi}{4}$$

問題 1.　半径 a の半球の表面積を求めよ.

半球は $z = f(x,y) = \sqrt{a^2-x^2-y^2}$ で定義される.

この $f(x,y)$ に対し

$$\frac{\partial f}{\partial x} = \frac{-x}{\sqrt{a^2-x^2-y^2}} \qquad \frac{\partial f}{\partial y} = \frac{-y}{\sqrt{a^2-x^2-y^2}}$$

となるので, 求める面積 S は

$$S = \iint_K \sqrt{1+\frac{x^2}{\cancel{1}\,a^2-x^2-y^2}+\frac{y^2}{\cancel{1}\,a^2-x^2-y^2}}\,dxdy \qquad K = \{x^2+y^2 \leqq 1\}$$

$$= \iint_K \frac{\cancel{1}\,a}{\sqrt{\cancel{1}\,a^2-x^2-y^2}}\,dxdy$$

ここで　$x = r\cos\theta$　$y = r\sin\theta$ とすると,

$$(x,y) \in K \iff \begin{cases} 0 \leqq r \leqq a \\ 0 \leqq \theta \leqq 2\pi \end{cases}$$

よって上の重積分は

$$= \iint_{K'} \frac{\cancel{1}\,a}{\sqrt{\cancel{1}\,a^2-r^2}} \cdot r\,drd\theta \qquad K' = [0,a]\times[0,2\pi]$$

$$= \int_0^a \left\{ \int_0^{2\pi} \frac{\cancel{r}\,a}{\sqrt{a^2-r^2}}\,d\theta \right\} dr = \int_0^a \frac{2\pi r a}{\sqrt{a^2-r^2}}\,dr$$

$$= \left[-2\pi a\sqrt{a^2-r^2}\right]_0^a = 2\pi a^2$$