Invariant functions with respect to the Whitehead-link

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(joint work with Haruko Nishi and Masaaki Yoshida)

1 Introduction

The Gauss hypergeometric equation $E(\alpha,\beta,\gamma)$

$$x(1-x)\frac{d^2f}{dx^2}+\{\gamma-(\alpha+\beta+1)x\}\frac{df}{dx}-\alpha\beta f=0$$

for $(\alpha,\beta,\gamma)=(1/2,1/2,1)$ induces an isomorphism

$per:\mathbb{C}-\{0,1\}\to\mathbb{H}/M$ (taking the ratio of solutions),

where $\mathbb{H}=\{\tau\in\mathbb{C}\mid \mathrm{Im}(\tau)>0\}$ and M is the monodromy group of $E(1/2,1/2,1)$. Note that M is the level 2 principal congruence subgroup of $SL_2(\mathbb{Z})$, which is isomorphic to the fundamental group $\pi_1(\mathbb{C}-\{0,1\})$. We can regard per as the period map for the family of marked elliptic curves (double covers of $\mathbb{P}^1$ branching at 4 points).

As generalizations, we have some period maps and systems of hypergeometric equations; each of them induces an isomorphism between a certain moduli space of algebraic varieties and the quotient space of a certain symmetric domain by its monodromy group.

For examples,

1. Appell's F_D with special parameters: from families of k-fold branched coverings of $\mathbb{P}^1$ to complex balls studied by Terada and Deligne-Mostow,

2. the period map from the family of cubic surfaces to the 4-dimensional complex ball studied by Allcock-Carlson-Toledo,

3. period maps from some families of certain $K3$ surfaces to complex balls embedded in symmetric domains of type IV lectured by Dolgachev and Kondo,

4. $E(3, 6; 1/2, \dots, 1/2)$: from the family of the double covers branching along 6-lines to the symmetric domain $\mathbb{D}$ of type I_{22} studied by Sasaki-Yoshida-Matsumoto(the speaker).

Interesting automorphic forms appear when we study the inverses of them !

In my talk, we construct automorphic functions on the real 3-dimensional upper half space $\mathbb{H}^3 = \{(z,t) \in \mathbb{C} \times \mathbb{R} \mid t > 0\}$, by observing the Whitehead link $L = L_0 \cup L_\infty$ in Figure 1.

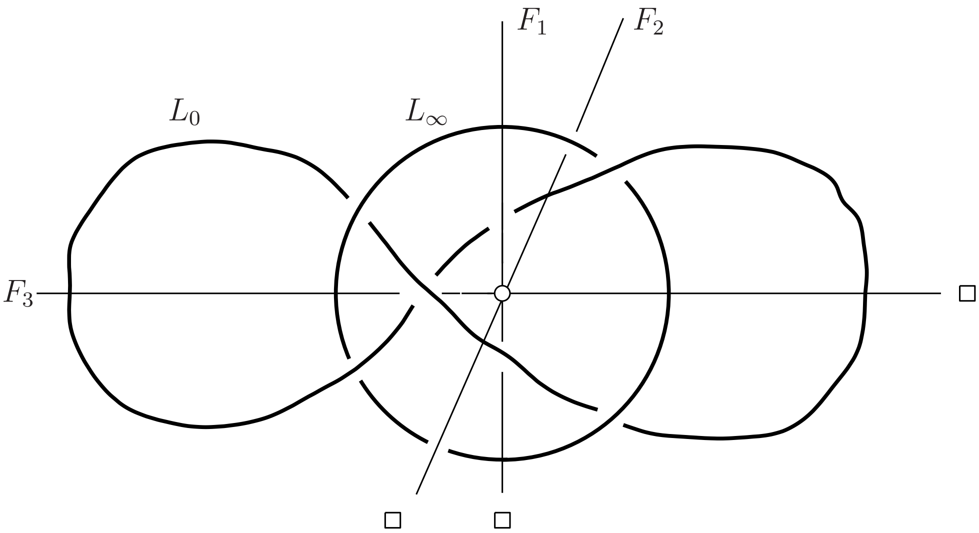


Figure 1: Whitehead link

The Whitehead-link-complement $S^3 - L$ is known to admit a hyperbolic structure: there is a discrete group $W \subset GL_2(\mathbb{C})$ acting on on $\mathbb{H}^3$, and a homeomorphism

$$\varphi : \mathbb{H}^3/W \xrightarrow{\cong} S^3 - L.$$

Note that the situation is quite similar to the inverse of per :

$$per^{-1} : \mathbb{H}/M \longrightarrow \mathbb{C} - \{0,1\}.$$

But one has never tried to make the homeomorphism φ explicit. We construct automorphic functions for W in terms of $\Theta\binom{a}{b}(\tau)$ on the symmetric domain $\mathbb{D}$ of type $I_{2,2}$ over the ring $\mathbb{Z}[i]$ appeared in Example 4, and express the homeomorphism φ in terms of these automorphic functions, which realize some branched coverings of real 3-dimensional orbifolds.

Our automorphic functions derive some properties with respect to the Whitehead link:

- We can express the space $S^3 - L$ as a part of a real algebraic set (we need some inequalities). We can regard L_0 and L_∞ as the exceptional curves arising from the cusps. I expect that some link invariants can be obtained algebraically by our expression.

- We can realize symmetries of the Whitehead link as actions of $(\mathbb{Z}/2\mathbb{Z})^2$ on these automorphic functions.

- Our automorphic functions give an arithmetical characterization of W.

  The group W is given by 2 generators ($W \simeq \pi_1(S^3 - L)$ which is generated by 2 elements). By the definition of W, for a given $g \in GL_2(\mathbb{C})$ it is difficult to know if g belongs to W or not.

2 A hyperbolic structure on the complement of the Whitehead link

Let $\mathbb{H}^3$ be the upper half space model

$$\mathbb{H}^3 = \{(z,t) \in \mathbb{C} \times \mathbb{R} \mid t > 0\}$$

of the 3-dimensional real hyperbolic space.

$GL_2(\mathbb{C})$ and an involution T act on $\mathbb{H}^3$ as

$$g \cdot (z,t) =$$
$$\left(\frac{g_{11}\bar{g}_{21}t^2 + (g_{11}z+g_{12})\overline{(g_{21}z+g_{22})}}{|g_{21}|^2t^2 + (g_{21}z+g_{22})\overline{(g_{21}z+g_{22})}}, \frac{|\det(g)|t}{|g_{21}|^2t^2 + (g_{21}z+g_{22})\overline{(g_{21}z+g_{22})}} \right),$$

$$T \cdot (z,t) = (\bar{z}, t),$$

where

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in GL_2(\mathbb{C}).$$

Put

$$GL_2^T(\mathbb{C}) := \{\langle GL_2(\mathbb{C}), T\rangle \mid T \cdot g = \bar{g} \cdot T\}$$

for $g \in GL_2(\mathbb{C})$.

