ζ関数の周期関係式

(Period relations for the zeta function)

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1. Introduction.

Riemann's zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \text{Re}(s) > 1.$$

It is known that $\zeta(s)$ can be extend to a meromorphic function on $\mathbb{C}$ with a simple pole only on $s = 1$ and that it satisfies the inversion formula

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s)\zeta(1-s).$$

Riemann's period relations

$$\sum_{k=1}^{g} \int_{A_k} \varphi \times \int_{B_k} \psi - \sum_{k=1}^{g} \int_{B_k} \varphi \times \int_{A_k} \psi = 0,$$

$$\mathrm{Im}\left(\sum_{k=1}^{g} \overline{\int_{A_k} \varphi} \times \int_{B_k} \varphi \right) > 0,$$

where φ and ψ are holomorphic 1-forms on a compact Riemann surface X of genus g, and $(A_1, \dots, A_g, B_1, \dots, B_g)$ is a symplectic basis of $H_1(X, \mathbb{Z})$ ($A_j \cdot B_k = -\delta_{jk}$ for the intersection from $\cdot$).

Riemann proved these relations by the Stokes theorem.

We can show these relations by the compatibility of the pairings:

- the integral between $H^1(X,\mathbb{C})$ and $H_1(X,\mathbb{C})$

$$H^1(X,\mathbb{C}) \times H_1(X,\mathbb{C}) \ni (\varphi, A) \mapsto \int_A \varphi \in \mathbb{C};$$

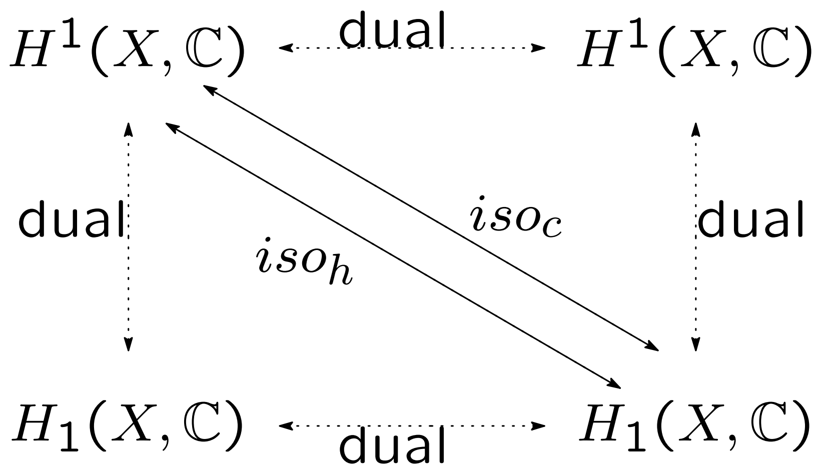
- the intersection form for $H_1(X,\mathbb{C})$

$$H_1(X,\mathbb{C}) \times H_1(X,\mathbb{C}) \ni (A, B) \mapsto A \cdot B \in \mathbb{C};$$

- the cup product for $H^1(X,\mathbb{C})$

$$H^1(X,\mathbb{C}) \times H^1(X,\mathbb{C}) \ni (\varphi, \psi) \mapsto \int_X \varphi \wedge \psi \in \mathbb{C}.$$

We have the diagram



iso_c is defined via $H^1(X,\mathbb{C})$; iso_h is defined via $H_1(X,\mathbb{C})$;

$$iso_c = iso_h.$$

This implies Riemann's period relations.

It is known that $\zeta(s)$ admits an integral representation. We regard it as the paring between an element of a kind of cohomology group and that of homology group.

In my talk, suppose that the compatibility of the pairings among these (co)homology groups holds. Then we can show that it implies the inversion formula for $\zeta(s)$.

I can not show the compatibility. I would like someone to help me.

2. An integral representation for $\zeta(s)$

The <u>Γ-function</u> is defined as

$$\Gamma(s) = \int_0^\infty e^{-x} x^s \frac{dx}{x}, \qquad \mathrm{Re}(s) > 0;$$

it satisfies the inversion formula

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

(This formula can be obtained also from the compatibility of the pairings among some kinds of (co)homology groups.)

