ARITHMETIC-GEOMETRIC MEANS FOR HYPERELLIPTIC CURVES AND CALABI-YAU VARIETIES

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ABSTRACT. In this paper, we define a generalized arithmetic-geometric mean μ_g among 2^g terms motivated by 2τ-formulas of theta constants. By using Thomae's formula, we give two expressions of μ_g when initial terms satisfy some conditions. One is given in terms of period integrals of a hyperelliptic curve C of genus g. The other is by a period integral of a certain Calabi-Yau g-fold given as a double cover of the g-dimensional projective space $\mathbf{P}^g$.

1. INTRODUCTION

Let $\{a_{n,0}\}_n$ and $\{a_{n,1}\}_n$ be positive real sequences defined by the recurrence relations

$$a_{n+1,0} = \frac{a_{n,0} + a_{n,1}}{2}, \quad a_{n+1,1} = \sqrt{a_{n,0}a_{n,1}}, \tag{1.1}$$

and initial terms $a_{0,0} = a_0$, $a_{0,1} = a_1$ with $0 < a_1 < a_0$. One can easily show that $\{a_{n,0}\}_n$ and $\{a_{n,1}\}_n$ have a common limit, which is called the arithmetic-geometric mean of a_0 and a_1, and is denoted by $\mu_1(a_0, a_1)$. By the homogeneity of the arithmetic and geometric means, we have $\mu_1(ca_0, ca_1) = c\mu_1(a_0, a_1)$ for any positive real number c.

On the other hand, two Jacobi's theta constants θ_0 and θ_1 satisfy the following 2τ-formulas:

$$\theta_0(2\tau)^2 = \frac{\theta_0(\tau)^2 + \theta_1(\tau)^2}{2}, \quad \theta_1(2\tau)^2 = \theta_0(\tau)\theta_1(\tau), \tag{1.2}$$

where

$$\theta_i(\tau) = \sum_{n\in\mathbf{Z}} \exp(\pi\sqrt{-1}(n^2\tau + in)), \qquad i = 0, 1,$$

and τ belongs to the upper half space $\mathbf{H}$. If we find an element $\tau \in \mathbf{H}$ such that $\theta_1(\tau)^2/\theta_0(\tau)^2 = a_1/a_0$ for given initial terms a_0 and a_1, then we have

$$\frac{a_0}{\mu_1(a_0, a_1)} = \frac{\theta_0(\tau)^2}{\mu_1(\theta_0(\tau)^2, \theta_1(\tau)^2)} = \frac{\theta_0(\tau)^2}{\mu_1(\theta_0(2^n\tau)^2, \theta_1(2^n\tau)^2)} = \theta_0(\tau)^2$$

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by (1.1), (1.2) and $\lim_{n\to\infty}\theta_i(2^n\tau)=1$. Moreover, the Jacobi's formula between $\theta_0(\tau)^2$ and an elliptic integral implies that

$$\frac{a_0}{\mu_1(a_0,a_1)}=\frac{2}{\pi}\int_0^1\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},\quad k=\frac{\sqrt{a_0^2-a_1^2}}{a_0}.$$

In this paper, we define a generalized arithmetic-geometric mean μ_g among 2^g terms $(\dots,a_I,\dots)$ $(I\in\mathbf{F}_2^g)$ motivated by the 2τ-formulas (2.3) of theta constants obtained by Theorem 2 in [3] p.139. By using Thomae's formula, we give two expressions of μ_g whose initial terms are given as (3.1) for some $2g+1$ real numbers p_j. One is given in terms of period integrals of the hyperelliptic curve C of genus g represented by the double cover of the complex projective line $\mathbf{P}^1$ branching at ∞ and $2g+1$ points p_j. The other is by a period integral of the Calabi-Yau g-fold which is the double cover of the g-dimensional projective space $\mathbf{P}^g$ branching along the dual hyperplanes of the images of ∞ and p_j $(j=1,\dots,2g+1)$ under the Veronese embedding of $\mathbf{P}^1$ into $\mathbf{P}^g$.

