Automorphic functions for the Borromean-rings-complement group

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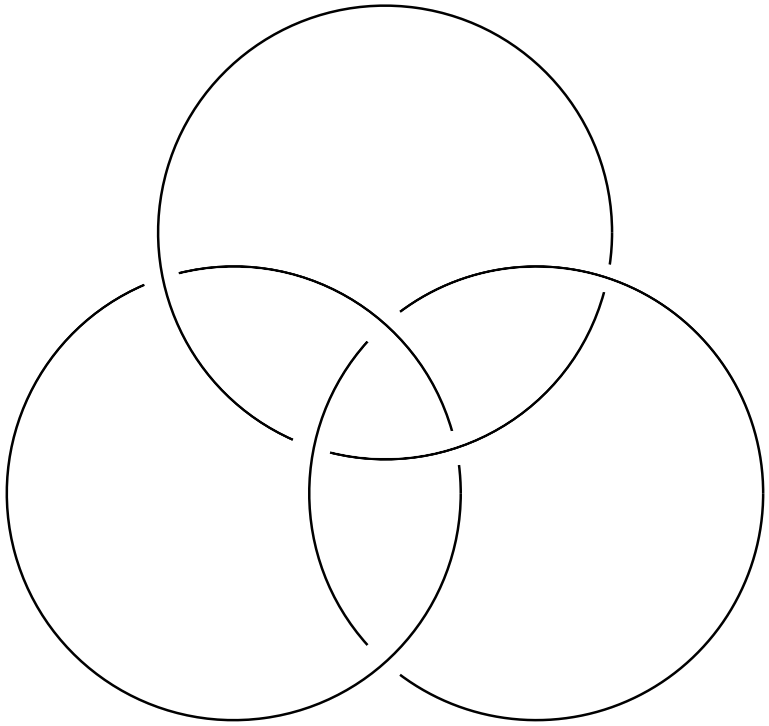
1. Introduction

The Borromean rings L in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ is given as



The Borromean-rings-complement $S^3 - L$ admits a hyperbolic structure: there is a group B in $GL_2(\mathbb{Z}[i])$ acting on the 3-dim. hyperbolic space $\mathbb{H}^3$, and there is a homeomorphism

$$\varphi : \mathbb{H}^3/B \xrightarrow{\cong} S^3 - L.$$

In this talk, we construct automorphic functions for B (analytic functions on $\mathbb{H}^3$ which are invariant under B), and express the homeomorphism φ in terms of these automorphic functions.

We realize the quotient space $\mathbb{H}^3/B$ as part of an affine algebraic variety in $\mathbb{R}^6$, and write down the defining equations.

2. A hyperbolic structure on the complement of the Borromean rings

The complement of the Borromean rings admits a hyperbolic structure, i.e.,

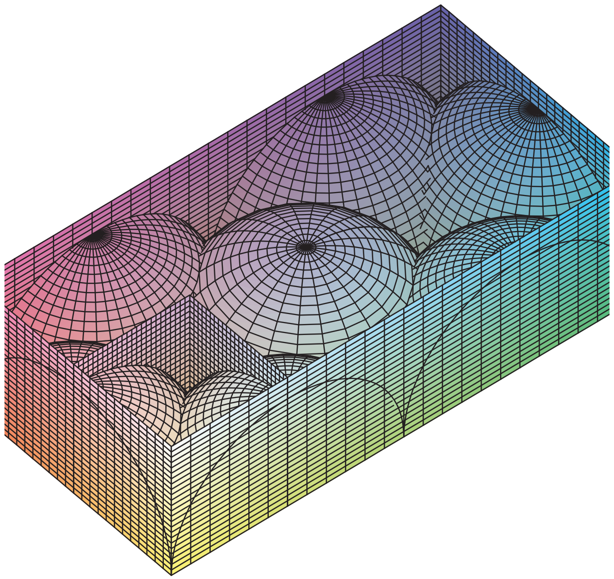
$$S^3 - L \simeq \mathbb{H}^3/B,$$

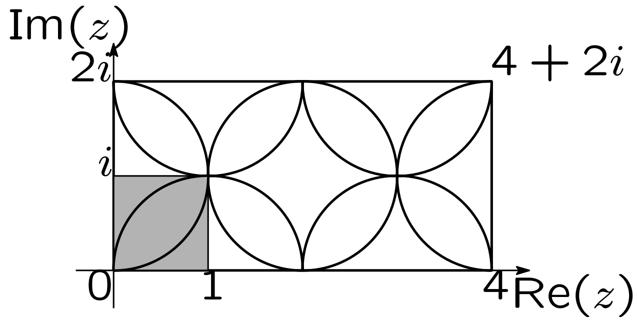
where $\mathbb{H}^3 = \{(z,t) \in \mathbb{C} \times \mathbb{R} \mid t > 0\}$, and B is a subgroup of $\Gamma = GL_2(\mathbb{Z}[i])$ generated by

$$g_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 2i \\ 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 2+i & 2i \\ -1 & -i \end{pmatrix},$$

and the scalar matrix iI_2.