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By putting $x = ny$, we have

$$\Gamma(s) = n^s \int_0^\infty e^{-ny} y^s \frac{dy}{y}, \qquad \frac{\Gamma(s)}{n^s} = \int_0^\infty e^{-ny} y^s \frac{dy}{y}.$$

Thus

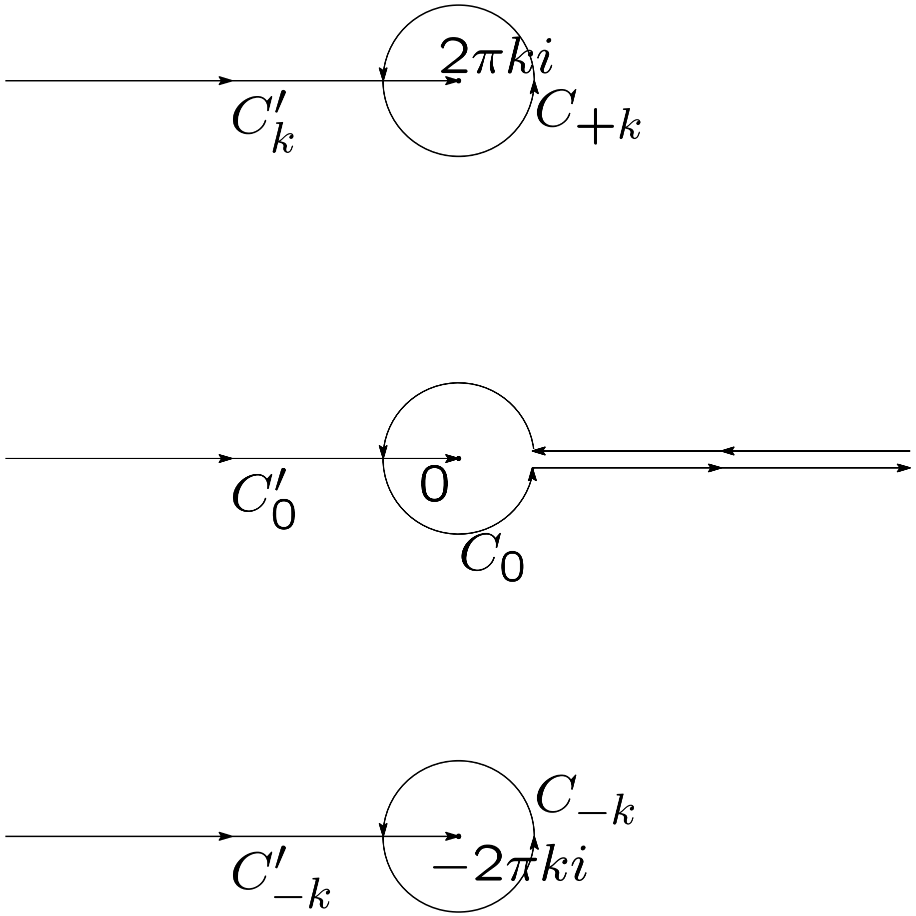
$$\sum_{n=1}^{\infty} \frac{\Gamma(s)}{n^s} = \int_0^\infty \left(\sum_{n=1}^{\infty} e^{-ny} \right) y^s \frac{dy}{y}$$

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{e^{-y}}{1 - e^{-y}} y^s \frac{dy}{y} = \int_0^\infty \frac{x^s}{e^x - 1} \frac{dx}{x}.$$

We have

$$(e^{2\pi i s} - 1)\Gamma(s)\zeta(s) = \int_{C_0} \frac{x^s}{e^x - 1} \frac{dx}{x}, \qquad (1)$$

where C_0 is given is Figure 1; this formula implies the analytic continuation of $\zeta(s)$.



(We assume that $\arg(x)$ is in the interval $[0, 2\pi]$.)