In 1876, Borchardt studied in [1] the case of $g=2$: the generalized arithmetic-geometric mean μ_2 of $a=(a_{00},a_{01},a_{10},a_{11})$ was given by the iteration of four means

$$\frac{a_{00}+a_{01}+a_{10}+a_{11}}{4},\quad \frac{\sqrt{a_{00}a_{01}}+\sqrt{a_{11}a_{10}}}{2},$$
$$\frac{\sqrt{a_{00}a_{10}}+\sqrt{a_{11}a_{01}}}{2},\quad \frac{\sqrt{a_{00}a_{11}}+\sqrt{a_{10}a_{01}}}{2},$$

and $\mu_2(a)$ was expressed in terms of period integrals of a hyperelliptic curve of genus 2. Mestre showed in [4] that $\mu_2(a)$ could be expressed in terms of μ_1 and some algebraic functions of a when

$$a_{00}>a_{01}=a_{10}>a_{11},\quad a_{00}a_{11}>a_{01}a_{10}.$$

2. Comparison to theta constants

We define a hyperelliptic curve C of genus g by

$$C: y^2=(x-p_1)\cdots(x-p_{2g+1}),$$

where p_j's are real numbers satisfying $p_1<\cdots<p_{2g+1}$. As in [6] p.76, we choose the cycles $A_1,\dots,A_g,B_1,\dots,B_g$ in the union of the following two sheets (I),(II) in Figure 1. Here $\mathbf{R}_+$ is the set of non-negative real numbers, the range of values of y is written, and the cycles in the sheet II are written in thick lines. Note that the cycles satisfy

$$A_i\cdot A_j=B_i\cdot B_j=0,\quad A_i\cdot B_j=\delta_{ij}$$

for $1\le i,j\le g$ under the intersection form.

We define holomorphic forms ω_j for $j=1,\dots,g$ as

$$\omega_j=\frac{x^{j-1}dx}{y}.$$

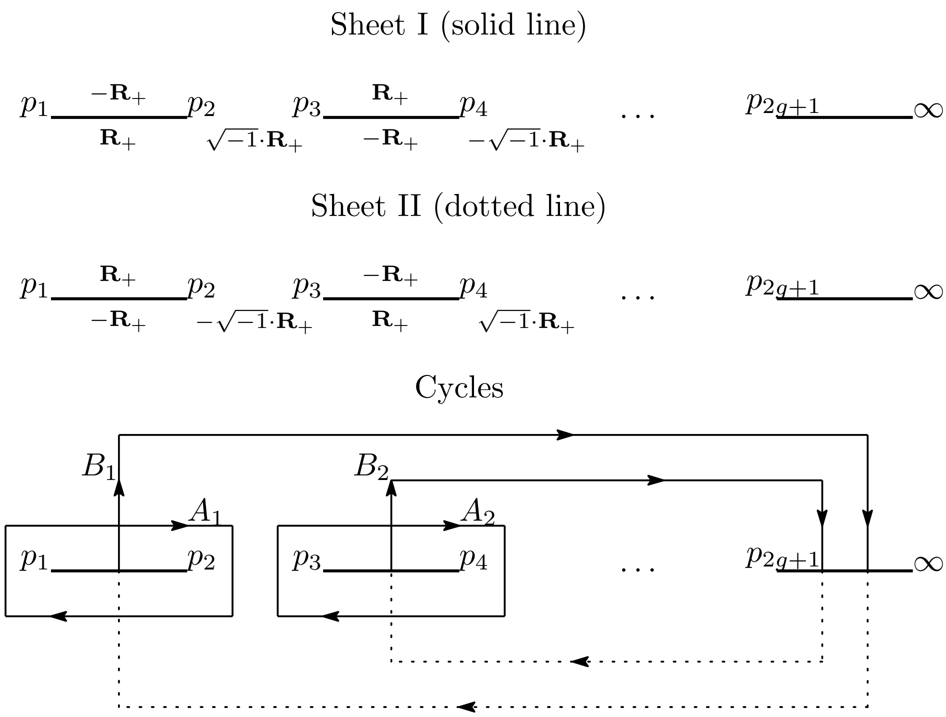

FIGURE 1. Symplectic basis

We define integrals $T_i^{(j)}$ by

$$T_i^{(j)} = \int_{p_i}^{p_{i+1}} \frac{x^{j-1}dx}{\sqrt{\prod_{k=1}^{i}(x-p_k)\prod_{k=i+1}^{2g+1}(p_k-x)}}$$

for $1 \le i \le 2g$ and $1 \le j \le g$. Then the integrals $T_i^{(j)}$ are positive real numbers. Using these integrals, we express the period integrals of C:

$$\int_{A_i} \omega_j = (-1)^i 2T_{2i-1}^{(j)}, \quad \int_{B_i} \omega_j = 2\sqrt{-1}(\sum_{k=i}^{g}(-1)^{k+1}T_{2k}^{(j)}).$$