The Whitehead-link-complement $S^3 - L$ admits a hyperbolic structure. We have a homeomorphism

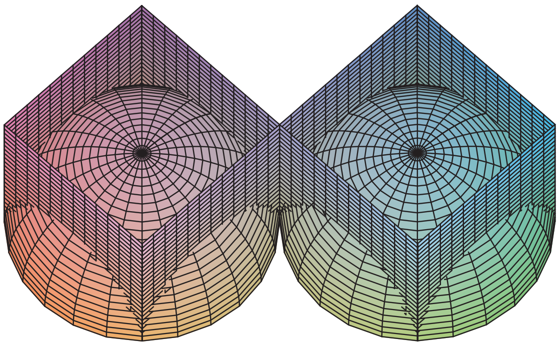
$$\varphi : \mathbb{H}^3/W \xrightarrow{\cong} S^3 - L,$$

where

$$W := \langle g_1, g_2 \rangle, \quad g_1 = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \; g_2 = \begin{pmatrix} 1 & 0 \\ 1+i & 1 \end{pmatrix}.$$

We call W the *Whitehead-link-complement group.*

A fundamental domain FD for W in $\mathbb{H}^3$ is in Figure 2.



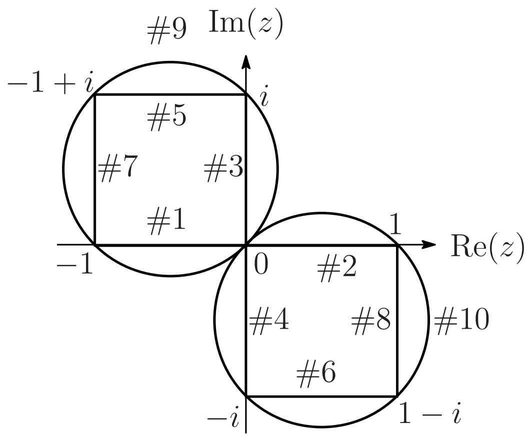


Figure 2: Fundamental domain FD of W in $\mathbb{H}^3$

The group W has two cusps.

$$(z,t) = (*, +\infty), \quad (0,0) \sim (\pm i, 0) \sim (\pm 1, 0) \sim (\mp 1 \pm i, 0).$$

Remark 1 *The monodromy groups of $E(\alpha,\beta,\gamma)$ for parameters satisfying*

$$\cos(2\pi\alpha) = \frac{1+i}{2}, \quad \beta = -\alpha, \quad \gamma \in \mathbb{Z}$$

are conjugate to W.

3 Discrete subgroups of $GL_2(\mathbb{C})$, especially Λ

We define some discrete subgroups of $GL_2(\mathbb{C})$:

$$\begin{aligned}
\Gamma &= GL_2(\mathbb{Z}[i]),\\
S\Gamma_0(1+i) &= \{g = (g_{jk}) \in \Gamma \mid \det(g) = \pm 1,\ g_{21} \in (1+i)\mathbb{Z}[i]\},\\
S\Gamma(1+i) &= \{g \in S\Gamma_0(1+i) \mid g_{12} \in (1+i)\mathbb{Z}[i]\},\\
\Gamma(2) &= \{g \in \Gamma \mid g_{11}-1, g_{12}, g_{21}, g_{22}-1 \in 2\mathbb{Z}[i]\},\\
\overline{W} &= TWT = \{\bar{g} \mid g \in W\},\\
\hat{W} &= W \cap \overline{W},\\
\breve{W} &= \langle W, \overline{W}\rangle.
\end{aligned}$$

Convention: We regard these groups as subgroups of the projectified group $PGL_2(\mathbb{C})$.

For $^\forall G$ in Γ, we denote $G^T = \langle G, T\rangle$ in $GL_2^T(\mathbb{C})$.

$\Gamma^T(2)$ is a Coxeter group generated by the eight reflections, of which mirrors form an octahedron in $\mathbb{H}^3$, see Figure 3.

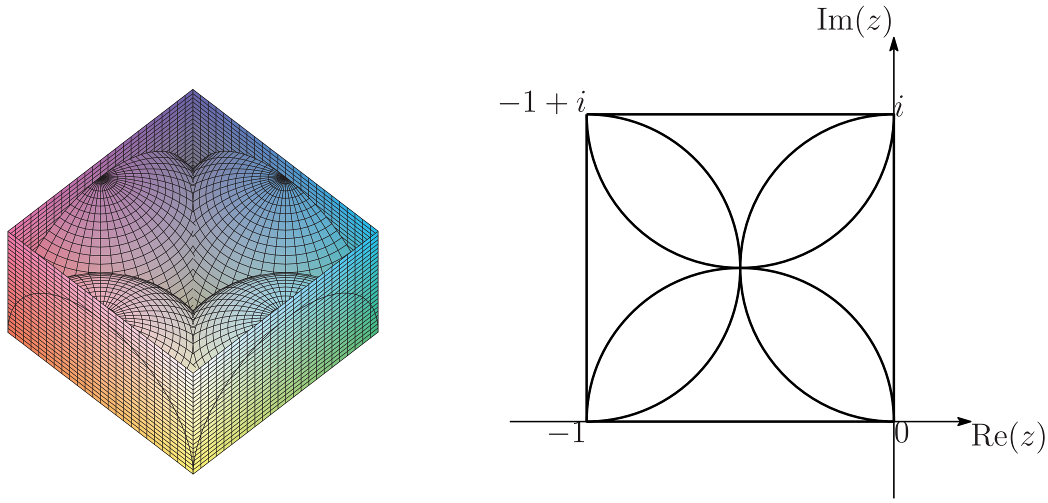


Figure 3: Weyl chamber of $\Gamma^T(2)$

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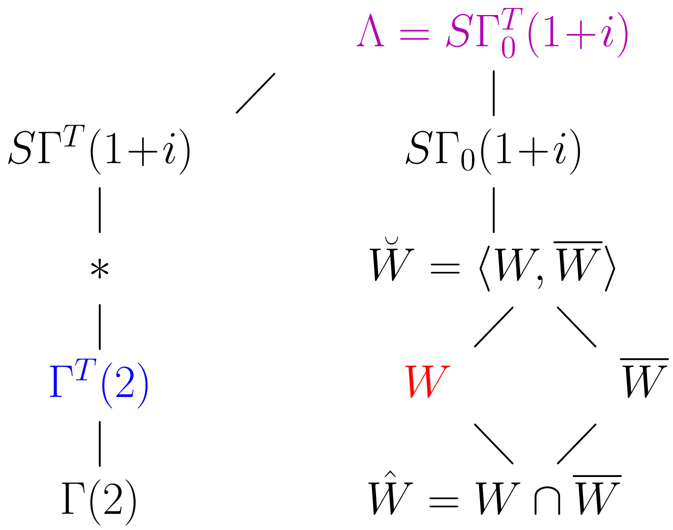
We put