Figure 1. Cycles

3. Twisted (co)homology groups

We put $X = \mathbb{C} - 2\pi i\mathbb{Z}$ and

$$u(x) = \frac{x^s}{e^x - 1}, \quad \omega = d\log u(x) = s\frac{dx}{x} - \frac{e^x dx}{e^x - 1}.$$

For the twisted differential operators $\nabla_\omega = d + \omega\wedge$ and $\nabla_{-\omega} = d - \omega\wedge$, we define the twisted de Rham cohomology groups as

$$H^1(X, \nabla_\omega) = \Omega^1(X)/\nabla_\omega(\Omega^0(X)),$$

$$H^1(X, \nabla_{-\omega}) = \Omega^1(X)/\nabla_{-\omega}(\Omega^0(X)),$$

where $\Omega^k(X)$ is the vector space of holomorphic k-forms on X.

Let $\mathcal{C}_k^+(X)$ be the vector space of locally finite linear combinations of $\sigma \otimes u_\sigma$'s over $\mathbb{C}$, where σ is a k-chain in X and u_σ is a branch of $u(x)$ on σ.

The twisted boundary operator ∂_ω is defined by

$$\partial_\omega(\sigma \otimes u_\sigma) = \partial(\sigma) \otimes u_\sigma|_{\partial\sigma},$$

where ∂ is the usual boundary operator and $u_\sigma|_{\partial\sigma}$ is the restriction of u_σ on $\partial\sigma$.

$\mathcal{C}_k^-(X)$ and $\partial_{-\omega}$ are similarly defined for the function $1/u(x)$.

We define the twisted homology groups as

$$H_1(X, \partial_\omega) = \ker(\partial_\omega : \mathcal{C}_1^+(X) \to \mathcal{C}_0^+(X))/\partial_\omega(\mathcal{C}_2^+(X)),$$

$$H_1(X, \partial_{-\omega}) = \ker(\partial_{-\omega} : \mathcal{C}_1^-(X) \to \mathcal{C}_0^-(X))/\partial_{-\omega}(\mathcal{C}_2^-(X)).$$

We regard the integral (1) as the pairing $\langle \varphi, \gamma_0 \rangle$ of

$\varphi = \frac{dx}{x} \in H^1(X, \nabla_\omega)$ and $\gamma_0 = C_0 \otimes u(x) \in H_1(X, \partial_\omega)$,

where the branch of $u(x)$ is assigned as the argument $\arg(x)$ of $x \in C_0$ is in the interval $[0, 2\pi]$.

$H^1(X, \nabla_{\pm\omega})$ and $H_1(X, \partial_{\pm\omega})$ are infinitely dimensional. We give some elements of them:

$$\frac{dx}{x^j} \in H^1(X, \nabla_{\pm\omega}), \quad j \in \mathbb{Z},$$

$$\gamma_k^+ = C_k \otimes u(x) \in H_1(X, \partial_\omega), \quad \gamma_k^- = C'_k \otimes u(x)^{-1} \in H_1(X, \partial_{-\omega}),$$

where $k \in \mathbb{Z}$, C_k and C'_k are given in Figure 1.

4. Periods

The pairings

$$\langle \varphi, \gamma_k^+ \rangle = \int_{C_k} \frac{x^s}{e^x - 1} \frac{dx}{x}$$

are called <u>periods</u>.

The residue theorem implies that

Fact 1

$$\langle \varphi, \gamma_k^+ \rangle = \begin{cases} (e^{2\pi i s} - 1)\Gamma(s)\zeta(s) & \text{if} \quad k = 0, \\ 2\pi i (2\pi k)^{s-1} e^{\pi i s}(-i e^{-\pi i s/2}) & \text{if} \quad k > 0, \\ 2\pi i (2\pi k)^{s-1} e^{\pi i s}(i e^{\pi i s/2}) & \text{if} \quad k < 0. \end{cases}$$

We put $\psi = \frac{dx}{x^3}$. The pairings

$$\langle \psi, \gamma_k^- \rangle = \int_{C'_k} (e^x - 1) x^{-s} \frac{dx}{x^3}$$

are called dual periods.