The fundamental domain for B is given as





$\mathbb{H}^3/B$ has three cusps c_j; they are represented by

$$c_1 : (z,t) = (*,\infty), \quad c_2 : (z,t) = (1+i,0) \sim (3+i,0),$$

$$c_3 : (z,t) = (0,0) \sim (2,0) \sim (4,0) \sim (2i,0) \sim (2+2i,0) \sim (4+2i,0).$$

By considering of its volume, we have $[\Gamma : B] = 48$.

Note that the generators g_j of B belongs to

$$\Gamma_1(2) = \{g = (g_{jk}) \in \Gamma \mid g_{12}, g_{11} - g_{22} \in 2\mathbb{Z}[i]\}.$$

Proposition 1

$$\langle \Gamma(2), B \rangle = \Gamma_1(2),$$

where $\Gamma(2) = \{g = (g_{jk}) \in \Gamma \mid g_{12}, g_{21}, g_{11} - g_{22} \in 2\mathbb{Z}[i]\}$.

Let T be the involution

$$T : (z, t) \mapsto (\bar{z}, t).$$

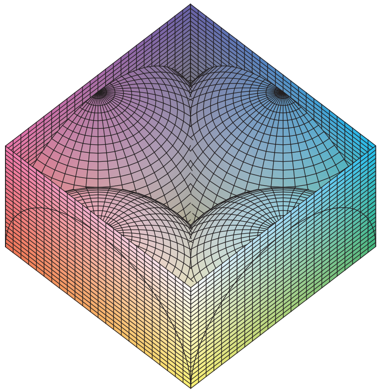
For a subgroup $G \in GL_2(\mathbb{C})$, the group generated by G and T with relations $gT = T\bar{g}$ for any $g \in G$ is denoted G^T.

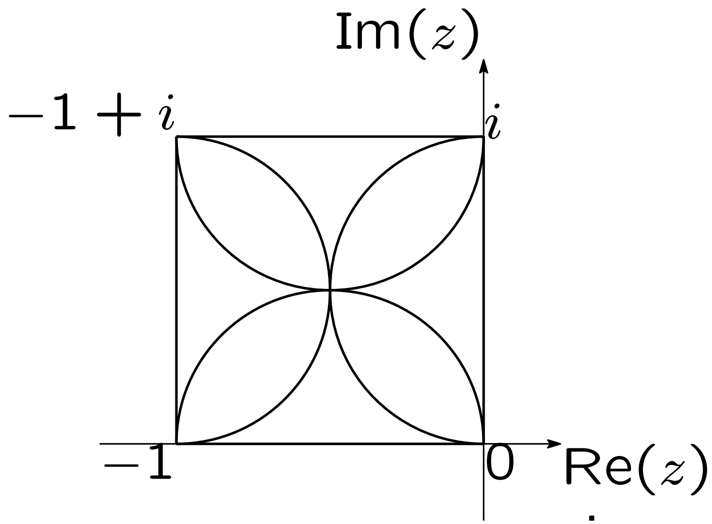
The group $\Gamma_1^T(2)$ is generated by the six reflections

$$\gamma_1 = T, \quad \gamma_2 = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} T, \quad \gamma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} T,$$

$$\gamma_4 = \begin{pmatrix} 1 & 2i \\ 0 & 1 \end{pmatrix} T, \quad \gamma_5 = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} T, \quad \gamma_6 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} T.$$

$\Gamma^T(2)$ is generated by the eight reflections with mirrors given in the figure





Note that the Weyl chamber bounded by these eight mirrors is an (ideal) octahedron in the hyperbolic space $\mathbb{H}^3$.