We set

$$A = (\int_{A_i} \omega_j)_{ij}, \quad B = (\int_{B_i} \omega_j)_{ij} \tag{2.1}$$

and consider the normalized period matrix τ by A-period:

$$\tau = BA^{-1}. \tag{2.2}$$

By Riemann's bilinear relations, $\det(A)$ is a non-zero real number and τ is a symmetric matrix whose imaginary part is positive definite. Note also that τ is purely imaginary.

Remark 2.1. *Since the Vandermonde matrix* $\det(x_i^{j-1})_{1\le i,j\le g}$ *is positive on* $p_{2i-1} \le x_i \le p_{2i}$, $(-1)^{g(g+1)/2}\det(A)$ *is positive.*

For $I = (i_1, \ldots, i_g) \in \mathbf{F}_2^g$, we define theta constants as

$$\theta_I(\tau) = \sum_{n \in \mathbf{Z}^g} \exp(\pi\sqrt{-1} \cdot n\tau\ {}^t n + \pi\sqrt{-1} \cdot n \cdot {}^t I).$$

Proposition 2.2. *Let M be a positive definite symmetric $g \times g$ real matrix. Then $\theta_I(\sqrt{-1}M)$ is positive for each $I \in \mathbf{F}_2^g$.*

Proof. By the inversion formula of the theta function in [5] p.195, we have

$$\sqrt{\det(M)} \cdot \theta_I(\sqrt{-1}M) = \sum_{n \in \mathbf{Z}^g} \exp\left(\sqrt{-1}\pi(n + \frac{I}{2})(\sqrt{-1}M^{-1})\ {}^t(n + \frac{I}{2})\right),$$

where $\sqrt{\det(M)}$ takes a positive value. Since each term of the right hand side is positive, the left hand side is positive. □

We consider variable $u = (u_I)_{I \in \mathbf{F}_2^g}$ whose coordinates are indexed by $\mathbf{F}_2^g$. The pair $(\theta_I(\tau))_I$ is denoted by $\theta(\tau)$. For $I \in \mathbf{F}_2^g$, we define quadratic polynomials $F_I(u)$ of 2^g variables $u = (u_I)_{I \in \mathbf{F}_2^g}$ by

$$F_I(u) = \frac{1}{2^g} \sum_{P \in \mathbf{F}_2^g} u_{I+P} u_P.$$

We remark that the coefficients of $2^g F_I(u)$ are in $\mathbf{Z}_{\geq 0}$. By Theorem 2 in [3] p.139, we have 2τ-formulas of theta constants

$$\theta_I(2\tau)^2 = F_I(\theta(\tau)) \tag{2.3}$$

for $I \in \mathbf{F}_2^g$.

Now prepare some combinatorial notations for the statement of Thomae's formula. For an index $I \in \mathbf{F}_2^g$, we define a subset S_I of $R = \{1, \ldots, 2g+1, \infty\}$ as follows. Let η_i be elements of $M(2, g, \mathbf{F}_2)$ defined as

$$\eta_{2i-1} = \begin{pmatrix} 0 & \cdots & 0 & \overset{i\text{-th}}{1} & 0 & \cdots & 0 \\ 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$\eta_{2i} = \begin{pmatrix} 0 & \cdots & 0 & \overset{i\text{-th}}{1} & 0 & \cdots & 0 \\ 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \end{pmatrix},$$

for $i = 1, \ldots, 2g+1$. Then a subset T_I of $R - \{2g+1, \infty\} = \{1, 2, \ldots, 2g\}$ is characterized by the equality

$$\begin{pmatrix} 0 \\ I \end{pmatrix} = \sum_{j \in T_I} \eta_j.$$

We set

$$S_I = \begin{cases} T_I & \text{if } \#T_I \text{ is even,} \\ T_I \cup \{2g+1\} & \text{if } \#T_I \text{ is odd.} \end{cases}$$

Let U be the set $\{1, 3, 5, \ldots, 2g+1\}$ and $R_1 \circ R_2$ be the symmetric difference of sets R_1 and R_2.