Let us evaluate $\langle \psi, \gamma_0^- \rangle$: the integral

$$\int_{-\infty}^{0} (e^x - 1) x^{-s} \frac{dx}{x^3},$$

converges when $-2 < \mathrm{Re}(s) < -1$;

$$\langle \psi, \gamma_0^- \rangle = \frac{1}{s+2} \left[\left[-(e^x - 1) x^{-s-2} \right]_{-\infty}^{0} + \int_{-\infty}^{0} e^x x^{-s-2} dx \right]$$

$$= \frac{1}{s+2} \int_{\infty}^{0} e^{-y} (-y)^{-s-2} d(-y) = \frac{e^{-\pi i s}}{s+2} \int_{0}^{\infty} e^{-y} y^{-s-2} dy$$

$$= \frac{e^{-\pi i s}}{s+2} \Gamma(-1-s) = \frac{e^{-\pi i s}}{s(s+1)(s+2)} \Gamma(1-s).$$

Remark 1 *1. Since*

$$\nabla_{-\omega}\left(\frac{1}{x^2}\right) = -(2+s)\frac{dx}{x^3} + \frac{e^x dx}{(e^x-1)x^3},$$

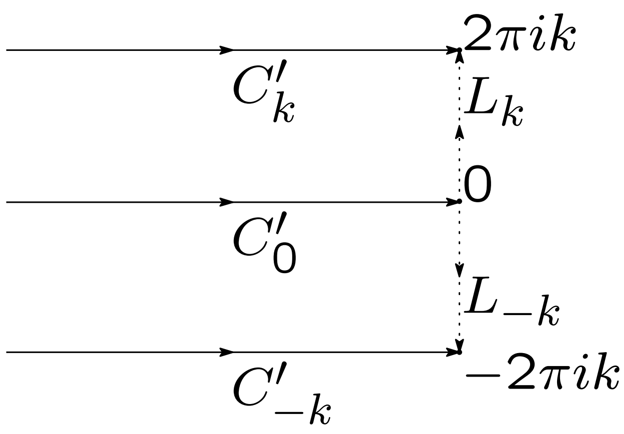
$\frac{dx}{x^3}$ *is cohomologous to* $\frac{1}{s+2}\frac{e^x dx}{(e^x-1)x^3}$, *which implies*

$$\langle \psi, \gamma_0^- \rangle = \int_{-\infty}^{0} (e^x - 1)x^{-s}\frac{dx}{x^3} = \frac{e^{-\pi i s}}{s(s+1)(s+2)}\Gamma(1-s).$$

2. By this equality, we can make the analytic continuation of $\langle \psi, \gamma_0^- \rangle$ *defined on* $-2 < \mathrm{Re}(s) < -1$ *to* $\mathbb{C} - \mathbb{Z}$.

We can not evaluate $\langle \varphi, \gamma_k^- \rangle$ explicitly for a non-zero k.

We regard C_k' as the sum of $C_0' \otimes u(x)^{-1}$ and $L_k \otimes u(x)^{-1}$, where L_k is the path from 0 to $2\pi i k$ and the branch of $u(x)^{-1}$ is assigned as $\arg(x) \in [0, 2\pi]$.



Fact 2

$$
\begin{aligned}
\langle \psi, \gamma_k^- \rangle &= \int_{-\infty}^{0} (e^x - 1) x^{-s} \frac{dx}{x^3} + \int_0^{2\pi ik} (e^x - 1) x^{-s} \frac{dx}{x^3} \\
&= \frac{e^{-\pi is}}{s+2} \Gamma(-1-s) + \frac{-ie^{-\pi is/2}}{s+2} \int_0^{2\pi k} e^{ix} x^{-1-s} \frac{dx}{x},
\end{aligned}
$$

$$
\begin{aligned}
\langle \psi, \gamma_{-k}^- \rangle &= \frac{e^{-\pi is}}{s+2} \Gamma(-1-s) + \frac{1}{s+2} \int_0^{-2\pi ik} e^x x^{-1-s} \frac{dx}{x} \\
&= \frac{e^{-\pi is}}{s+2} \Gamma(-1-s) + \frac{ie^{-3\pi is/2}}{s+2} \int_0^{2\pi k} e^{-ix} x^{-1-s} \frac{dx}{x}.
\end{aligned}
$$