We have the following inclusion relations:

$$
\begin{tikzcd}
\textcolor{magenta}{\Gamma_1^T(2)} \arrow[dd, dash, "(\mathbb{Z}_2)^2"'] \arrow[dr, dash, "\mathbb{Z}_2"] & & \\
& \Gamma_1(2) \arrow[dd, dash, "(\mathbb{Z}_2)^2"'] \arrow[ddr, dash] & \\
\textcolor{blue}{\Gamma^T(2)} \arrow[dr, dash, "\mathbb{Z}_2"] & & \\
& \Gamma(2) & \textcolor{red}{B}
\end{tikzcd}
$$

3. Automorphic functions for $\Gamma^T(2)$

$\mathbb{H}^3$ can be embedded into the hermitian symmetric domain $\mathbb{D} = \{\tau \in M_{2,2}(\mathbb{C}) \mid (\tau - \tau^*)/2i$ is positive definite$\}$ of type $I_{2,2}$ by

$$\jmath : \mathbb{H}^3 \ni (z,t) \mapsto \frac{i}{t} \begin{pmatrix} t^2 + |z|^2 & z \\ \bar{z} & 1 \end{pmatrix} \in \mathbb{D}.$$

Through this embedding, $GL_2(\mathbb{C})$ and T act on $\mathbb{D}$ as

$$\jmath(g \cdot (z,t)) = \frac{1}{|\det(g)|} g\, \jmath(z,t)\, g^*, \quad \jmath(T \cdot (z,t)) = {}^t\jmath(z,t).$$

Theta functions $\Theta\binom{a}{b}$ on $\mathbb{D}$ are defined as

$$\Theta\binom{a}{b}(\tau) = \sum_{n \in \mathbb{Z}[i]^2} \mathbf{e}[(n+a)\tau(n+a)^* + 2\mathrm{Re}(nb^*)],$$

where $\mathbf{e}[x] = \exp(\pi i x)$, $\tau \in \mathbb{D}$, $a, b \in \mathbb{Q}[i]^2$.

Fact 1 *1. For $k \in \mathbb{Z}$ and $m, n \in \mathbb{Z}[i]^2$, we have*

$$\begin{aligned} \Theta\binom{i^k a}{i^k b}(\tau) &= \Theta\binom{a}{b}(\tau), \\ \Theta\binom{a+m}{b+n}(\tau) &= \mathbf{e}[-2\mathrm{Re}(mb^*)]\Theta\binom{a}{b}(\tau). \end{aligned}$$

2. We have

$$\begin{aligned} \Theta\binom{a}{b}(g\tau g^*) &= \Theta\binom{ag}{b(g^*)^{-1}}(\tau) \quad \text{for } g \in \Gamma, \\ \Theta\binom{a}{b}(T \cdot \tau) &= \Theta\binom{\bar{a}}{\bar{b}}(\tau). \end{aligned}$$

The pull back of $\Theta\binom{a}{b}(\tau)$ by $\jmath : \mathbb{H}^3 \to \mathbb{D}$ is denoted $\Theta\binom{a}{b}(z,t)$.
For $a, b \in (\frac{\mathbb{Z}[i]}{2})^2$, we use the convention:

$$\Theta\binom{a}{b}(z,t) = \Theta\begin{bmatrix}2a\\2b\end{bmatrix}(z,t) = \Theta\begin{bmatrix}2a\\2b\end{bmatrix}.$$

Set

$$x_0 = \Theta\begin{bmatrix}0,0\\0,0\end{bmatrix},\ x_1 = \Theta\begin{bmatrix}1+i,1+i\\1+i,1+i\end{bmatrix},\ x_2 = \Theta\begin{bmatrix}1+i,0\\0,1+i\end{bmatrix},\ x_3 = \Theta\begin{bmatrix}0,1+i\\1+i,0\end{bmatrix}.$$

Note that x_0 is positive and invariant under the action of Γ^T.

One of the main results in [MY] is the following.

Fact 2 x_1, x_2, x_3 *are invariant under the action of* $\Gamma^T(2)$.

The map

$$\mathbb{H}^3 \ni (z,t) \mapsto \frac{1}{x_0}(x_1, x_2, x_3) \in \mathbb{R}^3$$

induces an isomorphism between $\mathbb{H}^3/\Gamma^T(2)$ *and the octahedron*

$$\{(t_1, t_2, t_3) \in \mathbb{R}^3 \mid |t_1| + |t_2| + |t_3| \leq 1\}$$

minus the six vertices $(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$.

