ある４項算術幾何平均と
３変数超幾何関数

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1. Introduction

$m(a, b)$: The arithmetic-geometric mean of a, b.

$F(\alpha, \beta, \gamma; x)$: The Gauss hypergeometric function.

Fact 1 *For* $0 < x \leq 1$,

$$\frac{1}{m(1, x)} = F(\frac{1}{2}, \frac{1}{2}, 1; 1 - x^2).$$

$\mu(a, b, c, d)$: An arithmetic-geometric mean of a, b, c, d.

$F_D(\alpha, \beta_1, \beta_2, \beta_3, \gamma; x_1, x_2, x_3)$: Lauricella's hypergeometric function.

Theorem 1 *For* $0 < x_3 \leq x_2 \leq x_1 \leq 1$,

$$\frac{1}{\mu(1, x_1, x_2, x_3)} = F_D(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 1; 1 - x_1^2, 1 - x_2^2, 1 - x_3^2)^2.$$

2. The arithmetic-geometric mean

For $a > b > 0$, we define two sequences $\{a_n\}$ and $\{b_n\}$ as

$$a_0 = a, \quad b_0 = b,$$

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}.$$

Fact 2 *$\{a_n\}$ and $\{b_n\}$ converge and*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

This common limit $m(a,b)$ is called the arithmetic-geometric mean of a, b.

Proof. It is clear that $b_0 < b_1 < a_1 < a_0$.

For any $n \in \mathbb{N}$,

$$b = b_0 < b_1 < \cdots < b_n < a_n < \cdots < a_1 < a_0 = a.$$

Since $\{a_n\}$ and $\{b_n\}$ are monotonous and bounded, they converge.

Put $\alpha = \lim_{n\to\infty} a_n$, $\beta = \lim_{n\to\infty} b_n$; and consider the limit of the equality

$$a_{n+1} = \frac{a_n + b_n}{2} \qquad \textit{as} \quad n \to \infty.$$

We have

$$\alpha = \frac{\alpha + \beta}{2},$$

which implies $\alpha = \beta$. $\square$

Remark 1 *The convergence of* $\{a_n\}$ *and* $\{b_n\}$ *is very rapid. We will show the rapidness by Maple.*

Fact 3

$$\begin{aligned} m(ta, tb) &= tm(a, b), \quad (t > 0) \\ m(\frac{a+b}{2}, \sqrt{ab}) &= m(a, b). \end{aligned}$$

By Fact 3,

$$m(1, x) = m(\frac{1+x}{2}, \sqrt{x}) = \frac{1+x}{2} m(1, \frac{2\sqrt{x}}{1+x}).$$

Fact 4 (Shifted relation)

$$m(1, \frac{2\sqrt{x}}{1+x}) = \frac{2}{1+x} m(1, x).$$

The study of the AGM from the elliptic integral, refer to [U].

3. Hypergeometric function

The Gauss hypergeometric function with parameters α, β, γ is defined by

$$F(\alpha, \beta, \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} z^n,$$

where $|z| < 1$, $\gamma \neq 0, -1, -2, \ldots$, and $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$.

$F(\alpha, \beta, \gamma; z)$ satisfies the hypergeometric differential equation:

$$z(1-z)f'' + \{\gamma - (\alpha+\beta+1)z\}f' - \alpha\beta f = 0.$$

Fact 5 (The Gauss quadratic transformation formula)

$$(1+z)^{2\alpha} F(\alpha, \alpha-\beta+\frac{1}{2}, \beta+\frac{1}{2}; z^2) = F(\alpha, \beta, 2\beta; \frac{4z}{(1+z)^2}).$$

Proof. The both sides of Fact 5 are holomorphic functions with value 1 at $z = 0$. By a straightforward calculation, their differential equations are coincident and singular at $z = 0$. $\square$

Corollary 1

$$F(\frac{1}{2}, \frac{1}{2}, 1; 1 - (\frac{2\sqrt{x}}{1+x})^2) = \frac{1+x}{2} F(\frac{1}{2}, \frac{1}{2}, 1; 1 - x^2).$$

Proof. Substitute $\alpha = \beta = \frac{1}{2}$ and $z = \frac{1-x}{1+x}$ into Fact 5. $\square$