$$\Lambda := \langle \Gamma^T(2), W \rangle.$$



Lemma 1 *1.* $\Gamma^T(2)$ *is normal in* Λ*;*

$\Lambda/\Gamma^T(2) \simeq$ *the dihedral group* D_8 *of order 8.*

2. $[\Lambda, W] = 8$, W *is not normal in* Λ*:* $TWT = \overline{W}$*.*

3. The domain bounded by the four walls

$$a : \mathrm{Im}(z) = 0, \quad b : \mathrm{Re}(z) = 0,$$

$$c : \mathrm{Im}(z) = \frac{1}{2}, \quad d : \mathrm{Re}(z) = -\frac{1}{2},$$

and by the hemisphere

$$\#9 : |z - \frac{-1+i}{2}| = \frac{1}{\sqrt{2}}.$$

is a fundamental domain of Λ*, see Figure 4.*

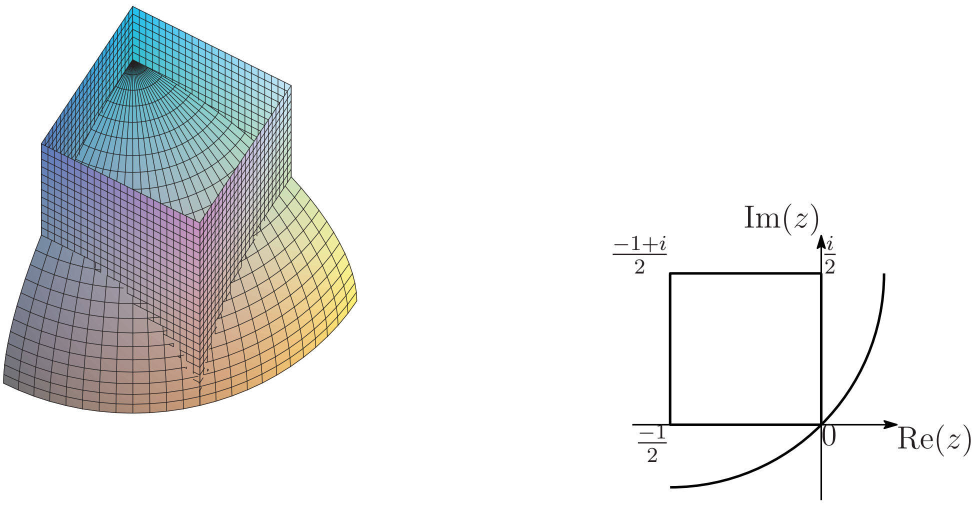


Figure 4: Fundamental domain of Λ

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4. $\Lambda = S\Gamma_0^T(1+i)$ *and* $[S\Gamma_0(1+i), W] = 4$.

(We will see $S\Gamma_0(1+i)/W = (\mathbb{Z}/2\mathbb{Z})^2$*.)*

$$
\begin{array}{ccccc}
\mathbb{H}^3/\Gamma^T(2) & & & & \mathbb{H}^3/W \\
& \diagdown & & & \big|\,\mathbb{Z}/(2\mathbb{Z}) \\
& & & & \mathbb{H}^3/\langle W, \overline{W} \rangle \\
& & \diagdown_{D_8} & & \big|\,\mathbb{Z}/(2\mathbb{Z}) \\
& & & & \mathbb{H}^3/S\Gamma_0(1+i) \\
& & \diagdown & \diagup_{\mathbb{Z}/(2\mathbb{Z})} & \\
& & & \mathbb{H}^3/\Lambda &
\end{array}
$$

Our strategy is following.

At first, we realize the quotient space $\mathbb{H}^3/\Gamma^T(2)$ by using theta functions $\Theta\binom{a}{b}(\tau)$ on $\mathbb{D}$. Next we construct D_8-invariant functions which realize $\mathbb{H}^3/\Lambda$. This step corresponds to the construction of the j-function from the λ-function. Finally, we construct the 3 double covers in the right line step by step. We must know the branch locus of each of double covers. We investigate the symmetry of the Whitehead link.

4 Symmetry of the Whitehead link

Orientation preserving homeomorphisms of S^3 keeping L fixed form a group $(\mathbb{Z}/2\mathbb{Z})^2$. The group consists of π-rotations with axes F_1, F_2 and F_3, and the identity.

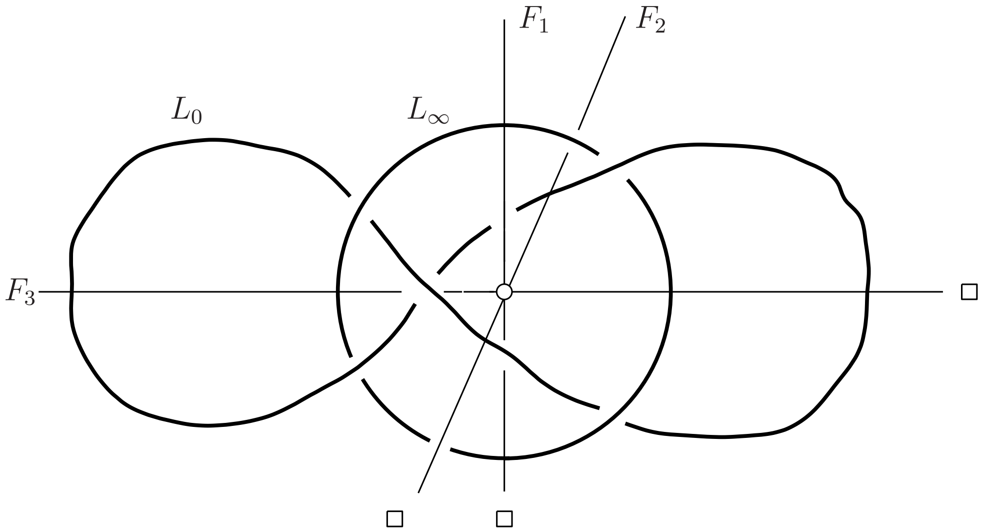


Figure 5: The Whitehead link with its symmetry axes

There is also a reflection of S^3 keeping a mirror (containing L) pointwise fixed.

These rotations and the reflection can be represent as elements of Λ. We give the axes and the mirror in the fundamental domain of $\mathbb{H}^3/W$.

Proposition 1 *The three π-rotations with axes F_1, F_2 and F_3, and the reflection can be represented by the transformations*

$$\gamma_1 : \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \ \gamma_2 : \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ \gamma_3 : \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T,$$

respectively, of $\mathbb{H}^3$ modulo W.

The fixed loci in FD, as well as in $\mathbb{H}^3/W$, of the rotations γ_1, γ_2 and γ_3 are also called the axes F_1, F_2 and F_3; they are depicted in FD as in Figure 6. A bullet $\bullet$ stands for a vertical line: the inverse image of the point under π.