4. Automorphic functions for $\Gamma_1^T(2)$

Lemma 1 *By the actions of* g_1*,* g_2*,* g_3*, the functions* x_1, x_2, x_3 *are transformed as*

$$
\begin{aligned}
(x_1, x_2, x_3) \cdot g_1 &= (x_1, x_2, x_3) \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}, \\
(x_1, x_2, x_3) \cdot g_2 &= (x_1, x_2, x_3), \\
(x_1, x_2, x_3) \cdot g_3 &= (x_1, x_2, x_3) \begin{pmatrix} & & -1 \\ & 1 & \\ -1 & & \end{pmatrix}.
\end{aligned}
$$

The functions $x_1 + x_3$ and $x_1 - x_3$ are invariant modulo sign under the action of $\Gamma_1^T(2)$.

Lemma 2 *By the actions of g_1, g_2, g_3 and T, their signs change as*

	g_1	g_2	g_3	T
$x_1 + x_3$	$+$	$+$	$-$	$+$
$x_1 - x_3$	$-$	$+$	$+$	$+$

Proposition 2 *The functions x_0, x_2, x_1x_3, $x_1^2 + x_3^2$ are invariant under the action of $\Gamma_1^T(2)$. The map*

$$\varphi_0 : \mathbb{H}^3 \ni (z, t) \mapsto \frac{1}{x_0^2}(x_0x_2, x_1x_3, x_1^2 + x_3^2) \in \mathbb{R}^3$$

induces an isomorphism between $\mathbb{H}^3/\Gamma_1^T(2)$ and $\varphi_0(\mathbb{H}^3)$.

5. Automorphic functions for B

Set $w_1 = \Theta\begin{bmatrix}1,0\\0,1\end{bmatrix}$, $w_2 = \Theta\begin{bmatrix}i,0\\0,1\end{bmatrix}$, $w_3 = \Theta\begin{bmatrix}1,1+i\\1+i,1\end{bmatrix}$, $\omega_4 = \Theta\begin{bmatrix}i,1+i\\1+i,1\end{bmatrix}$.
By using Fact 1, we have the following.

Lemma 3 *The functions $w_1, \ldots, w_4$ are invariant modulo sign under the action of $\Gamma_1^T(2)$.*

Especially, by the actions of g_1, g_2, g_3, T, their signs change as

	g_1	g_2	g_3	T
w_1	$+$	$+$	$-$	$+$
w_2	$+$	$-$	$+$	$+$
w_3	$-$	$+$	$+$	$+$
w_4	$+$	$-$	$+$	$-$

This lemma implies the following Proposition.

Proposition 3 *The functions* $f_1 = w_2 w_4$, $f_2 = (x_1 + x_3)w_1$, $f_3 = (x_1 - x_3)w_3$ *are invariant under the action of* B.

By the actions of

$$\gamma_1 = T, \quad \gamma_2 = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} T, \quad \gamma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} T,$$

we have

	γ_1	γ_2	γ_3
f_1	$-$	$+$	$+$
f_2	$+$	$-$	$+$
f_3	$+$	$+$	$-$

Let Iso_j be the subgroup of $\Gamma_1^T(2)$ consisting of elements keeping f_j invariant for $j = 1, 2, 3$, and Iso_0 the subgroup of $\Gamma_1^T(2)$ consisting of elements keeping $f_1 f_2 f_3$ invariant.