Proposition 2.3 ([6] p.120, [2]). *Let A be the period matrix of C in (2.1). Then we have*

$$\frac{(2\pi)^{2g}\theta_I(\tau)^4}{\det(A)^2} = \prod_{i<j,i,j\in S_I\circ U}(p_j-p_i)\prod_{i<j,i,j\notin S_I\circ U}(p_j-p_i). \tag{2.4}$$

Here we used the fact that $\theta_I(\tau)$ is a real number to determine the sign of Thomae's formula in [6].

3. Statement and proof of the main theorem

Definition 3.1 (AGM sequences).

(1) *For an element $u=(u_I)_I\in\mathbf{R}_+^{2^g}$, we define the termwise root $\sqrt{u}$ of u by $(\sqrt{u_I})_I$.*

(2) *Let $a=(a_I)_I$ be an element in $(\mathbf{R}_+)^{2^g}$. We define $a_k=(a_{k,I})_I$ inductively by the relation*

$$a_{0,I}=a_I,\quad a_{k+1,I}=F_I(\sqrt{a_k}).$$

A proof of the following proposition will be left to readers.

Proposition-Definition 3.2 (Generalized arithmetic-geometric mean). *For an element $a=(a_I)_I$ in $(\mathbf{R}_+)^{2^g}$, the limits $\lim_{k\to\infty}a_{k,I}$ exist and are independent of indexes I. This common limit is called the generalized arithmetic-geometric mean of $(a_I)_I$ and denoted by $\mu_g(a_I)$.*

Problem 3.3. *Is it possible to express the generalized arithmetic-geometric mean $\mu_g(a_I)$ of $a=(a_I)_I\in(\mathbf{R}_+)^{2^g}$ in terms of period integrals of a family of varieties parametrized by a ?*

Theorem 3.4. *Let $p_1<\cdots<p_{2g+1}$ be real numbers. We define a_I by*

$$a_I=\sqrt{\prod_{i<j,i,j\in S_I\circ U}(p_j-p_i)\prod_{i<j,i,j\notin S_I\circ U}(p_j-p_i)}. \tag{3.1}$$

Then we have

$$\mu_g(a_I)=\frac{(2\pi)^g}{|\det(A)\,|},$$

where A is the period matrix of C in (2.1).

Proof. By the initial condition, we have

$$a_{0,I}=\frac{(2\pi)^g\theta_I(\tau)^2}{|\det(A)|}.$$

We show that

$$a_{n,I}=\frac{(2\pi)^g\theta_I(2^n\tau)^2}{|\det(A)|},$$

by induction on n. Since $\theta_I(2^n\tau)$ is a positive real number by Proposition 2.2 for each I, we have

$$\begin{aligned} a_{n+1,I} =& F(\sqrt{a_n}) \\ =& \frac{(2\pi)^g \cdot F(\theta(2^n\tau))}{|\det(A)|} \qquad \text{(by the induction hypothesis)} \\ =& \frac{(2\pi)^g \cdot \theta_I(2^{n+1}\tau)^2}{|\det(A)|} \qquad \text{(by the formula (2.3))} \end{aligned}$$

Therefore we have

$$\lim_{n\to\infty} a_{n,I} = \frac{(2\pi)^g}{|\det(A)|}.$$

□

4. Period of Calabi-Yau variety of certain type

We study a relation between the generalized arithmetic-geometric mean of the last section and a period of a Gorenstein Calabi-Yau variety of a certain type.