Fact 1 For $0 < x \leq 1$,

$$\frac{1}{m(1,x)} = F(\frac{1}{2}, \frac{1}{2}, 1; 1 - x^2).$$

Proof. Make $\{a_n\}$ and $\{b_n\}$ for initial terms $a = 1$, $b = x$ and put

$$F(x) = F(\frac{1}{2}, \frac{1}{2}, 1; 1 - x^2), \quad f(x) = m(1, x)F(x).$$

By Fact 4 and Corollary 1,

$$f(\frac{b_1}{a_1}) = f(\frac{2\sqrt{x}}{1+x}) = m(1, \frac{2\sqrt{x}}{1+x})F(\frac{2\sqrt{x}}{1+x})$$
$$= \left(\frac{2}{1+x}m(1,x)\right)\left(\frac{1+x}{2}F(x)\right) = m(1,x)F(x) = f(x) = f(\frac{b_0}{a_0}).$$

For any $n \in \mathbb{N}$,

$$f(\frac{b_n}{a_n}) = f(\frac{b_0}{a_0}) = f(x).$$

Let $n \to \infty$ then $f(\frac{b_n}{a_n}) \to 1$, since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ and $m(1,1) = F(1) = 1$. Thus $m(1,x)F(x) = 1$. $\square$

4. Some results for AGM's

J.M. Borwein and P.B. Borwein considered modified arithmetic-geometric means $m_3(a,b)$ and $m_4(a,b)$ as the common limits of sequences defined by the recurrence relations

$$a_{n+1} = \frac{a_n + 2b_n}{3}, \quad b_{n+1} = \sqrt[3]{b_n \frac{a_n^2 + a_n b_n + b_n^2}{3}},$$

and

$$a_{n+1} = \frac{a_n + 3b_n}{4}, \quad b_{n+1} = \sqrt{b_n \frac{a_n + b_n}{2}}.$$

Fact 6

$$\begin{aligned} \frac{1}{m_3(1,x)} &= F(\frac{1}{3}, \frac{2}{3}, 1; 1 - x^3), \\ \frac{1}{m_4(1,x)} &= F(\frac{1}{4}, \frac{3}{4}, 1; 1 - x^2)^2. \end{aligned}$$

As a generalization of $m_3(a,b)$, K. Koike and H. Shiga give an arithmetic-geometric mean among three terms and express it by Appell's hypergeometric function $F_1(\frac{1}{3},\frac{1}{3},\frac{1}{3},1;z_1,z_2)$.

They studied another arithmetic-geometric mean among three terms and Appell's hypergeometric functions F_1 with different parameters.

5. Verification 1 by Maple

Gauss AGM, Borwein AGM

6. An arithmetic-geometric mean among four terms

Define four sequences as follows:

$$a_0 = a,\ b_0 = b,\ c_0 = c,\ d_0 = d, \quad a \geq b \geq c \geq d > 0,$$

$$a_{n+1} = \frac{a_n + b_n + c_n + d_n}{4}, \quad b_{n+1} = \frac{\sqrt{(a_n + d_n)(b_n + c_n)}}{2},$$
$$c_{n+1} = \frac{\sqrt{(a_n + c_n)(b_n + d_n)}}{2}, \quad d_{n+1} = \frac{\sqrt{(a_n + b_n)(c_n + d_n)}}{2}.$$

Lemma 1 *$\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ converge to a common value $\mu(a, b, c, d)$, which is called an arithmetic-geometric mean among four terms a, b, c, d.*

Proof. $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ satisfy

$$a \geq a_{n-1} \geq a_n \geq b_n \geq c_n \geq d_n \geq d_{n-1} \geq d$$

for any $n \in \mathbb{N}$. Since the sequences $\{a_n\}$ and $\{d_n\}$ are monotonous and bounded, they converge.

Since

$$\begin{aligned} & a_{n+1} - d_{n+1} \\ = \ & \frac{1}{4}(\sqrt{a_n + b_n} - \sqrt{c_n + d_n})^2 \leq \frac{1}{4}(\sqrt{2a_n} - \sqrt{2d_n})^2 \\ = \ & \frac{1}{2}(a_n + d_n - 2\sqrt{a_n d_n}) \leq \frac{a_n - d_n}{2} \leq \frac{a-d}{2^{n+1}}, \end{aligned}$$

we have $\lim_{n\to\infty}(a_n - d_n) = 0$.

