A Heun differential equation derived from the Gauss hypergeometric differential equation

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1. Introduction.

We study the differential equation $H(\alpha, \beta)$ for the function $f(x^3)$ under the condition $\gamma = 2/3$, where $f(y)$ is a solution of the Gauss H.G.D.E $E(\alpha, \beta, \gamma)$:

$$y(1-y)\frac{d^2}{dy^2}f(y) + \{\gamma - (\alpha+\beta+1)y\}\frac{d}{dy}f(y) - \alpha\beta f(y) = 0.$$

$H(\alpha, \beta)$ has four regular singular points $x = 1, \omega, \omega^2$ and ∞, where $\omega = \frac{-1+\sqrt{-3}}{2}$; this is a Heun differential equation.

We first show that the periods for the family $\{C(x) \mid x \in \mathbb{C} - \{1, \omega, \omega^2\}\}$ of cubic curves

$$C(x) = \{[t_0, t_1, t_2] \in \mathbb{P}^2 \mid t_0^3 + t_1^3 + t_2^3 - 3xt_0t_1t_2 = 0\} \subset \mathbb{P}^2$$

of the Hesse normal form satisfy $H(1/3, 1/3)$.

We next give a monodromy representation of $H(\alpha, \beta)$.

Finally, we find parameters α, β and fundamental solutions of $H(\alpha, \beta)$ such that the monodromy group of these solutions coincides with a representation of the fundamental group of the Borromean-rings-complement.

2. The Heun equation derived from the Gauss H.G.D.E.

Let f be a solution of H.G.D.E. $E(\alpha,\beta,\gamma)$ and $\imath$ be the map $\mathbb{C} \ni x \mapsto y = x^3 \in \mathbb{C}$. We study the differential equation for the function $h(x) = f(x^3) = \imath^*(f)$.

Since we have

$$\frac{d}{dx}h(x) = 3x^2\frac{d}{dy}f(y), \qquad \frac{d^2}{dx^2}h(x) = 6x\frac{d}{dy}f(y) + 9x^4\frac{d^2}{dy^2}f(y),$$

$\frac{d}{dy}f(y)$ and $\frac{d^2}{dy^2}f(y)$ are expressed as

$$\frac{1}{3x^2}\frac{d}{dx}h(x), \qquad \frac{1}{9x^4}\frac{d^2}{dx^2}h(x) - \frac{2}{9x^5}\frac{d}{dx}h(x),$$

respectively.

Thus $h(x)$ satisfies the differential equation

$$x^3(1-x^3)[\frac{1}{9x^4}\frac{d^2}{dx^2}h(x)-\frac{2}{9x^5}\frac{d}{dx}h(x)]$$
$$+\{\gamma-(\alpha+\beta+1)x^3\}[\frac{1}{3x^2}\frac{d}{dx}h(x)]-\alpha\beta h(x)=0,$$

which is equivalent to

$$x(1-x^3)\frac{d^2h(x)}{dx^2}+\{(3\gamma-2)-(3\alpha+3\beta+1)x^3\}\frac{dh(x)}{dx}-9\alpha\beta x^2h(x)=0.$$

When $\gamma=2/3$, this equation reduces to

$$H(\alpha,\beta):\ (1-x^3)\frac{d^2}{dx^2}h(x)-(3\alpha+3\beta+1)x^2\frac{d}{dx}h(x)-9\alpha\beta xh(x)=0,$$

which has four regular singular points $x=1,\omega,\omega^2$ and ∞.
Hence, $H(\alpha,\beta)$ is a Heun differential equation.

3. Periods of cubic curves of the Hesse normal form.

Any non-singular cubic curve in $\mathbb{P}^2$ can be transformed into the Hesse normal form

$$C(x) = \{[t_0, t_1, t_2] \in \mathbb{P}^2 \mid t_0^3 + t_1^3 + t_2^3 - 3x t_0 t_1 t_2 = 0\},$$

$x \in \mathbb{C} - \{1, \omega, \omega^2\}$, by a projective transformation.