Definition 4.1 (Calabi-Yau varieties)**.** *A variety X only with Gorenstein singularities is called a Calabi-Yau variety if the dualizing sheaf of X is trivial and X has a global crepant resolution.*

Let $\mathbf{P} = \mathbf{P}^g$ be the g dimensional projective space and $H_1 \cdots H_{2g+2}$ be hyperplanes of $\mathbf{P}$. There is a unique line bundle $\mathcal{L}$ on $\mathbf{P}$ and a unique isomorphism $\varphi : \mathcal{L}^{\otimes 2} \simeq O_X(-\sum_{i=1}^{2g+2} H_i)$ up to a non-zero constant. Using the isomorphism φ, we define a double covering $X = \mathcal{S}pec(\mathcal{O}_X \oplus \mathcal{L})$, where the multiplication on $\mathcal{L} \otimes \mathcal{L} \to \mathcal{O}_X$ is given by the isomorphism φ.

By the following Proposition 4.2, X becomes a Calabi-Yau variety, since it admits a global crepant resolution.

Proposition 4.2.

(1) *If $\cup_{i=1}^{2g+2} H_i$ is normal crossing, then the variety X has only Gorenstein singularities. Also it admits a global crepant resolution.*

(2) *Under the above hypotheses, the dualizing sheaf is isomorphic to the structure sheaf.*

Proof. (1) Locally on $\mathbf{P}$, the variety X is defined by the equation $\eta^2 = \xi_1 \cdots \xi_g$, where $\xi_1, \ldots, \xi_g$ are local coordinates. Therefore this variety U is an affine toric variety defined by $Spec(\check{\sigma} \cap M^*)$, where

$$M^* = \mathbf{Z}^g + (\frac{1}{2}, \ldots, \frac{1}{2})\mathbf{Z} \subset \mathbf{Q}^g, \quad \check{\sigma} = (\mathbf{R}_+)^g.$$

Let σ be the dual simplex of $\check{\sigma}$ and M be the dual lattice of M. Since σ is generated by elements contained primitive hyperplanes, X is Gorenstein. We can construct a global crepant resolution as follows. We make a refinement of the simplex σ into a regular fan $\cup_{\mathbf{w}\in\rho_g}\sigma_{\mathbf{w}}$ indexed by the set ρ_g of “unfair tournament” of $\{1, \ldots, g\}$. A sequence $\mathbf{w} = (w_1, \ldots, w_{g-1})$ is an element of the set ρ_g if it satisfies the following properties:

(i) w_1 is equal to 1 or 2 and
(ii) w_i is equal to w_{i-1} or $i+1$ for $2 \le i \le g-1$.

For an element $\mathbf{w}$ of ρ_g, we define $\sigma_{\mathbf{w}}$ as a cone generated by

$$B_{\mathbf{w}} = \{u_1 = e_1 + e_2, u_2 = e_{w_1} + e_3, u_3 = e_{w_2} + e_4 \dots, u_{g-1} = e_{w_{g-2}} + e_g, \\ u_g = 2e_{w_{g-1}}\},$$

where e_i is the standard basis of $\mathbf{Z}^g \supset M$. Since the set $B_{\mathbf{w}}$ is a free base of M, the fan $\cup_{\mathbf{w}\in\rho_g}\sigma_{\mathbf{w}}$ is regular and it defines a smooth toric variety $\tilde{X}$. The coordinates associated to $\mathbf{Z}^g \subset M^*$ are written as $\xi_1, \dots, \xi_g$. (η corresponds to $\frac{1}{2}(1, \dots, 1)$.) Let $z_1, \dots, z_g$ be the coordinates associated to the dual base $B_{\mathbf{w}}$ of M. Then we have

$$z_1^{u_1} \cdots z_g^{u_g} = \xi_1^{e_1} \cdots \xi_g^{e_g}.$$

Thus $\xi_1^{\frac{1}{2}} \cdots \xi_g^{\frac{1}{2}} = z_1 \cdots z_g$. Therefore the pull back of the rational differential form ω_X to the affine toric variety associated to $\sigma_{\mathbf{w}}$ is a non-zero constant multiple of $dz_1 \wedge \cdots \wedge dz_g$, which shows that the map $\tilde{X} \to X$ is a crepant resolution. Since the local crepant resolutions depend only on the choice of order of the components of the branching divisor, they are patched together into a global crepant resolution.