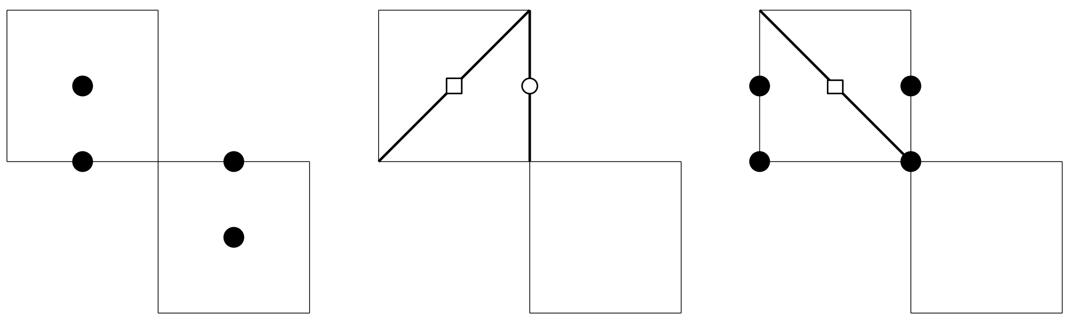
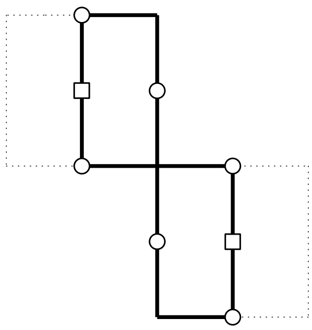
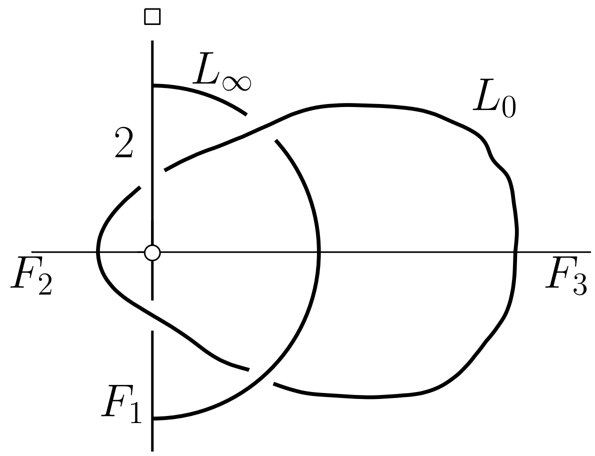


Figure 6: The fixed loci of γ_1, γ_2, γ_3

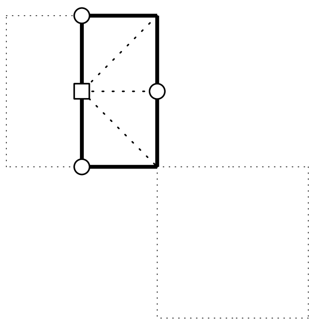
5 Orbit spaces under $\breve{W}, S\Gamma_0(1+i)$ and Λ

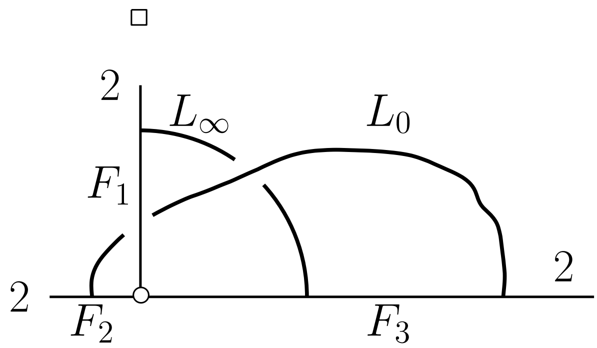
A fundamental domain for $\breve{W}$ and the orbifold $\mathbb{H}^3/\breve{W}$



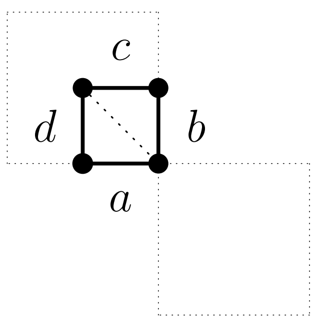


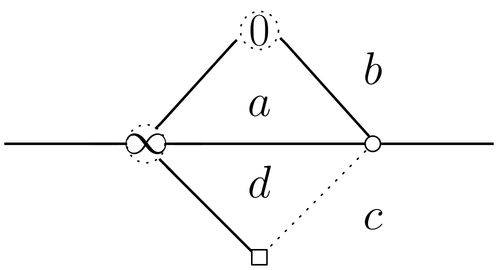
A fundamental domain for $S\Gamma_0(1+i)$ and the orbifold $\mathbb{H}^3/S\Gamma_0(1+i)$





A fundamental domain for Λ and the boundary of $\mathbb{H}^3/\Lambda$





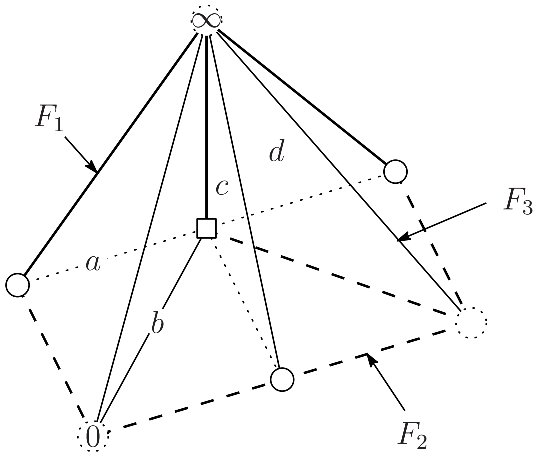


Figure 7: A better picture of the fundamental domain for $S\Gamma_0(1+i)$

Proposition 2

- *The branch locus of the double cover $\mathbb{H}^3/S\Gamma_0(1+i)$ of $\mathbb{H}^3/\Lambda$ is the union of the walls a, b, c, d.*
- *That of the double cover $\mathbb{H}^3/\breve{W}$ of $\mathbb{H}^3/S\Gamma_0(1+i)$ is the union of the axes F_2 and F_3 (the axes F_2 and F_3 are equivalent in the space $\mathbb{H}^3/\breve{W}$).*
- *That of the double cover $\mathbb{H}^3/W$ of $\mathbb{H}^3/\breve{W}$ is the axis F_1.*

6 Theta functions on $\mathbb{D}$

The symmetric domain $\mathbb{D}$ of type $I_{2,2}$ is defined as

$$\mathbb{D} = \left\{ \tau \in M_{2,2}(\mathbb{C}) \mid \frac{\tau - \tau^*}{2i} \text{ is positive definite} \right\}.$$

The group

$$U_{2,2}(\mathbb{C}) = \left\{ h \in GL_4(\mathbb{C}) \mid gJg^* = J = \begin{pmatrix} O & -I_2 \\ I_2 & O \end{pmatrix} \right\}$$

and an involution T act on $\mathbb{D}$ as

$$h \cdot \tau = (h_{11}\tau + h_{12})(h_{21}\tau + h_{22})^{-1}, \quad T \cdot \tau = {}^t\tau,$$

where $h = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \in U_{2,2}(\mathbb{C})$, and h_{jk} are 2×2 matrices.

We define some discrete subgroups of $U_{2,2}(\mathbb{C})$:

$$\begin{aligned} U_{2,2}(\mathbb{Z}[i]) &= U_{2,2}(\mathbb{C}) \cap GL_4(\mathbb{Z}[i]), \\ U_{2,2}(1+i) &= \{h \in U_{2,2}(\mathbb{Z}[i]) \mid h \equiv I_4 \bmod (1+i)\}. \end{aligned}$$

Theta functions $\Theta\binom{a}{b}(\tau)$ on $\mathbb{D}$ are defined as

$$\Theta\begin{pmatrix} a \\ b \end{pmatrix}(\tau) = \sum_{n\in\mathbb{Z}[i]^2} \mathbf{e}[(n+a)\tau(n+a)^* + 2\mathrm{Re}(nb^*)],$$

where $\tau \in \mathbb{D}$, $a, b \in \mathbb{Q}[i]^2$ and $\mathbf{e}[x] = \exp[\pi i x]$. By definition, we have the following fundamental properties.

