Some transformation formulas for Lauricella's hypergeometric functions F_D

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Abstract

In this paper, we give some functional equations with a parameter c for Lauricella's hypergeometric functions; they can be regarded as multivariable versions of the Gauss quadratic transformation formula for the hypergeometric function. These functional equations for $c = 1$ are utilized for the study of arithmetic-geometric means of several terms.

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1 Introduction

It is known that the hypergeometric function $F(\alpha, \beta, \gamma; z)$ satisfies the Gauss quadratic transformation formula:

$$(1+z)^{2\alpha} F(\alpha, \alpha-\beta+\frac{1}{2}, \beta+\frac{1}{2}; z^2) = F(\alpha, \beta, 2\beta; \frac{4z}{(1+z)^2}).$$

When $\alpha = \beta = \frac{1}{2}$, this equality reduces to

$$\frac{1+z}{2} F(\frac{1}{2}, \frac{1}{2}, 1; 1-z^2) = F(\frac{1}{2}, \frac{1}{2}, 1; 1-(\frac{2\sqrt{z}}{1+z})^2),$$

which implies that the reciprocal of the arithmetic-geometric mean of 1 and $x \in (0,1)$ coincides with $F(\frac{1}{2}, \frac{1}{2}, 1; 1-x^2)$, refer to [HKM].

In this paper, we give some functional equations with a parameter c for Lauricella's hypergeometric functions F_D; they can be regarded as multivariable versions of the Gauss quadratic transformation formula. Our functional equations for $c = 1$ are given in [KS1],[KS2] and [KM], and they imply the expressions of arithmetic-geometric means of several terms by Lauricella's hypergeometric functions F_D. We also show that each of our functional equations admits no other parameters when we specify the transformations of variables of F_D and an admissible factor to the product of power functions associated with singular locus of F_D.

By considering restrictions of variables for our theorems, we obtain three transformation formulas for the hypergeometric function $F(\alpha, \beta, \gamma; z)$. We remark that they are not listed in [E] and [G], and that one of them was recently found in [BBG].

For proofs of our theorems, we utilize yang [O1], which is a package of computer algebra system Risa/Asir for the ring of differential-difference operators.

2 Lauricella's hypergeometric function F_D

Lauricella's hypergeometric function F_D of m variables $z_1, \dots, z_m$ with parameters $\alpha, \beta_1, \dots, \beta_m, \gamma$ is defined as

$$F_D(\alpha, \beta, \gamma; z) = \sum_{n_1,\dots,n_m \geq 0}^{\infty} \frac{(\alpha, \sum_{j=1}^m n_j) \prod_{j=1}^m (\beta_j, n_j)}{(\gamma, \sum_{j=1}^m n_j) \prod_{j=1}^m (1, n_j)} \prod_{j=1}^m z_j^{n_j},$$

where $z = (z_1, \dots, z_m)$ satisfies $|z_j| < 1$ $(j = 1, \dots, m)$, $\beta = (\beta_1, \dots, \beta_m)$, $\gamma \neq 0, -1, -2, \dots$ and $(\alpha, n) = \alpha(\alpha+1)\cdots(\alpha+n-1) = \Gamma(\alpha+n)/\Gamma(\alpha)$. This function admits the integral representation of Euler type:

$$F_D(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha}(1-t)^{\gamma-\alpha} \prod_{j=1}^m (1 - z_j t)^{-\beta_j} \frac{dt}{t(1-t)}. \quad (1)$$

When $m = 1$, $F_D(\alpha, \beta, \gamma; z)$ coincides with the Gauss hypergeometric function $F(\alpha, \beta, \gamma; z)$, and when $m = 2$, $F_D(\alpha, \beta, \gamma; z)$ is Appell's hypergeometric function $F_1(\alpha, \beta_1, \beta_2, \gamma; z_1, z_2)$.

Fact 1 (Proposition 9.1.4 in [IKSY]) *The function $F_D(\alpha,\beta,\gamma;z)$ satisfies the integrable Pfaffian system*

$$d\hat{f} = \Omega_{\hat{f}}(z)\hat{f}, \quad \Omega_{\hat{f}}(z) = \sum_{1\le i<j\le m+2} A_{ij}\, d\log(z_i - z_j),$$

where $\hat{f} = {}^t(f_0, f_1, \dots, f_m)$, $f_0 = F_D(\alpha,\beta,\gamma;z)$, $f_i = z_i \frac{\partial f_0}{\partial z_i}$ $(1 \le i \le m)$, $z_{m+1} = 0$, $z_{m+2} = 1$, *and* $(m+1)\times(m+1)$*-matrices* A_{ij} *are given as*

$$A_{ij} = \begin{array}{cc} & \begin{array}{ccc} 0\text{-}th & i\text{-}th & j\text{-}th \end{array} \\ \begin{array}{c} 0\text{-}th \\ i\text{-}th \\ j\text{-}th \end{array} & \begin{pmatrix} & & \\ & -\beta_j & \beta_i \\ & \beta_j & -\beta_i \end{pmatrix} \end{array} \quad (1 \le i < j \le m),$$

$$A_{i,m+1} = \begin{array}{cc} & \begin{array}{ccc} 0\text{-}th & i\text{-}th & \end{array} \\ \begin{array}{c} 0\text{-}th \\ \\ \\ \\ i\text{-}th \\ \\ \\ \\ \end{array} & \begin{pmatrix} & 1 & \\ & -\beta_1 & \\ O & \vdots & O \\ & -\beta_{i-1} & \\ & 1-\gamma+\sum\limits_{1\le k\le m}^{k\ne i}\beta_k & \\ & -\beta_{i+1} & \\ O & \vdots & O \\ & -\beta_m & \end{pmatrix} \end{array} \quad (1 \le i \le m),$$