(2) Let $\xi_1, \dots, \xi_g$ be inhomogeneous coordinates of $\mathbf{P}$ with the infinite hyperplane H_{g+2} and $l_i = l_i(\xi)$ be inhomogeneous linear forms defining the hyperplane H_i for $i = 1, \dots, 2g+1$. Then defining equation of the double covering X can be written as

$$\eta^2 = \prod_{i=1}^{2g+1} l_i(\xi).$$

As is shown in the proof of (1),

$$\omega_X = \frac{1}{\eta} d\xi_1 \wedge \cdots \wedge d\xi_g \tag{4.1}$$

is a global generator of the dualizing sheaf of X. □

For real numbers $p_1 < \cdots < p_{2g+1}$, we define linear forms l_i by

$$l_i = \xi_1 - p_i\xi_2 + p_i^2\xi_3 + \cdots + (-1)^{g-1}p_i^{g-1}\xi_g + (-1)^g p_i^g$$

and set $H_i = \{l_i = 0\}$. By using the Vandermonde matrix, we see that $\cup_{i=1}^{2g+2} H_i$ is a normal crossing divisor.

We define a subset Δ of $\mathbf{R}^g$ as

$$\Delta = \{(x_1, \dots, x_g) \mid (-1)^{i-1} l_{2i-1}(x_1, \dots, x_g) \ge 0 \text{ for } i = 1, \dots, g+1, \text{ and} \\ (-1)^i l_{2i}(x_1, \dots, x_g) \ge 0 \text{ for } i = 1, \dots, g\}.$$

We set

$$\omega_X = \frac{1}{\eta} d\xi_1 \wedge \cdots \wedge d\xi_g, \text{ and } \gamma_\pm = \{(\xi, \eta) \in X \mid \xi \in \Delta, \pm\eta \ge 0\}.$$

Then $\gamma = \gamma_+ - \gamma_-$ defines a g-chain in X. We have the following relation between the generalized arithmetic-geometric mean and a period of the Calabi-Yau variety X. The following theorem is obtained by Theorem 2 in [7].

Theorem 4.3. *Let $(a_I)_I$ be an element of $\mathbf{R}_+^g$ defined in (3.1). Under the above notation, we have*

$$\mu(a_I) = \frac{2\pi^g}{\int_\gamma \omega_X}.$$

Proof. Let C_j be a copy of the curve C given by $y_j = \prod_{i=1}^{2g-1}(x_j - p_i)$. We define a map $\pi : C_1 \times \cdots \times C_g \to X$ by sending $((x_1, y_1), \ldots, (x_g, y_g))$ to the point whose ξ_k-coordinate and η-coordinate are the $(g+1-k)$-th elementary symmetric function of $x_1, \ldots, x_g$ and $\prod_{i=1}^g y_i$, respectively. Then we have

$$\pi^* \omega_X = \sum_{\sigma \in \mathcal{S}_g} \mathrm{sgn}(\sigma) \boxtimes_{i=1}^g \omega_{\sigma(i)}.$$

Since $\pi_*(A_1 \times \cdots \times A_g) = (-1)^{g(g+1)/2} 2^{g-1} \gamma$, we have

$$2^{g-1} \int_\gamma \omega_X = \mid \det(A) \mid .$$

By Theorem 3.4, we have the theorem. □

5. Genus two case

In this section, we will give a detailed study for the case of $g = 2$. Refer to [1] and [4] for the original results by Borchardt and recent related works by Mestre, respectively. We begin with $(a_{00}, a_{01}, a_{10}, a_{11})$ as initial data for AGM sequences. The recursive relations for $a_{k,I}$ $(I \in \mathbf{F}_2^2, k = 0, 1, \cdots)$ are given as $a_{0,I} = a_I$ and $a_{k+1,I} = F_I(\sqrt{a_{k,00}}, \cdots, \sqrt{a_{k,11}})$, where

$$F_{00}(u_{00}, u_{01}, u_{10}, u_{11}) = \frac{1}{4}(u_{00}^2 + u_{01}^2 + u_{10}^2 + u_{11}^2),$$
$$F_{01}(u_{00}, u_{01}, u_{10}, u_{11}) = \frac{1}{2}(u_{00}u_{01} + u_{11}u_{10}),$$
$$F_{10}(u_{00}, u_{01}, u_{10}, u_{11}) = \frac{1}{2}(u_{00}u_{10} + u_{11}u_{01}),$$
$$F_{11}(u_{00}, u_{01}, u_{10}, u_{11}) = \frac{1}{2}(u_{00}u_{11} + u_{10}u_{01}).$$