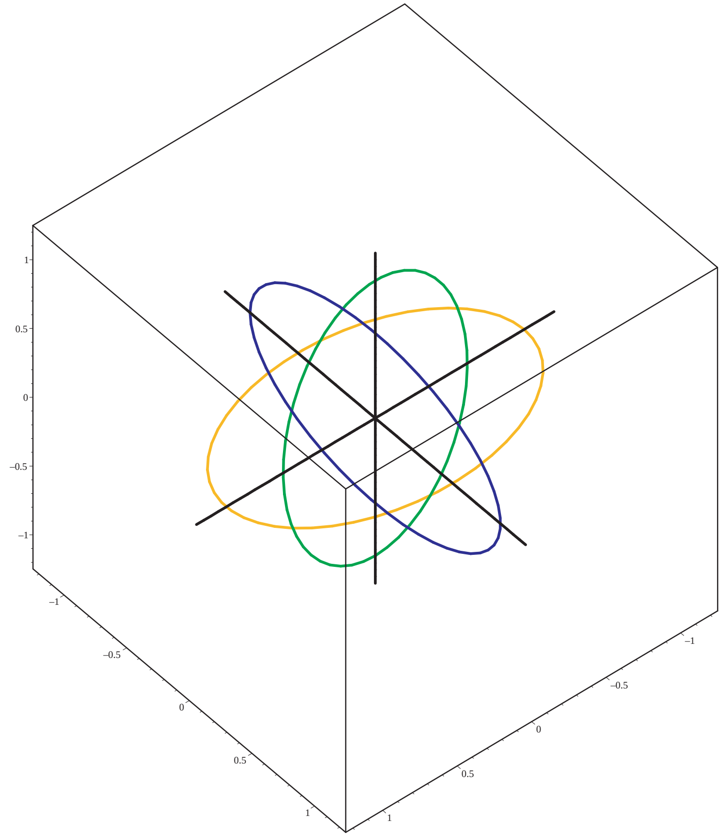
Proposition 4 *We have*

$$\Gamma_1(2) = \mathrm{Iso}_0, \quad B = \mathrm{Iso}_1 \cap \mathrm{Iso}_2 \cap \mathrm{Iso}_3.$$

The group B is normal in $\Gamma_1^T(2)$; $\Gamma_1^T(2)/B \simeq (\mathbb{Z}_2)^3$.

Proof. The group $\Gamma_1^T(2)$ is generated by the group B and the reflections γ_1, γ_2 and γ_3. Since the index $[\Gamma_1^T(2) : B]$ is eight, we have $B = \mathrm{Iso}_1 \cap \mathrm{Iso}_2 \cap \mathrm{Iso}_3$ and $\Gamma_1^T(2)/B \simeq (\mathbb{Z}_2)^3$ by Proposition 3. Proposition 3 also shows that the function $f_1 f_2 f_3$ is invariant under the action of $\Gamma_1(2)$, and that it changes its sign by T. $\square$

Remark 1 $\Gamma_1^T(2)/B \simeq (\mathbb{Z}_2)^3$ *corresponds to some symmetries of the Borromean rings L. See the figure.*



We can assume that any element $g \in \Gamma_1(2)$ takes the form $I_2 + \begin{pmatrix} 2p & 2q \\ r & 2s \end{pmatrix}$, otherwise multiply i to g. For example, $ig_3 = I_2 + \begin{pmatrix} -2+2i & -2 \\ -i & 0 \end{pmatrix}$.

Theorem 1 *The element* $g = I_2 + \begin{pmatrix} 2p & 2q \\ r & 2s \end{pmatrix} \in \Gamma_1(2)$ *belongs to* B *if and only if* $\mathrm{Re}(q) + \mathrm{Im}(r) \equiv 0$ *and*

$$\tfrac{1+(-1)^{\mathrm{Re}(r)+\mathrm{Im}(r)}}{2}\mathrm{Re}(q) + \tfrac{1-(-1)^{\mathrm{Re}(r)+\mathrm{Im}(r)}}{2}\mathrm{Im}(q)$$

$$\equiv \mathrm{Re}(p+s) + \mathrm{Im}(p+s),$$

modulo 2.

Proposition 5 *We have*

$$4w_1^2 = 4\Theta\begin{bmatrix}1,0\\0,1\end{bmatrix}^2 = (x_0+x_1+x_2+x_3)(x_0-x_1+x_2-x_3),$$

$$4w_2^2 = 4\Theta\begin{bmatrix}i,0\\0,1\end{bmatrix}^2 = (x_0+x_1+x_2-x_3)(x_0-x_1+x_2+x_3),$$

$$4w_3^2 = 4\Theta\begin{bmatrix}1,1+i\\1+i,1\end{bmatrix}^2 = (x_0+x_1-x_2-x_3)(x_0-x_1-x_2+x_3),$$

$$4w_4^2 = 4\Theta\begin{bmatrix}i,1+i\\1+i,1\end{bmatrix}^2 = (x_0+x_1-x_2+x_3)(x_0-x_1-x_2-x_3).$$

Proof. Use Theorem 1 in [M2] and Lemma 3.2 in [MY]. $\square$

Theorem 2 *The map*

$$\varphi : \mathbb{H}^3 \ni (z,t) \mapsto \frac{1}{x_0^2}(x_0x_2, x_1x_3, x_1^2 + x_3^2, f_1, f_2, f_3) \in \mathbb{R}^6$$

induces an isomorphism between $\mathbb{H}^3/B$ and $\varphi(\mathbb{H}^3)$.

f_j^2 *are expressed in terms of* $\Gamma_1^T(2)$*-invariant functions:*

$$\begin{aligned} 16f_1^2 &= (x_0^2 - x_2^2)^2 - 2(x_0^2 + x_2^2)(x_1^2 + x_3^2) + (x_1^2 + x_3^2)^2 \\ &\quad -4(x_1x_3)^2 - 8(x_0x_2)(x_1x_3), \\ 4f_2^2 &= (x_1^2 + x_3^2 + 2x_1x_3)((x_0 + x_2)^2 - (x_1^2 + x_3^2) - 2x_1x_3), \\ 4f_3^2 &= (x_1^2 + x_3^2 - 2x_1x_3)((x_0 - x_2)^2 - (x_1^2 + x_3^2) + 2x_1x_3). \end{aligned}$$

These relations together with the image of the map φ_0 determine the image of the map φ.