Thus $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ have a common limit. $\square$

Remark 2 *We have*

$$\begin{aligned}(4a_{n+1})^2-(4b_{n+1})^2 &= (a_n-b_n-c_n+d_n)^2,\\(4a_{n+1})^2-(4c_{n+1})^2 &= (a_n-b_n+c_n-d_n)^2,\\(4a_{n+1})^2-(4d_{n+1})^2 &= (a_n+b_n-c_n-d_n)^2.\end{aligned}$$

Lemma 2

$$\mu(ta,tb,tc,td)=t\mu(a,b,c,d) \qquad (t>0),$$

$$\begin{aligned}&\mu(a,b,c,d)\\=\ &\mu\left(\frac{a+b+c+d}{4},\frac{\sqrt{(a+d)(b+c)}}{2},\frac{\sqrt{(a+c)(b+d)}}{2},\frac{\sqrt{(a+b)(c+d)}}{2}\right).\end{aligned}$$

Lemma 3 (Shifted relation) *Let* (y_1, y_2, y_3) *be the image of* (x_1, x_2, x_3) *by the map* φ

$$\varphi(x_1, x_2, x_3) = (\varphi_1(x), \varphi_2(x), \varphi_3(x))$$
$$= \left(\frac{2\sqrt{(1+x_3)(x_1+x_2)}}{1+x_1+x_2+x_3}, \frac{2\sqrt{(1+x_2)(x_1+x_3)}}{1+x_1+x_2+x_3}, \frac{2\sqrt{(1+x_1)(x_2+x_3)}}{1+x_1+x_2+x_3}\right).$$

Then μ *satisfies*

$$\frac{4}{1+x_1+x_2+x_3}\mu(1, x_1, x_2, x_3) = \mu(1, y_1, y_2, y_3) \qquad (1)$$

for $0 < x_3 \leq x_2 \leq x_1 \leq 1$.

Remark 3 *J.F. Mestre studied in [M] a different arithmetic-geometric mean among four terms related to periods of hyper-elliptic curves of genus* 2*.*

7. Lauricella's hypergeometric function

Lauricella's hypergeometric function F_D of 3-variables z_1, z_2, z_3 with parameters $\alpha, \beta_1, \beta_2, \beta_3, \gamma$ is defined as

$$F_D(\alpha,\beta,\gamma;z) = \sum_{n_1,n_2,n_3 \geq 0}^{\infty} \frac{(\alpha)_{n_1+n_2+n_3} \prod_{j=1}^{3}(\beta_j)_{n_j}}{(\gamma)_{n_1+n_2+n_3} \prod_{j=1}^{3} n_j!} \prod_{j=1}^{3} z_j^{n_j},$$

where $z = (z_1, z_2, z_3)$ satisfies $|z_j| < 1$ $(j = 1, 2, 3)$, $\beta = (\beta_1, \beta_2, \beta_3)$, $\gamma \neq 0, -1, -2, \ldots$.

Fact 7 This function satisfies the Pfaffian system given as

$$df = \sum_{1 \le i < j \le 5} A_{ij} d \log(z_i - z_j) f,$$

where $f = {}^t(f_0, f_1, f_2, f_3)$, $f_0 = F_D(\alpha, \beta, \gamma; z)$,

$f_i = z_i \frac{\partial f_0}{\partial z_i}$ $(i = 1, 2, 3)$, $z_4 = 0$, $z_5 = 1$ and

$$A_{ij} = \begin{array}{c} \\ i \\ j \\ \\ \end{array} \!\!\begin{pmatrix} & i & & j & \\ & -\beta_j & & \beta_i & \\ & \beta_j & & -\beta_i & \\ & & & & \end{pmatrix} \quad (1 \le i < j \le 3),$$

$$A_{14} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1+\beta_2+\beta_3-\gamma & 0 & 0 \\ 0 & -\beta_2 & 0 & 0 \\ 0 & -\beta_3 & 0 & 0 \end{pmatrix},$$

$$A_{24} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -\beta_1 & 0 \\ 0 & 0 & 1+\beta_1+\beta_3-\gamma & 0 \\ 0 & 0 & -\beta_3 & 0 \end{pmatrix},$$

$$A_{34} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\beta_1 \\ 0 & 0 & 0 & -\beta_2 \\ 0 & 0 & 0 & 1+\beta_1+\beta_2-\gamma \end{pmatrix},$$

$$A_{15} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\alpha\beta_1 & \gamma - \alpha - \beta_1 - 1 & -\beta_1 & -\beta_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_{25} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha\beta_2 & -\beta_2 & \gamma - \alpha - \beta_2 - 1 & -\beta_2 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_{35} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha\beta_3 & -\beta_3 & -\beta_3 & \gamma - \alpha - \beta_3 - 1 \end{pmatrix}.$$