Fact 1 *1. If* $b \in \frac{1}{1+i}\mathbb{Z}[i]^2$, *then* $\Theta\binom{a}{ib}(\tau) = \Theta\binom{a}{b}(\tau)$.

If $b \in \frac{1}{2}\mathbb{Z}[i]^2$, *then* $\Theta\binom{a}{-b}(\tau) = \Theta\binom{a}{b}(\tau)$.

2. For $k \in \mathbb{Z}$ *and* $m, n \in \mathbb{Z}[i]^2$, *we have*

$$\Theta\begin{pmatrix} i^k a \\ i^k b \end{pmatrix}(\tau) = \Theta\begin{pmatrix} a \\ b \end{pmatrix}(\tau),$$

$$\Theta\begin{pmatrix} a+m \\ b+n \end{pmatrix}(\tau) = \mathbf{e}[-2\mathrm{Re}(mb^*)]\Theta\begin{pmatrix} a \\ b \end{pmatrix}(\tau).$$

3. If $(1+i)ab^* \notin \mathbb{Z}[i]$ *for* $a, b \in \frac{1}{1+i}\mathbb{Z}[i]^2$, *then* $\Theta\binom{a}{b}(\tau) = 0$.

It is known that any action of $U_{2,2}(\mathbb{Z}[i])$ on $\tau \in \mathbb{D}$ can be decomposed into the following transformations:

(1) $\tau \mapsto \tau + s$, where $s = (s_{jk})$ is a 2×2 hermitian matrix over $\mathbb{Z}[i]$;

(2) $\tau \mapsto g\tau g^*$, where $g \in GL_2(\mathbb{Z}[i])$;

(3) $\tau \mapsto -\tau^{-1}$.

Fact 2 *By* T *and these actions,* $\Theta\binom{a}{b}(\tau)$ *is changed into*

$$\Theta\begin{pmatrix} a \\ b \end{pmatrix}(T \cdot \tau) = \Theta\begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix}(\tau),$$

$$\Theta\begin{pmatrix} a \\ b \end{pmatrix}(\tau + s) = \mathbf{e}[asa^*]\Theta\begin{pmatrix} a \\ b + as + \frac{1+i}{2}(s_{11}, s_{22}) \end{pmatrix}(\tau),$$

$$\Theta\begin{pmatrix} a \\ b \end{pmatrix}(g\tau g^*) = \Theta\begin{pmatrix} ag \\ b(g^*)^{-1} \end{pmatrix}(\tau) \quad \text{for } g \in GL_2(\mathbb{Z}[i]),$$

$$\Theta\begin{pmatrix} a \\ b \end{pmatrix}(-\tau^{-1}) = -\det(\tau)\mathbf{e}[2\mathrm{Re}(ab^*)]\Theta\begin{pmatrix} -b \\ a \end{pmatrix}(\tau).$$

In order to get the last equality, use the multi-variable version of the Poisson summation formula. We show the 3rd equality.

$$\begin{aligned} &\Theta\begin{pmatrix} a \\ b \end{pmatrix}(g\tau g^*) \\ =& \sum_{n\in\mathbb{Z}[i]^2} \mathbf{e}[(n+a)(g\tau g^*)(n+a)^* + 2\mathrm{Re}(n(gg^{-1})b^*)] \\ =& \sum_{n\in\mathbb{Z}[i]^2} \mathbf{e}[(ng+ag)\tau(ng+ag)^* + 2\mathrm{Re}(ng(b(g^*)^{-1})^*)] \\ =& \Theta\begin{pmatrix} ag \\ b(g^*)^{-1} \end{pmatrix}(\tau), \end{aligned}$$

since $m = ng$ rnus over $\mathbb{Z}[i]^2$ for any $g \in GL_2(\mathbb{Z}[i])$.

Proposition 3 *If $a, b \in \frac{1}{1+i}\mathbb{Z}[i]^2$ then $\Theta^2\binom{a}{b}(\tau)$ is a modular from of weight 2 with character* det *for $U_{2,2}(1+i)$, i.e.,*

$$\Theta^2\begin{pmatrix} a \\ b \end{pmatrix}(T\cdot\tau) = \Theta^2\begin{pmatrix} a \\ b \end{pmatrix}(\tau),$$

$$\Theta^2\begin{pmatrix} a \\ b \end{pmatrix}(h\cdot\tau) = \det(h)\det(h_{21}\tau + h_{22})^2\Theta^2\begin{pmatrix} a \\ b \end{pmatrix}(\tau),$$

for any $h = (h_{jk}) \in U_{2,2}(1+i)$.

By following the proof of Jacobi' identity for lattices $L_1 = \mathbb{Z}[i]^2$, $L_2 = \mathbb{Z}[i]^2 A$, $L = \langle L_1, L_2\rangle$, where

$$A = \frac{1+i}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad AA^* = I_2,\ A^2 = iI_2,$$

we have quadratic relations among theta functions $\Theta\binom{a}{b}(\tau)$.

Theorem 1

$$4\Theta\binom{a}{b}(\tau)^2 = \sum_{e,f\in\frac{1+i}{2}\mathbb{Z}[i]^2/\mathbb{Z}[i]^2} \mathbf{e}[2\mathrm{Re}((1+i)be^*)] \Theta\binom{e+(1+i)a}{f+(1+i)b}(\tau)\Theta\binom{e}{f}(\tau).$$

For $a, b \in (\frac{\mathbb{Z}[i]}{1+i}/\mathbb{Z}[i])^2$, there are 10 non-vanishing $\Theta\binom{a}{b}(\tau)$.

Corollary 1 *The ten $\Theta\binom{a}{b}(\tau)^2$ satisfy the same linear relations as the Plücker relations for the $(3,6)$-Grassmann manifold, which is the linear relations among the 10 products $D_{ijk}(X)D_{lmn}(X)$ of the Plücker coordinates, where*

$$X = \begin{pmatrix} x_{11} & \dots & x_{16} \\ x_{21} & \dots & x_{26} \\ x_{31} & \dots & x_{36} \end{pmatrix}, \quad D_{ijk}(X) = \det \begin{pmatrix} x_{1i} & x_{1j} & x_{1k} \\ x_{2i} & x_{2j} & x_{2k} \\ x_{3i} & x_{3j} & x_{3k} \end{pmatrix}$$

and $\{i,j,k,l,m,n\} = \{1,\dots,6\}$.

There are 5 linearly independent $\Theta^2\binom{a}{b}(\tau)$.