Since $C(x)$ is a Riemann surface of genus 1, there exists a nowhere vanishing holomorphic 1-from

$$\varphi = \frac{t_0 dt_1 - t_1 dt_0}{t_2^2 - x t_0 t_1}$$

for any $x \in \mathbb{C} - \{1, \omega, \omega^2\}$.

We take an element c of $H_1(C(0), \mathbb{Z})$ for $x = 0$; we can make the continuation $c(x) \in H_1(C(x), \mathbb{Z})$ of the cycle c along a path in $\mathbb{C} - \{1, \omega, \omega^2\}$ by the local triviality of the family $\{C(x)\}$. The integral $p(x) = \int_{c(x)} \varphi$ is called **a period of $C(x)$**.

Proposition 1 The period $p(x) = \int_{c(x)} \varphi$ of $C(x)$ satisfies the differential equation $H(1/3, 1/3)$.

Proof. Set $(u, v) = (t_1/t_0, t_2/t_0)$ and

$$q = q(x; u, v) = u^3 + v^3 + 1 - 3xuv;$$

the curve $C(x)$ is expressed as $q(x; u, v) = 0$.

Note that

$$p(x) = \int_{c(x)} \frac{du}{v^2 - xu}.$$

By the local triviality of the family $\{C(x)\}$, we have

$$\frac{d}{dx}\int_{c(x)} \psi(x; u, v)du = \int_{c(x)} \{\frac{\partial}{\partial x}\psi + \frac{\partial}{\partial v}\psi \frac{\partial v(x, u)}{\partial x}\}du,$$

where $\psi du = \psi(x; u, v)du$ is a meromorphic 1-form on $C(x)$, and we regard the variable v as the implicit function of x and u by the equality $q(x; u, v) = 0$.

Differentiating the equality

$$q(x;u,v) = u^3 + v(x,u)^3 - 3xuv(x,u) = 0$$

with respect to x, we have

$$3v(x,u)^2 \frac{\partial v(x,u)}{\partial x} - 3uv(x,u) - 3xu\frac{\partial v(x,u)}{\partial x} = 0,$$

which is equivalent to

$$\frac{\partial v(x,u)}{\partial x} = \frac{uv(x,u)}{v^2(x,u) - xu}.$$

Thus $\frac{d}{dx}\int_{c(x)} \psi(x;u,v)du$ is given as

$$\int_{c(x)} \{(\frac{\partial}{\partial x} + \frac{uv}{v^2 - xu}\frac{\partial}{\partial v})\psi\}du.$$

Hence we have

$$\begin{aligned}\frac{d}{dx}p(x) &= \int_{c(x)} \frac{-u(v^2+xu)}{(v^2-xu)^3}du,\\ \frac{d^2}{dx^2}p(x) &= \int_{c(x)} \frac{2xu^3(5v^2+xu)}{(v^2-xu)^5}du.\end{aligned}$$

We show that the 1-from $\eta(x;u,v)du$ is exact, where

$$[(1-x^3)\frac{d^2}{dx^2} - 3x^2\frac{d}{dx} - x]p(x) = x\int_{c(x)} \eta(x;u,v)du,$$

$$\eta(x;u,v) = \frac{2xu^4-(9x^3-10)u^3v^2-9x^2u^2v^4+7xuv^6-v^8}{(v^2-xu)^5}.$$

Since $dq = q_u du + q_v dv = 0$, we have

$$dv = -\frac{q_u}{q_v} du = -\frac{u^2 - xv}{v^2 - xu} du.$$

For a meromorphic function $F = \frac{(u^3-1)uv}{(v^2-xu)^3}$ on $C(x)$, dF is

$$\frac{\partial}{\partial u} F du + \frac{\partial}{\partial v} F dv = \{(\frac{\partial}{\partial u} - \frac{u^2 - xv}{v^2 - xu} \frac{\partial}{\partial v}) F\} du,$$

and $\eta + dF$ is

$$\frac{xu^4 + 5u^3v^2 + 3x^2u^2v + 4xuv^3 - v^5}{(v^2 - xu)^5} q(x; u, v) du,$$

which vanishes on $C(x)$. $\square$

4. Monodromy representation.

Fact 1 (Theorem 6.1 in [K)] If none of $\alpha, \beta, \gamma-\alpha$ and $\gamma-\beta$ is an integer, then there exists a fundamental system $\mathrm{f}(y) = \begin{pmatrix} f_0(y) \\ f_1(y) \end{pmatrix}$ of $E(\alpha, \beta, \gamma)$ such that the monodromy group with respect to this system is generated by