In the following, we assume that $a_{00} > a_{10} > a_{11} > a_{01}$ and $a_{00}a_{01} > a_{10}a_{11}$. First we define positive real numbers $k_1 > k_2$ and $0 < l_2 < l_1 < 1$ such that

$$(a_{00} + a_{01})^2 - (a_{10} + a_{11})^2 = k_1^2, \quad (a_{00} - a_{01})^2 - (a_{10} - a_{11})^2 = k_2^2,$$

$$a_{00}+a_{01}=\frac{1+l_1^2}{1-l_1^2}k_1,\quad a_{10}+a_{11}=\frac{2l_1}{1-l_1^2}k_1,$$
$$a_{00}-a_{01}=\frac{1+l_2^2}{1-l_2^2}k_2,\quad a_{10}-a_{11}=\frac{2l_2}{1-l_2^2}k_2,$$

We set

$$p_1=0,\quad p_2=\frac{1}{(1-{l_2}^2)\,(1-{l_1}^2)},$$
$$p_3=\frac{2(l_1\,l_2+1)\,a_{00}}{(1-{l_1}^2)\,(1-{l_2}^2)\,(k_1+k_2)\,(1-l_1\,l_2)},$$
$$p_4=\frac{2(l_1\,l_2+1)\,a_{01}}{(1-{l_1}^2)\,(1-{l_2}^2)\,(k_1-k_2)\,(1-l_1\,l_2)},$$
$$p_5=\frac{4a_{00}a_{01}}{(k_1-k_2)\,(k_1+k_2)\,(1-{l_2}^2)\,(1-{l_1}^2)}.$$

Then we have

$$(5.1)\qquad \begin{aligned}(a_{00}^2:a_{01}^2:a_{10}^2:a_{11}^2)=((p_3-p_1)(p_5-p_1)(p_5-p_3)(p_4-p_2):\\ (p_4-p_1)(p_5-p_1)(p_5-p_4)(p_3-p_2):\\ (p_3-p_2)(p_5-p_2)(p_5-p_3)(p_4-p_1):\\ (p_4-p_2)(p_5-p_2)(p_5-p_4)(p_3-p_1)).\end{aligned}$$

Therefore by Theorem 3.4, we have

$$\begin{aligned}\lim_{n\to\infty}a_{n,00}&=\frac{4\pi^2a_{00}}{\mid\det(A)\mid\sqrt{(p_3-p_1)(p_5-p_1)(p_5-p_3)(p_4-p_2)}}\\ &=\frac{8\pi^2}{\mid\det(A)\mid}\cdot(1-l_1^2)^2(1-l_2^2)^2\sqrt{\frac{(a_{00}a_{01}-a_{10}a_{11})^3(1-l_1l_2)^3}{a_{00}a_{01}a_{10}a_{11}(l_1^2-l_2^2)(1+l_1l_2)}}.\end{aligned}$$

where A is the period matrix of C in (2.1).

Using the result of §4, we have

$$\mid\det(A)\mid=4\cdot\int_\Delta\frac{d\xi_1\wedge d\xi_2}{\sqrt{\prod_{i=1}^5(\xi_1-p_i\xi_2+p_i^2)}},$$

where Δ is a domain in $\mathbf{R}^2$ defined by $l_1\geq 0,-l_2\geq 0,-l_3\geq 0,l_4\geq 0$ and $l_5\geq 0$. This is a period integral of the covering X of $\mathbf{P}^2$ defined by

$$\eta^2=\prod_{i=1}^5(\xi_1-p_i\xi_2+p_i^2).$$

We notice that the variety X is the (nodal) Kummer surface of the Jacobian of C.

Remark 5.1. *When*

$$a_{00}>a_{01}=a_{10}>a_{11},\quad a_{00}a_{11}>a_{01}a_{10},$$

$\mu_2(a)$ can be expressed in terms of the arithmetic-geometric mean μ_1 and expressions $p_2, \dots, p_5$ by a (see [4]*).*

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