Remark 2 *τ can be regarded as periods of the $K3$-surface coming from the double cover of $\mathbb{P}^2$ branching along 6 lines given by the 6 columns of X.*

7 Embedding of $\mathbb{H}^3$ into $\mathbb{D}$

We embed $\mathbb{H}^3$ into $\mathbb{D}$ by

$$\imath : \mathbb{H}^3 \ni (z,t) \mapsto \frac{i}{t}\begin{pmatrix} t^2+|z|^2 & z \\ \bar{z} & 1 \end{pmatrix} \in \mathbb{D};$$

we define a homomorphism

$$\jmath : GL_2(\mathbb{C}) \ni g \mapsto \begin{pmatrix} g/\sqrt{|\det(g)|} & O \\ O & (g^*/\sqrt{|\det(g)|})^{-1} \end{pmatrix} \in U_{2,2}(\mathbb{C}).$$

They satisfy

$$\begin{aligned} \imath(g\cdot(z,t)) &= \jmath(g)\cdot\imath(z,t) \quad \text{for any } g \in GL_2(\mathbb{C}), \\ \imath(T\cdot(z,t)) &= T\cdot\imath(z,t), \\ -(\imath(z,t))^{-1} &= \left(\jmath\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\cdot T\right)\cdot\imath(z,t). \end{aligned}$$

We denote the pull back of $\Theta\binom{a}{b}(\tau)$ under the embedding $\imath : \mathbb{H}^3 \to \mathbb{D}$ by $\Theta\binom{a}{b}(z,t)$.

By definition, we have the following.

Fact 3 *1. For $a, b \in \frac{1}{2}\mathbb{Z}[i]^2$, each $\Theta\binom{a}{b}(z,t)$ is real valued. If $2\mathrm{Re}(ab^*) + 2\mathrm{Im}(ab^*) \notin \mathbb{Z}$ then $\Theta\binom{a}{b}(z,t) \equiv 0$.*

2. For $a, b \in \frac{1}{1+i}\mathbb{Z}[i]^2$, each $\Theta\binom{a}{b}(z,t)$ is invariant under the action of $\Gamma^T(2)$.

3. The function $\Theta = \Theta\binom{00}{00}(z,t)$ is positive and invariant under the action of Γ^T.

8 Automorphic functions for $\Gamma^T(2)$ and an embedding of $\mathbb{H}^3/\Gamma^T(2)$

Set

$$\Theta\begin{bmatrix} p \\ q \end{bmatrix} = \Theta\begin{bmatrix} p \\ q \end{bmatrix}(z,t) = \Theta\begin{pmatrix} \frac{p}{2} \\ \frac{q}{2} \end{pmatrix}(z,t), \quad p,q \in \mathbb{Z}[i]^2$$

and

$$x_0 = \Theta\begin{bmatrix} 0,0 \\ 0,0 \end{bmatrix}, \quad x_1 = \Theta\begin{bmatrix} 1+i, 1+i \\ 1+i, 1+i \end{bmatrix},$$
$$x_2 = \Theta\begin{bmatrix} 1+i, 0 \\ 0, 1+i \end{bmatrix}, \quad x_3 = \Theta\begin{bmatrix} 0, 1+i \\ 1+i, 0 \end{bmatrix}.$$

Theorem 2 *The map*

$$\mathbb{H}^3 \ni (z,t) \mapsto \frac{1}{x_0}(x_1, x_2, x_3) \in \mathbb{R}^3$$

induces an isomorphism between $\mathbb{H}^3/\Gamma^T(2)$ *and the octahedron*

$$Oct = \{(t_1,t_2,t_3) \in \mathbb{R}^3 \mid |t_1| + |t_2| + |t_3| \leq 1\}$$

minus the six vertices $(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$.

9 Automorphic functions for Λ and an embedding of $\mathbb{H}^3/\Lambda$

Proposition 4 *g_1, g_2 induce transformations of x_1, x_2, x_3:*

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot g_1 = \begin{pmatrix} & -1 & \\ -1 & & \\ & & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot g_2 = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

This a representation of the dihedral group D_8 of order 8.

Theorem 3 $x_1^2 + x_2^2,\ \ x_1^2 x_2^2,\ \ x_3^2,\ \ x_1 x_2 x_3$ *are Λ-invariant. The map*

$$\lambda : \mathbb{H}^3 \ni (z,t) \longmapsto \begin{matrix} & (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \\ = & (\xi_1^2 + \xi_2^2,\ \ \xi_1^2 \xi_2^2,\ \ \xi_3^2,\ \ \xi_1 \xi_2 \xi_3) \end{matrix} \in \mathbb{R}^4,$$

where $\xi_j = x_j / x_0$,

induces an embedding of $\mathbb{H}^3/\Lambda$ into the subdomain of the variety $\lambda_2 \lambda_3 = \lambda_4^2$.

10 Automorphic functions for W

Set

$$y_1 = \Theta\begin{bmatrix} 0, 1 \\ 1+i, 0 \end{bmatrix}, \quad y_2 = \Theta\begin{bmatrix} 1+i, 1 \\ 1+i, 0 \end{bmatrix},$$

$$z_1 = \Theta\begin{bmatrix} 0, 1 \\ 1, 0 \end{bmatrix}, \quad z_2 = \Theta\begin{bmatrix} 1+i, 1 \\ 1, 1+i \end{bmatrix}.$$

We define functions as

$$\begin{aligned} \Phi_1 &= x_3 z_1 z_2, \\ \Phi_2 &= (x_2 - x_1) y_1 + (x_2 + x_1) y_2, \\ \Phi_3 &= (x_1^2 - x_2^2) y_1 y_2. \end{aligned}$$

Theorem 4 *Φ_1, Φ_2 and Φ_3 are W-invariant. By the actions $g = I_2 + 2\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \Gamma(2)$ and T,*

$$\begin{aligned} \Phi_1 \cdot g &= \mathbf{e}[\mathrm{Re}((1+i)p + (1-i)s)]\Phi_1, \\ \Phi_2 \cdot g &= \mathbf{e}[\mathrm{Re}(r(1-i))]\Phi_2, \\ \Phi_3 \cdot g &= \Phi_3. \end{aligned}$$

$$\Phi_1 \cdot T = \Phi_1, \quad \Phi_3 \cdot T = -\Phi_3.$$

Remark 3 $\Phi_2 \cdot T = (x_2 - x_1) y_1 - (x_2 + x_1) y_2$. *This is not invariant under W but invariant under $\overline{W}$.*

Let Iso_j be the isotropy subgroup of $\Lambda = S\Gamma_0^T(1+i)$ for Φ_j.

Theorem 5 *We have*

$$S\Gamma_0(1+i) = \mathrm{Iso}_3, \quad \breve{W} = \mathrm{Iso}_1 \cap \mathrm{Iso}_3, \quad W = \mathrm{Iso}_1 \cap \mathrm{Iso}_2 \cap \mathrm{Iso}_3.$$

By this theorem, we have $S\Gamma_0(1+i)/W \simeq (\mathbb{Z}/2\mathbb{Z})^2$.