Remark 4 *For this fact, we refer to the proof of Proposition 9.1.4 in [IKSY], but the A_{ij} and $A_{i,n+1}$ are wrong. K. Ohara informed us of the correct Pfaffian system given by [O].*

8. Main theorem

Theorem 1. For $0 < x_3 \le x_2 \le x_1 \le 1$,

$$\frac{1}{\mu(1, x_1, x_2, x_3)} = F_D(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 1; 1 - x_1^2, 1 - x_2^2, 1 - x_3^2)^2.$$

Proposition 1 *The function* $F(z_1, z_2, z_3) = F_D(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 1; z_1, z_2, z_3)$ *satisfies*

$$\frac{1+x_1+x_2+x_3}{4} F(1 - x_1^2, 1 - x_2^2, 1 - x_3^2)^2$$

$$= F\left(\left(\frac{1-x_1-x_2+x_3}{1+x_1+x_2+x_3}\right)^2, \left(\frac{1-x_1+x_2-x_3}{1+x_1+x_2+x_3}\right)^2, \left(\frac{1+x_1-x_2-x_3}{1+x_1+x_2+x_3}\right)^2\right)^2.$$

Proof. We can show that the Pfaffian systems obtained by the functions in the both sides of the above equality coincide.

Proof of Theorem 1.

Make four sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ for $a = 1, b = x_1, c = x_2, d = x_3$. Lemma 3 and Proposition 1 imply that

$$\begin{aligned} &\mu(1, x_1, x_2, x_3)F(1 - x_1^2, 1 - x_2^2, 1 - x_3^2)^2 \\ = \; &\mu(1, y_1, y_2, y_3)F(1 - y_1^2, 1 - y_2^2, 1 - y_3^2)^2. \end{aligned}$$

Thus for any $n \in \mathbb{N}$, we have

$$\begin{aligned} &\mu(1, x_1, x_2, x_3)F(1 - x_1^2, 1 - x_2^2, 1 - x_3^2)^2 \\ = \; &\mu(1, \frac{b_n}{a_n}, \frac{c_n}{a_n}, \frac{d_n}{a_n})F(1 - (\frac{b_n}{a_n})^2, 1 - (\frac{c_n}{a_n})^2, 1 - (\frac{d_n}{a_n})^2)^2. \end{aligned}$$

Since $\lim_{n \to \infty} \frac{b_n}{a_n} = \lim_{n \to \infty} \frac{c_n}{a_n} = \lim_{n \to \infty} \frac{d_n}{a_n} = 1$ and

$\mu(1, 1, 1, 1) = F(0, 0, 0) = 1$,

We have $\mu(1, x_1, x_2, x_3)F(1 - x_1^2, 1 - x_2^2, 1 - x_3^2)^2 = 1$. □

Theorem 2

$$F_D\left(\frac{a}{4}, \frac{a+2}{12}, \frac{a+2}{12}, \frac{a+2}{12}, \frac{a+2}{3}; 1-z_1^2, 1-z_2^2, 1-z_3^2\right)$$

$$= \left(\frac{1+z_1+z_2+z_3}{4}\right)^{\frac{-a}{2}} F_D\left(\frac{a}{4}, \frac{a+2}{12}, \frac{a+2}{12}, \frac{a+2}{12}, \frac{a+5}{6}; z_1', z_2', z_3'\right),$$

where

$$z_1' = \left(\frac{1-z_1-z_2+z_3}{1+z_1+z_2+z_3}\right)^2,$$

$$z_2' = \left(\frac{1-z_1+z_2-z_3}{1+z_1+z_2+z_3}\right)^2,$$

$$z_3' = \left(\frac{1+z_1-z_2-z_3}{1+z_1+z_2+z_3}\right)^2.$$

9. Verification 2 by Maple

4-terms AGM

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