$$\begin{pmatrix} 1 & 0 \\ -(1-e^{-2\pi i\beta}) & e^{-2\pi i\gamma} \end{pmatrix}, \quad \begin{pmatrix} 1 & 1-e^{-2\pi i\alpha} \\ 0 & e^{-2\pi i(\alpha+\beta-\gamma)} \end{pmatrix}.$$

These matrices are given by the continuation of $\mathrm{f}(y)$ along a loop encircling the point $y=0$ once in the positive sence and along a loop encircling the point $y=1$ once in the positive sence, respectively.

By putting $\gamma = 2/3$ for the matrices in Fact 1, we set

$$\rho_0 = \begin{pmatrix} 1 & 0 \\ -(1-e^{-2\pi i\beta}) & \omega \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} 1 & 1-e^{-2\pi i\alpha} \\ 0 & \omega^2 e^{-2\pi i(\alpha+\beta)} \end{pmatrix}.$$

Note that the eigenvalues of ρ_0 are 1 and ω and that

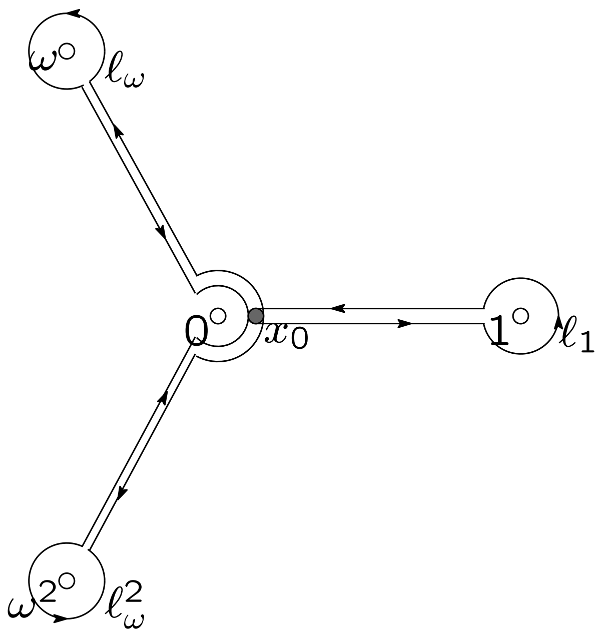
$$\rho_0^3 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Proposition 2 If none of $\alpha, \beta, 2/3-\alpha$ and $2/3-\beta$ is an integer, then there exists a fundamental system of $H(\alpha,\beta)$ such that the monodromy group with respect to this system is generated by

$$\rho_1, \quad \rho_0\rho_1\rho_0^{-1}, \quad \rho_0^2\rho_1\rho_0^{-2}.$$

Proof. Under the condition for parameters in this proposition, $\mathrm{h}(x) = \begin{pmatrix} f_0(x^3) \\ f_1(x^3) \end{pmatrix}$ is a fundamental system of solutions of $H(\alpha,\beta)$. We take a base point x_0 as a small positive real number ε.

Let ℓ_1,ℓ_ω and ℓ_{ω^2} be loops as in the following figure.



When x varies along ℓ_1, $y = x^3$ turns the point $y = 1$ once in the positive sence. Thus $\mathrm{h}(x)$ changes into $\rho_1 \mathrm{h}(x)$ by the continuation along the loop ℓ_1.

Since $y = x^3$ turns the point $y = 0$ once in the positive sence when x varies along the arc with center at 0 in the loop ℓ_ω, $\mathrm{h}(x)$ changes into $\rho_0 \mathrm{h}(x)$ by the continuation along this arc. Thus $\mathrm{h}(x)$ changes into $\rho_0 \rho_1 \rho_0^{-1} \mathrm{h}(x)$ by the continuation along the loop ℓ_ω.