Theorem 6 *An element* $g = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in S\Gamma_0(1+i)$ *satisfying* $\mathrm{Re}(s) \equiv 1 \bmod 2$ *belongs to* $\breve{W}$ *if and only if*

$$\frac{\mathrm{Re}(p) + \mathrm{Im}(s) - (-1)^{\mathrm{Re}(q)+\mathrm{Im}(q)}(\mathrm{Im}(p) + \mathrm{Re}(s))}{2} \equiv \frac{((-1)^{\mathrm{Re}(r)} + 1)\mathrm{Im}(q) + (\mathrm{Re}(q) + \mathrm{Im}(q))(\mathrm{Re}(r) + \mathrm{Im}(r))}{2} \bmod 2.$$

The element $g \in \breve{W}$ *belongs to* W *if and only if*

$$\mathrm{Re}(p+q) + \frac{\mathrm{Re}(r) - (-1)^{\mathrm{Re}(q)+\mathrm{Im}(q)}\mathrm{Im}(r)}{2} \equiv 1 \bmod 2.$$

The element $g \in W$ *belongs to* $\hat{W} = W \cap \bar{W}$ *if and only if* $r \in 2\mathbb{Z}[i]$.

11 Embeddings of the quotient spaces

Put

$$
\begin{aligned}
f_{00} &= (x_2^2 - x_1^2)y_1y_2 = \Phi_3,\\
f_{01} &= (x_2^2 - x_1^2)z_1z_2z_3z_4,\\
f_{11} &= x_3z_1z_2 = \Phi_1,\\
f_{12} &= x_1x_2z_1z_2,\\
f_{13} &= x_3(x_2^2 - x_1^2)z_3z_4,\\
f_{14} &= x_1x_2(x_2^2 - x_1^2)z_3z_4,\\
f_{20} &= (x_2 - x_1)z_2z_3 + (x_2 + x_1)z_1z_4,\\
f_{21} &= z_1z_2\{(x_2 - x_1)z_1z_3 + (x_2 + x_1)z_2z_4\},\\
f_{22} &= (x_2^2 - x_1^2)\{(x_2 - x_1)z_1z_4 + (x_2 + x_1)z_2z_3\},\\
f_{30} &= (x_2 - x_1)y_1 + (x_2 + x_1)y_2 = \Phi_2,\\
f_{31} &= (x_2 - x_1)z_1z_3 - (x_2 + x_1)z_2z_4,\\
f_{32} &= z_3z_4\{-(x_2 - x_1)z_1z_4 + (x_2 + x_1)z_2z_3\},
\end{aligned}
$$

where (x_0, x_1, x_2, x_3) and (z_1, z_2, z_3, z_4) are

$$
\Theta\begin{bmatrix}0,0\\0,0\end{bmatrix},\ \Theta\begin{bmatrix}1+i,1+i\\1+i,1+i\end{bmatrix},\ \Theta\begin{bmatrix}1+i,0\\0,1+i\end{bmatrix},\ \Theta\begin{bmatrix}0,1+i\\1+i,0\end{bmatrix},
$$

$$
\Theta\begin{bmatrix}0,1\\1,0\end{bmatrix},\quad \Theta\begin{bmatrix}1+i,1\\1,1+i\end{bmatrix},\quad \Theta\begin{bmatrix}0,i\\1,0\end{bmatrix},\quad \Theta\begin{bmatrix}1+i,i\\1,1+i\end{bmatrix}.
$$

Proposition 5 *We have*

$$
\begin{aligned}
4z_1^2 &= (x_0+x_1+x_2+x_3)(x_0-x_1-x_2+x_3),\\
4z_2^2 &= (x_0+x_1-x_2-x_3)(x_0-x_1+x_2-x_3),\\
4z_3^2 &= (x_0+x_1-x_2+x_3)(x_0-x_1+x_2+x_3),\\
4z_4^2 &= (x_0+x_1+x_2-x_3)(x_0-x_1-x_2-x_3).
\end{aligned}
$$

Proposition 6 f_{jp} *are* W*-invariant. These change the signs by the actions of* γ_1*,* γ_2 *and* γ_3 *as in the table*

	γ_1	γ_2	γ_3
f_{0j}	$+$	$+$	$+$
f_{1j}	$+$	$-$	$-$
f_{2j}	$-$	$+$	$-$
f_{3j}	$-$	$-$	$+$

Theorem 7 *The analytic sets* V_1*,* V_2, V_3 *of the ideals*

$$
I_1 = \langle f_{11}, f_{12}, f_{13}, f_{14}\rangle, \quad I_2 = \langle f_{21}, f_{22}\rangle, \quad I_3 = \langle f_{31}, f_{32}\rangle
$$

are $F_2 \cup F_3$*,* $F_1 \cup F_3$*,* $F_1 \cup F_2$*.*

Corollary 2 *The analytic set* V_{jk} *of the ideals* $\langle I_j, I_k\rangle$ *is* F_l *for* $\{j,k,l\} = \{1,2,3\}$*.*

Theorem 8 *The map*

$$\varphi_0 : \mathbb{H}^3/S\Gamma_0(1+i) \ni (z,t) \mapsto (\lambda_1, \dots, \lambda_4, \eta_{01}) \in \mathbb{R}^5$$

is injective, where $\eta_{01} = f_{01}/x_0^6$. *Its image* $\mathrm{Image}(\varphi_0)$ *is determined by the image* $\mathrm{Image}(\lambda)$ *under* $\lambda : \mathbb{H}^3 \ni (z,t) \mapsto (\lambda_1, \dots, \lambda_4)$ *and the relation*

$$\begin{aligned} & 256 f_{01}^2 \\ = \; & (\lambda_1^2 - 4\lambda_2) \prod_{\varepsilon_3=\pm 1} (\lambda_3^2 - 2(x_0^2 + \lambda_1)\lambda_3 + \varepsilon_3 8 x_0 \lambda_4 \\ & \qquad + x_0^4 - 2x_0^2\lambda_1 + \lambda_1^2 - 4\lambda_2), \end{aligned}$$

as a double cover of $\mathrm{Image}(\lambda)$ *branching along its boundary.*

F_1, F_2 and F_3 can be illustrated as in Figure 8. Each of the two cusps $\bar{\infty}$ and $\bar{0}$ is shown as a hole. These holes can be deformed into sausages as in Figure 9.

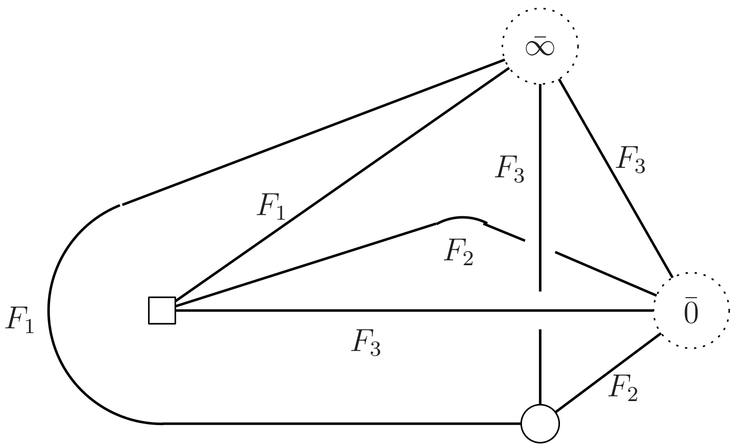


Figure 8: Orbifold singularities in $\mathrm{Image}(\varphi_0)$ and the cusps $\bar{\infty}$ and $\bar{0}$