Remark 2 *The Kummer confluent hypergeometric function is defined by*

$$F(\alpha,\gamma;z)=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\gamma)_n}\frac{z^n}{n!},$$

where $\gamma\neq 0,-1,-2,\ldots$ *and* $(\alpha)_n=\alpha(\alpha+1)\cdots(\alpha+n-1)$.

It satisfies

$$F(\alpha,\gamma;z)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}z^{1-\gamma}\int_0^z e^t t^{\alpha-1}(z-t)^{\gamma-\alpha-1}dt,$$

where $\mathrm{Re}(\gamma)>\mathrm{Re}(\alpha)>0$.

For $\alpha=1-s,\ \gamma=2-s,\ z=2\pi ik,$ *we have*

$$\begin{aligned}F(1-s,2-s;2\pi ik) &= (1-s)(2\pi ik)^{-1+s}\int_0^{2\pi ik}e^t t^{-s}dt\\ &= \sum_{n=0}^{\infty}\frac{1-s}{n+1-s}\frac{(2\pi ik)^n}{n!}.\end{aligned}$$

5. Intersection form

We define the intersection form between $H_1(X, \partial_\omega)$ and $H_1(X, \partial_{-\omega})$:

$$\langle \gamma, \gamma' \rangle = \sum_{P \in \sigma_j \cap \sigma'_k} (c_j \times c'_k) \times (\sigma_j \cdot \sigma'_k) \times (u_{\sigma_j}(P) \times u^{-1}_{\sigma'_k}(P)),$$

where $\gamma = \sum_j c_j \sigma_j \otimes u_{\sigma_j}$ and $\gamma' = \sum_k c'_k \sigma'_k \otimes u^{-1}_{\sigma'_k}$.

We have

$$\langle \gamma_j^+, \gamma_k^- \rangle = \delta_{jk};$$

i.e., the intersection matrix $I_h = (\langle \gamma_j^+, \gamma_k^- \rangle)_{j,k}$ becomes

$$\begin{pmatrix} \ddots & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix}.$$

6. Cup product

We define the cup product between $H^1(X, \nabla_\omega)$ and $H^1(X, \nabla_{-\omega})$:

$$\langle \eta^+, \eta^- \rangle = \int_X \eta^+ \wedge \eta^-,$$

where $\eta^+ \in H^1(X, \nabla_\omega)$ and $\eta^- \in H^1(X, \nabla_{-\omega})$.

It seems that $\langle \eta^+, \eta^- \rangle = 0$ for $\eta^+ = \frac{dx}{x^j}$, $\eta^- = \frac{dx}{x^k}$, as $dx \wedge dx = 0$.

The integral $\int_X \eta^+ \wedge \eta^-$ is indefinite of type $0 \times \infty$, since both of η^+ and η^- have poles at $x = 0$ or at $x = \infty$.

Proposition 1 *If* $k + j \geq 4$ *then*

$$\langle \frac{dx}{x^j}, \frac{dx}{x^k} \rangle = 2\pi i \mathrm{Res}_{x=0} \left(F(x) \frac{dx}{x^k} \right),$$

where $F(x)$ *is the meromorphic function around* $x = 0$ *satisfying* $\nabla_\omega(F(x)) = \frac{dx}{x^j}$.

Remark 3 *My student Mr. Shimada expresses the value* $\langle \frac{dx}{x^j}, \frac{dx}{x^k} \rangle$ *in terms of Bernoulli's numbers. Especially,*

$$\langle \frac{dx}{x}, \frac{dx}{x^3} \rangle = 2\pi i \frac{s+3}{12(s-1)s(s+1)}.$$

Problem 1 *1. Give a formula for the cup product* $\langle \frac{dx}{x^j}, \frac{dx}{x^k} \rangle$ *for general* $j, k \in \mathbb{Z}$.