Similarly, $\mathrm{h}(x)$ changes into $\rho_0^2 \rho_1 \rho_0^{-2} \mathrm{h}(x)$ by the continuation along a certain loop ℓ_{ω^2}.

Since $\pi_1(\mathbb{C} - \{1, \omega, \omega^2\}, x_0)$ is generated by the three loops ℓ_1, ℓ_ω and ℓ_{ω^2}, the monodromy group with respect to $\mathrm{h}(x)$ is generated by ρ_1, $\rho_0 \rho_1 \rho_0^{-1}$ and $\rho_0^2 \rho_1 \rho_0^{-2}$. $\square$

The monodromy group of the fundamental system $\mathrm{h}(x)$ of the differential equation $H(1/3, 1/3)$ is generated by

$$m_{1+j} = m_0^j m_1 m_0^{-j}, \quad (j = 0, 1, 2),$$

where

$$m_0 = \begin{pmatrix} 1 & 0 \\ -1+\omega^2 & \omega \end{pmatrix}, \quad m_1 = \begin{pmatrix} 1 & 1-\omega^2 \\ 0 & 1 \end{pmatrix}.$$

For the matrix $P = \begin{pmatrix} 0 & \omega^2 \\ -1+\omega^2 & -1 \end{pmatrix}$, Pm_jP^{-1} $(j = 0, 1, 2, 3)$ are

$$\omega^2 \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix},$$

respectively.

The group generated by Pm_jP^{-1} $(j = 1, 2, 3)$ coincides with the level 3 principal congruence subgroup

$$\Gamma(3) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) | a-1, b, c, d-1 \in 3\mathbb{Z} \right\}.$$

The group generated by $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ is conjugate to the congruence subgroup

$$\Gamma_0(3) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) | a-1, c, d-1 \in 3\mathbb{Z} \right\},$$

since $\Gamma(3)$ is normal in $GL_2(\mathbb{Z})$, $\Gamma_0(3)/\Gamma(3) \simeq \mathbb{Z}/(3\mathbb{Z})$, and $(QP)m_0(QP)^{-1} = \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix}$ belongs to $\Gamma_0(3)$, where $Q = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \in GL_2(\mathbb{Z})$.

We have the commutative diagram:

$$\begin{array}{ccc} \mathbb{C} - \{0, 1, \omega, \omega^2\} & \xrightarrow{\tilde{h}} & \mathbb{H}/\Gamma(3) \\ \imath \downarrow & & pr \downarrow \\ \mathbb{C} - \{0, 1\} & \xrightarrow{\tilde{f}} & \mathbb{H}/\Gamma_0(3), \end{array}$$

where $\mathbb{H}$ is the upper half space, the map $\imath$ is $x \mapsto y = x^3$, the map pr is the natural projection, the maps $\tilde{h}$ and $\tilde{f}$ are given by the ratio of the fundamental solutions of $(QP)\mathrm{h}(x)$ and $(QP)\mathrm{f}(y)$, respectively.

5. A representation of the fundamental group of the Borromean-rings-complement

It is shown in [W] that the fundamental group of the Borromean-rings-complement is isomorphic to the subgroup B of $SL_2(\mathbb{Z}[i])$ generated by three elements

$$g_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 2i \\ 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 2+i & 2i \\ -1 & -i \end{pmatrix}.$$

Lemma 1 We have

$$g_0^3 = I, \quad g_2 = g_0 g_1 g_0^{-1}, \quad g_3 = g_0^2 g_1 g_0^{-2},$$

where

$$g_0 = \begin{pmatrix} -1 & -1-i \\ \frac{1-i}{2} & 0 \end{pmatrix} \in SL_2(\mathbb{C}).$$

Proof. We can easily show this lemma by direct computations. We here explain how to find the matrix g_0.