Proof. By Proposition 5, f_j vanishes only on the mirror of the reflection γ_j for $j = 1, 2, 3$. Note that the space $\mathbb{H}^3/B$ is the eight fold covering of $\mathbb{H}^3/\Gamma_1^T(2)$ branching along the union of the mirrors of γ_1, γ_2 and γ_3, which corresponds to the zero locus of $f_1 f_2 f_3$. Thus the map φ realize this covering. Use Proposition 5 to express f_j^2 in terms of $x_0, \ldots, x_3$. $\square$

6. Differential equations with monodromy B

Let $H(\alpha, \beta)$ be the differential equation for $f(x^3)$ under the condition $\gamma = 2/3$, where $f(y)$ is a solution of the Gauss hypergeometric differential equation

$$y(1-y)f''(y) + \{\gamma - (\alpha+\beta+1)y\}f'(y) - \alpha\beta f(y) = 0.$$

We have

$$H(\alpha,\beta): \ (1-x^3)h''(x) - (3\alpha+3\beta+1)x^2h'(x) - 9\alpha\beta x h(x) = 0,$$

which has four regular singular points $x = 1, \omega, \omega^2$ and ∞. Hence, $H(\alpha, \beta)$ is a Heun differential equation.

Remark 2 *The periods of cubic curves*

$$C(x) = \{[t_0, t_1, t_2] \in \mathbb{P}^2 \mid t_0^3 + t_1^3 + t_2^3 - 3xt_0t_1t_2 = 0\},$$

$(x \in \mathbb{C} - \{1, \omega, \omega^2\})$ *of the Hesse normal form satisfy the differential equation* $H(1/3, 1/3)$.

Theorem 3 *The monodromy group of* $H(\alpha, \beta)$ *for* α *and* β *satisfying*

$$e^{2\pi i\alpha} = i\omega\zeta, \qquad e^{2\pi i\beta} = i\omega\zeta'$$

is conjugate to the group B, *where* $\zeta = \frac{1\pm\sqrt{5}}{2}$ *and* $\zeta' = \frac{1\mp\sqrt{5}}{2}$.

References

[F] E. Freitag, Modulformen zweiten Grades zum rationalen und Gaußschen Zahlkörper, *Sitzungsber. Heidelb. Akad. Wiss.*, **1** (1967), 1–49.

[M1] K. Matsumoto, Theta functions on the bounded symmetric domain of type $I_{2,2}$ and the period map of 4-parameter family of K3 surfaces, *Math. Ann.*, **295** (1993), 383–408.

[M2] K. Matsumoto, Algebraic relations among some theta functions on the bounded symmetric domain of type $I_{r,r}$, to appear in *Kyushu J. Math.*

[M3] K. Matsumoto, A Heun differential equation derived from the Gauss hypergeometric differential equation, preprint 2005.

[MNY] K. Matsumoto, H. Nishi and M. Yoshida, Automorphic functions for the Whitehead-link-complement group, preprint 2005.

[MSY] K. Matsumoto, T. Sasaki and M. Yoshida, The monodromy of the period map of a 4-parameter family of K3 surfaces and the Aomoto-Gel'fand hypergeometric function of type (3,6), *Internat. J. of Math.*, **3** (1992), 1–164.

[MY] K. Matsumoto and M. Yoshida, Invariants for some real hyperbolic groups, *Internat. J. of Math.*, **13** (2002), 415–443.

[T] W. Thurston, *Geometry and Topology of 3-manifolds,* Lecture Notes, Princeton Univ., 1977/78.

[W] N. Wielenberg, The structure of certain subgroups of the Picard group, *Math. Proc. Cambridge Philos. Soc.*, **84** (1978), no. 3, 427–436.

[Y] M. Yoshida, *Hypergeometric Functions, My Love,* Aspects of Mathematics, E32, Friedr Vieweg & Sohn, Braunschweig, 1997.