(Evaluate the integral $\int_X \eta^+ \wedge \eta^-$ *around* $x = \infty$*.)*

2. *Find nice cohomology groups so that the cup product is well-defined; and establish comparison theorems between these cohomology groups.*

3. *Show the compatibility of the parings between* $H^1(X, \nabla_{\pm\omega})$ *and* $H_1(X, \partial_{\pm\omega})$ *(among the integral, the intersection form, the cup product).*

7. Twisted period relations

Suppose that the compatibility of the parings between $H^1(X, \nabla_{\pm\omega})$ and $H_1(X, \partial_{\pm\omega})$ holds.

Then we expect that the twisted period relations

$$P^{+}\,{}^tI_h^{-1}\,{}^tP^{-} = I_{ch}$$

holds, where

$$P^{+} = (\langle \eta_j^{+}, \gamma_k^{+} \rangle)_{j,k}, \quad P^{-} = (\langle \eta_j^{-}, \gamma_k^{-} \rangle)_{j,k},$$

$$I_h = (\langle \gamma_j^{+}, \gamma_k^{-} \rangle)_{j,k}, \quad I_{ch} = (\langle \eta_j^{+}, \eta_k^{-} \rangle)_{j,k},$$

$$\eta_j^{\pm} \in H^1(X, \nabla_{\pm\omega}), \quad \gamma_j^{\pm} \in H_1(X, \partial_{\pm\omega}).$$

Since $\langle \gamma_j^{+}, \gamma_k^{-} \rangle = \delta_{jk}$, we have

$$P^{+}\,{}^tP^{-} = I_{ch}.$$

Let us see that the inner product of

$$(\ldots, \langle\varphi, \gamma_{-k}^{+}\rangle, \ldots, \langle\varphi, \gamma_0^{+}\rangle, \ldots, \langle\varphi, \gamma_k^{+}\rangle, \ldots),$$
$$(\ldots, \langle\psi, \gamma_{-k}^{-}\rangle, \ldots, \langle\psi, \gamma_0^{-}\rangle, \ldots, \langle\psi, \gamma_k^{-}\rangle, \ldots).$$

coincides with the cup product $\langle\varphi, \psi\rangle$ for $\varphi = \frac{dx}{x}$ and $\psi = \frac{dx}{x^3}$.

We have

$$\begin{aligned}
\langle\varphi, \gamma_0^{+}\rangle\langle\psi, \gamma_0^{-}\rangle &= \left[(e^{2\pi is} - 1)\Gamma(s)\zeta(s)\right]\left[\frac{e^{-\pi is}}{s(s+1)(s+2)}\Gamma(1-s)\right] \\
&= 2i\sin(\pi s)\frac{\pi}{\sin(\pi s)}\frac{\zeta(s)}{s(s+1)(s+2)} \\
&= \frac{2\pi i\zeta(s)}{s(s+1)(s+2)}.
\end{aligned}$$

(Here the inversion formula for Γ-function is used).

$$\begin{aligned}&\langle\varphi,\gamma_k^+\rangle\langle\psi,\gamma_k^-\rangle\\ = \;& \frac{2\pi i(2\pi k)^{s-1}}{s+2}\left[\frac{e^{-\pi is/2}}{is(s+1)}\Gamma(1-s)-\int_0^{2\pi k}e^{ix}x^{-1-s}\frac{dx}{x}\right],\end{aligned}$$

$$\begin{aligned}&\langle\varphi,\gamma_{-k}^+\rangle\langle\psi,\gamma_{-k}^-\rangle\\ = \;& \frac{2\pi i(2\pi k)^{s-1}}{s+2}\left[\frac{-e^{\pi is/2}}{is(s+1)}\Gamma(1-s)-\int_0^{2\pi k}e^{-ix}x^{-1-s}\frac{dx}{x}\right].\end{aligned}$$

