解答例

問1. $(x-1)(y-1)$ の項の係数は

$$\frac{1}{2!}\cdot {}_2C_1 \frac{\partial^2 x^y}{\partial x \partial y}(1,1)$$

$$= \frac{\partial^2 x^y}{\partial x \partial y}(1,1)$$

$$\frac{\partial x^y}{\partial x} = y x^{y-1} = y e^{(y-1)\log x} \text{ より}$$

$$\frac{\partial^2 x^y}{\partial x \partial y} = x^{y-1} + y\cdot(\log x)\cdot e^{(y-1)\log x}$$

$$= x^{y-1} + y x^{y-1}\log x$$

これに $(x,y)=(1,1)$ を代入して

求める係数は $\underline{\underline{1}}$

問2.

複雑な方の辺を簡単な辺の方に変形する.

のが等式を証明するコツの1つ.

公式

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

ここで $x = r\cos\theta \quad y = r\sin\theta$ より

$$(\text{上式}) = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y} r\cos\theta$$

よって

$$\frac{\partial^2 f}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y} r\cos\theta\right)$$

$$= \left\{\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial x \partial y}\frac{\partial y}{\partial \theta}\right\}(-r\sin\theta)$$

$$+ \frac{\partial f}{\partial x}(-r\cos\theta)$$

$$+ \left\{\frac{\partial^2 f}{\partial x \partial y}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial \theta}\right\} r\cos\theta$$

$$+ \frac{\partial f}{\partial y}(-r\sin\theta)$$

$$= \frac{\partial^2 f}{\partial x^2}\underline{(+r^2\sin^2\theta\quad)} + 2\frac{\partial^2 f}{\partial x \partial y}(-r^2\cos\theta\sin\theta)$$

$$+ \frac{\partial^2 f}{\partial y^2}\underline{(+r^2\cos^2\theta)}$$

$$+ \frac{\partial f}{\partial x}(-r\cos\theta) + \frac{\partial f}{\partial y}(-r\sin\theta)$$

次に $\dfrac{\partial^2 f}{\partial r^2}$ を計算する.

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$

$$\frac{\partial^2 f}{\partial r^2} = \frac{\partial}{\partial r}\left\{\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right\}$$

$$= \left\{\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 f}{\partial x\partial y}\cdot\frac{\partial y}{\partial r}\right\}\cos\theta$$

$$+ \left\{\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial r}\right\}\sin\theta$$

$$= \frac{\partial^2 f}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 f}{\partial x\partial y}\sin\theta\cos\theta$$

$$+ \frac{\partial^2 f}{\partial y^2}\sin^2\theta$$

よって $\dfrac{1}{r^2}\dfrac{\partial^2 f}{\partial \theta^2} + \dfrac{\partial^2 f}{\partial r^2}$

$$= \frac{\partial f}{\partial x}\left(-\frac{1}{r}\cos\theta\right) + \frac{\partial f}{\partial y}\left(-\frac{1}{r}\sin\theta\right)$$

$$+ \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} - \frac{1}{r}\frac{\partial f}{\partial r}$$

問3.

まず連立方程式

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$$

を解く. これは

$$\begin{cases} 3x^2 - 3y = 0 & \cdots ① \\ 3y^2 - 3x = 0 & \cdots ② \end{cases}$$

① より $y = x^2$ ② に代入して

$$3(y^4 - y) = 0$$

$$\Leftrightarrow 3y(y-1)(y^2+y+1) = 0$$

$y = 0$ または 1

$x = 0$ または 1

$x=-1$ は ② をみたさない.

次に $(x, y) = (0, 0), (1, 1)$ に対し

$$\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2} - \left\{\frac{\partial^2 f}{\partial x\partial y}\right\}^2$$

を計算する.

$\dfrac{\partial^2 f}{\partial x^2} = 6x \qquad \dfrac{\partial^2 f}{\partial y^2} = 6y$

$\dfrac{\partial^2 f}{\partial x \partial y} = -3$

これを使って計算すると

$\dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2} - \left\{ \dfrac{\partial^2 f}{\partial x \partial y} \right\}$

$= 6 \cdot 0 \cdot 6 \cdot 0 - (-3)^2 = -9 < 0$

$(x, y) = (0, 0)$

$= 6 \cdot 1 \cdot 6 \cdot 1 - (-3)^2 = 27 > 0$

よって $(x, y) = (1, 1)$ で極小となる.

問4.

$F(x, y, \lambda) = x^3 - xy + y^3 - \lambda(x^2 + y^2 - 1)$

この関数に対し

$\dfrac{\partial F}{\partial x} = \dfrac{\partial F}{\partial y} = \dfrac{\partial F}{\partial \lambda} = 0$

で定められる連立方程式を解く.

$\dfrac{\partial F}{\partial x} = 3x^2 - y - 2\lambda x = 0$ … ①

$\dfrac{\partial F}{\partial y} = -x + 3y^2 - 2\lambda y = 0$ … ②

$\dfrac{\partial F}{\partial \lambda} = x^2 + y^2 - 1 = 0$ … ③

① より $y = 3x^2 - 2\lambda x$

② に代入して

$-x + 3(3x^2 - 2\lambda x)^2 - 2\lambda(3x^2 - 2\lambda x) = 0.$

$+27x^4 - 36\lambda x^3 + 12\lambda^2 x^2 - 6\lambda x^2 + 4\lambda^2 x - x = 0$

$x\{27x^3 - 36\lambda^2 x^2 + (12\lambda^2 - 6\lambda)x + 4\lambda^2\} = 0$

まず $x = 0$ の場合を考える.

すると ① より $y = 0$.

これは ③ をみたさない.

①〜③ の解は別紙参照.