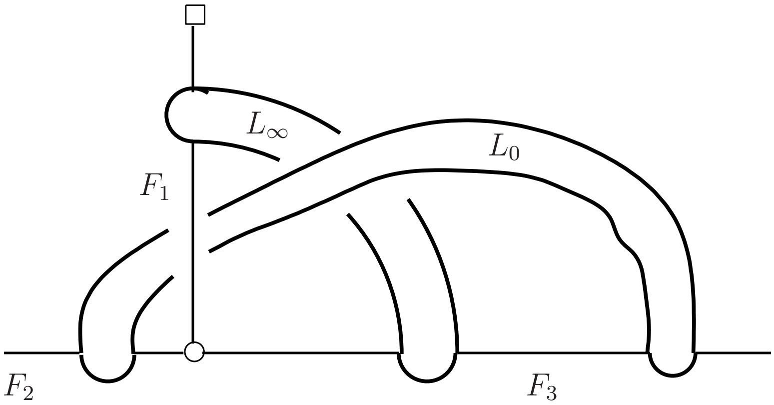


Figure 9: The cusp-holes are deformed into two sausages

Theorem 9 *The map*

$$\varphi_1 : \mathbb{H}^3/\breve{W} \ni (z,t) \mapsto (\varphi_0, \eta_{11}, \dots, \eta_{14}) \in \mathbb{R}^9$$

is injective, where $\eta_{1j} = f_{1j}/x_0^{\deg(f_{1j})}$. *The products* $f_{1p}f_{1q}$ $(1 \le p \le q \le 4)$ *can be expressed as polynomials of* x_0, $\lambda_1, \dots, \lambda_4$ *and* f_{01}. *The image* $\mathrm{Image}(\varphi_0)$ *together with these relations determines the image* $\mathrm{Image}(\varphi_1)$ *under the map* φ_1.

The boundary of a small neighborhood of the cusp $\varphi_1(0)$ is a torus, which is the double cover of that of the cusp $\varphi_0(0)$; note that two F_2-curves and two F_3-curves stick into $\varphi_0(0)$. The boundary of a small neighborhood of the cusp $\varphi_1(\infty)$ remains to be a 2-sphere; note that two F_1-curves and two F_3-curves stick into $\varphi_0(\infty)$, and that four F_1-curves stick into $\varphi_1(\infty)$.

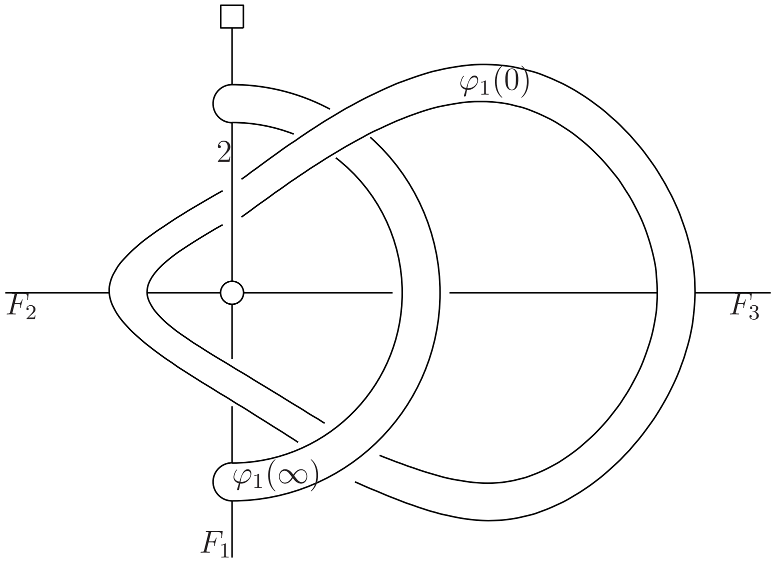


Figure 10: The double covers of the cusp holes

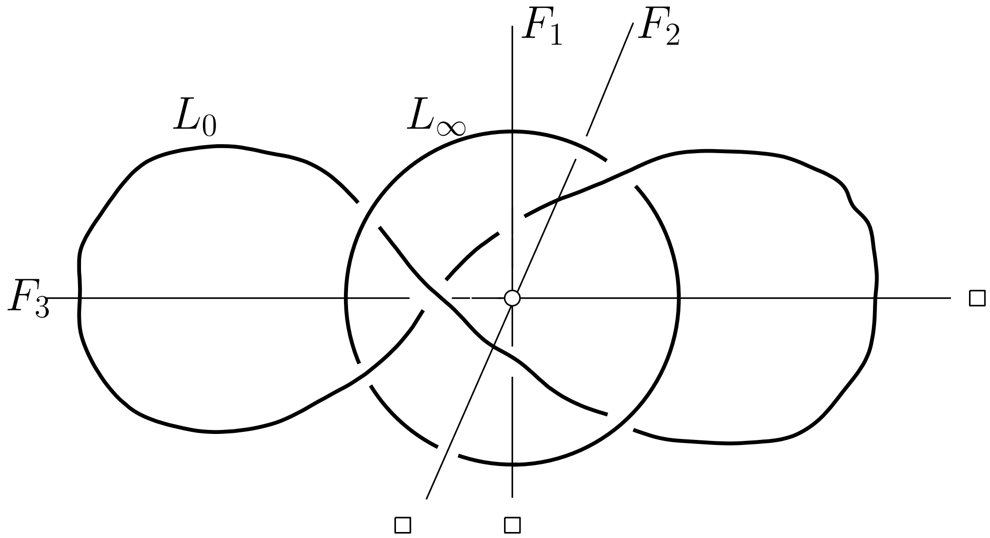
Theorem 10 *The map*

$$\varphi : \mathbb{H}^3/W \ni (z,t) \mapsto (\varphi_1, \eta_{21}, \eta_{22}, \eta_{31}, \eta_{32}) \in \mathbb{R}^{13}$$

is injective, where $\eta_{ij} = f_{ij}/x_0^{\deg(f_{ij})}$. *The products* $f_{2q}f_{2r}$ $f_{3q}f_{3r}$ *and* $f_{1p}f_{2q}f_{3r}$ $(p = 1, \ldots, 4,\ \ q, r = 1, 2)$ *can be expressed as polynomials of* x_0, $\lambda_1, \ldots, \lambda_4$ *and* f_{01}. *The image* $\mathrm{Image}(\varphi_1)$ *together with these relations determines the image* $\mathrm{Image}(\varphi)$ *under the map* φ.

The boundary of a small neighborhood of the cusp $\varphi(\infty)$ is a torus, which is the double cover of that of the cusp $\varphi_1(\infty)$; recall that four F_1-curves stick into $\varphi_1(\infty)$. The boundary of a small neighborhood of the cusp $\varphi(0)$ is a torus, which is the unbranched double cover of that of the cusp $\varphi_1(0)$, a torus.

Eventually, the sausage and the doughnut in Figure 10 are covered by two linked doughnuts, tubular neighborhoods of the curves L_0 and L_∞ of the Whitehead link.



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