Thus we have

$$\begin{aligned}&\langle\varphi,\gamma_k^+\rangle\langle\psi,\gamma_k^-\rangle+\langle\varphi,\gamma_{-k}^+\rangle\langle\psi,\gamma_{-k}^-\rangle\\ = \;& \frac{4\pi i(2\pi k)^{s-1}}{s+2}\left[-\frac{\sin(\pi s/2)\Gamma(1-s)}{s(s+1)}-\int_0^{2\pi k}(\cos x)x^{-1-s}\frac{dx}{x}\right].\end{aligned}$$

For $\mathrm{Re}(s) < 0$, let us compute

$$\langle \varphi, \gamma_0^+ \rangle \langle \psi, \gamma_0^- \rangle + \sum_{k=1}^{\infty} \left[\langle \varphi, \gamma_k^+ \rangle \langle \psi, \gamma_k^- \rangle + \langle \varphi, \gamma_{-k}^+ \rangle \langle \psi, \gamma_{-k}^- \rangle \right] :$$

$$\frac{2\pi i \zeta(s)}{s(s+1)(s+2)} - \frac{4\pi i (2\pi)^{s-1}}{s(s+1)(s+2)} \sum_{k=1}^{\infty} \sin(\pi s/2) \Gamma(1-s) k^{s-1}$$
$$- \frac{4\pi i (2\pi)^{s-1}}{s+2} \sum_{k=1}^{\infty} \int_0^{2\pi k} k^{s-1} (\cos x) x^{-1-s} \frac{dx}{x}$$
$$= \frac{2\pi i}{s(s+1)(s+2)} [\zeta(s) - 2(2\pi)^{s-1} \sin(\pi s/2) \Gamma(1-s) \zeta(1-s)]$$
$$- \frac{4\pi i (2\pi)^{s-1}}{s+2} \int_0^{2\pi} \left(\sum_{k=1}^{\infty} \frac{\cos kx}{k^2} \right) x^{-1-s} \frac{dx}{x}$$
$$= \frac{2\pi i}{s(s+1)(s+2)} [\zeta(s) - 2(2\pi)^{s-1} \sin(\pi s/2) \Gamma(1-s) \zeta(1-s)]$$
$$+ \frac{2\pi i (s+3)}{12(s-1)s(s+1)}.$$

Here note that

$$\sum_{k=1}^{\infty} \frac{\cos kx}{k^2} = \frac{1}{4}(x-\pi)^2 - \frac{\pi^2}{12},$$

$$\int_0^{2\pi} \left(\frac{1}{4}(x-\pi)^2 - \frac{\pi^2}{12}\right) x^{-1-s}\frac{dx}{x} = -\frac{2^{-s}\pi^{1-s}}{12}\frac{(s+2)(s+3)}{s(s-1)(s+1)}.$$

Recall that

$$\langle \frac{dx}{x}, \frac{dx}{x^3} \rangle = \frac{2\pi i(s+3)}{12(s-1)s(s+1)},$$

which implies the inversion formula for $\zeta(s)$:

$$\zeta(s) = 2(2\pi)^{s-1}\sin(\pi s/2)\Gamma(1-s)\zeta(1-s).$$

Remark 4 *The series*

$$\sum_{k=1}^{\infty} \frac{\cos kx}{k^{2m}}, \quad \sum_{k=1}^{\infty} \frac{\sin kx}{k^{2m-1}}, \quad (m \in \mathbb{N})$$

can be expressed in terms of Bernoulli's polynomials.

Mr. Shimada computed $\langle \frac{dx}{x^j}, \frac{dx}{x^k} \rangle$ *for* $j \leq k$ *by Bernoulli's polynomials under the assumption that the compatibility of parings holds.*

If $j \leq k, \quad j+k \geq 4$ *then the evaluations of* $\langle \frac{dx}{x^j}, \frac{dx}{x^k} \rangle$ *by the different ways coincide.*