The matrices g_1, g_2 and g_3 can be expressed as

$$g_j = I - v_j \, {}^t v_j J \quad (j = 1, 2, 3),$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ v_2 = \begin{pmatrix} 1+i \\ 0 \end{pmatrix}, \ v_3 = \begin{pmatrix} 1+i \\ -1 \end{pmatrix}.$$

Since any element $g \in SL_2(\mathbb{C})$ satisfies ${}^t gJg = J$, we have

$$g g_j g^{-1} = I - g(v_j \, {}^t v_j J) g^{-1} = I - (g v_j) \, {}^t (g v_j) J.$$

Thus if the matrix g satisfies $g(v_1, v_2) = (v_2, v_3)$ then $g_2 = g g_1 g^{-1}$, $g_3 = g^2 g_1 g^{-2}$. We put $g_0 = -(v_2, v_3)(v_1, v_2)^{-1}$ so that $g_0^3 = I$. □

Theorem 1 The monodromy group of $H(\alpha, \beta)$ for α and β satisfying

$$e^{2\pi i\alpha} = i\omega\zeta, \qquad e^{2\pi i\beta} = i\omega\zeta'$$

is conjugate to the group B, where $\zeta = \frac{1\pm\sqrt{5}}{2}$ and $\zeta' = \frac{1\mp\sqrt{5}}{2}$.

Proof. In fact, for parameters in Theorem 1 and the matrix

$$P = \begin{pmatrix} 0 & 1+i \\ \omega - i\zeta & \omega \end{pmatrix},$$

we have

$$P\rho_0P^{-1} = g_0, \quad P\rho_1P^{-1} = g_1.$$

Proposition 2 and Lemma 1 imply this theorem.

We explain our method to find these parameters and the matrix P.

Recall that

$$\rho_0 = \begin{pmatrix} 1 & 0 \\ -(1-e^{-2\pi i\beta}) & \omega \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} 1 & 1-e^{-2\pi i\alpha} \\ 0 & \omega^2 e^{-2\pi i(\alpha+\beta)} \end{pmatrix},$$

$$g_0 = \begin{pmatrix} -1 & -1-i \\ \frac{1-i}{2} & 0 \end{pmatrix}, \quad g_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

If g_1 is conjugate to ρ_1 then the Jordan normal form $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of g_1 must coincide with that of ρ_1. Thus we have the condition $\omega^2 e^{-2\pi i(\alpha+\beta)} = 1$. We eliminate α in ρ_1 by this condition, and put $b = e^{-2\pi i\beta}$; ρ_1 becomes

$$\begin{pmatrix} 1 & 1-\omega/b \\ 0 & 1 \end{pmatrix} = I - v\,{}^t v J, \quad v = \begin{pmatrix} \sqrt{1-\omega/b} \\ 0 \end{pmatrix}.$$

Note that

$$P_1^{-1} g_0 P_1 = \omega P_2^{-1} \rho_0 P_2 = \begin{pmatrix} \omega & \\ & \omega^2 \end{pmatrix}$$

for

$$P_1 = \begin{pmatrix} 1+i & 1+i \\ \omega^2 & \omega \end{pmatrix}, \quad P_2 = \begin{pmatrix} \sqrt{3}i & 0 \\ \omega(b-1) & 1 \end{pmatrix}.$$

We have

$$\omega P(z) \rho_0 P(z)^{-1} = g_0,$$

where z is a variable in $\mathbb{C} - \{0\}$ and

$$P(z) = \frac{1}{\sqrt{(1+i)z}} P_1 Z P_2^{-1} \in SL_2(\mathbb{C}), \quad Z = \begin{pmatrix} z & \\ & 1 \end{pmatrix}.$$

By the equality $P(z)v = v_1$, which implies $P(z)\rho_1 P(z)^{-1} = g_1$, we have two algebraic equations with variables b and z.

The first equation reduces to

$$(b-\omega)(z-\omega(b-1)) = 0.$$

If $b = \omega$ then ρ_1 becomes I; thus z should be $\omega(b-1)$.

By eliminating z from the second equation by this identity, we have the quadratic equation

$$b^2 - i\omega^2 b + \omega = 0,$$

of which solutions are $i\omega^2 \frac{1\pm\sqrt{5}}{2}$. Note that their inverses are $i\omega \frac{1\mp\sqrt{5}}{2}$.

The matrix P is given by $\sqrt{(1+i)z}P(z)$ for $b = \exp(-2\pi i\beta) = i\omega^2\zeta$ and $z = \omega(i\omega^2\zeta - 1)$. $\square$

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