$$A_{i,m+2} = \begin{array}{cc} & \begin{array}{cccccccc} 0\text{-}th & & & & i\text{-}th & & & \end{array} \\ \begin{array}{c} 0\text{-}th \\ i\text{-}th \\ \\ \end{array} & \begin{pmatrix} & & O & & & & O & \\ -\alpha\beta_i & -\beta_i & \cdots & -\beta_i & \gamma-\alpha-\beta_i-1 & -\beta_i & \cdots & -\beta_i \\ & & O & & & & O & \end{pmatrix} \end{array}$$
$$(1 \le i \le m).$$

(Here we correct A_{ij} and $A_{i,m+1}$.) The singular locus of this Pfaffian system

is $\bigcup_{1\le i<j\le m+2} L_{ij} \subset \mathbb{C}^m$, *where*

$$L_{ij} = \{\ell_{ij} = z_i - z_j = 0\} \tag{2}$$

and $L_{m+1,m+2}$ *is regarded as the hyperplane at infinity.*

3 Transformation formulas

Theorem 1 *The hypergeometric function* F_D *of* 2 *variables satisfies the transformation formula*

$$\begin{aligned} & \left(\frac{1+z_1+z_2}{3}\right)^c F_D(\frac{c}{3}, \frac{c+1}{6}, \frac{c+1}{6}, \frac{c+1}{2}; 1-z_1^3, 1-z_2^3) \\ = \quad & F_D(\frac{c}{3}, \frac{c+1}{6}, \frac{c+1}{6}, \frac{c+5}{6}; z_1', z_2'), \end{aligned} \tag{3}$$

$$\xi : (z_1, z_2) \mapsto (z_1', z_2') = \left((\frac{1+\omega z_1 + \omega^2 z_2}{1+z_1+z_2})^3, (\frac{1+\omega^2 z_1 + \omega z_2}{1+z_1+z_2})^3 \right)$$

where $\omega = \frac{-1+\sqrt{-3}}{2}$, $z = (z_1, z_2)$ *is in a small neighborhood* U *of* $(1,1)$, *and the value of* $(\frac{1+z_1+z_2}{3})^c$ *at* $(z_1, z_2) = (1,1)$ *is* 1. *This functional equation admits no other parameters when we specify the transformations of variables of the both sides of* F_D *in (3) and an admissible factor to* $\Delta(z) = a\prod_{i=1}^k p_i(z)^{c_i}$, *where* $p_1(z), \dots, p_k(z)$ *are the irreducible factors of the product* $\prod_{1\le i<j\le m+2} \xi^* \ell_{ij}$ *of the pull-back of* ℓ_{ij} *in (2) for* $m = 2$ *under the map* ξ.

Theorem 2 *The hypergeometric function* F_D *of* 2 *variables satisfies the transformation formula*

$$\begin{aligned} & \left(\frac{1+\sqrt{z_1 z_2}}{2}\right)^{2c-1} F_D(\frac{c}{2}, \frac{2c-1}{4}, \frac{2c-1}{4}, c; 1-z_1^2, 1-z_2^2) \\ = \quad & F_D(\frac{c}{2}, \frac{2c-1}{4}, \frac{2c-1}{4}, \frac{c+1}{2}; z_1', z_2'), \end{aligned} \tag{4}$$

$$\begin{aligned} & \xi : (z_1, z_2) \mapsto (z_1', z_2'), \\ & z_1' = \frac{(\sqrt{(1-z_1^2)(1-z_2^2)} - \sqrt{-1}(z_1 - z_2))^2}{(1+\sqrt{z_1 z_2})^4}, \\ & z_2' = \frac{(\sqrt{(1-z_1^2)(1-z_2^2)} + \sqrt{-1}(z_1 - z_2))^2}{(1+\sqrt{z_1 z_2})^4}, \end{aligned}$$

where $z=(z_1,z_2)$ is in a small neighborhood U of $(1,1)$, the values of $\sqrt{z_1z_2}$ and $(\frac{1+\sqrt{z_1z_2}}{2})^{2c-1}$ at $(z_1,z_2)=(1,1)$ are 1, and the value of $\sqrt{(1-z_1^2)(1-z_2^2)}$ in the expression of z_2' is same as that of z_1'. Though z_1' and z_2' are exchanged by the choice of branches of $\sqrt{(1-z_1^2)(1-z_2^2)}$, the right hand side of (4) is single-valued by the coincidence of the parameters β_1 and β_2 of F_D. This functional equation admits no other parameters when we specify the transformations of variables of the both sides of F_D in (3) and an admissible factor to $\Delta(z)=a\prod_{i=1}^k p_i(z)^{c_i}$, where $p_1(z),\dots,p_k(z)$ are the irreducible factors of the product $\prod_{1\le i<j\le m+2}\xi^*\ell_{ij}$ of the pull-back of ℓ_{ij} in (2) for $m=2$ under the map ξ.

Theorem 3 *The hypergeometric function F_D of 3 variables satisfies the transformation formula*

$$\begin{aligned}&\left(\frac{1+z_1+z_2+z_3}{4}\right)^{\frac{c}{2}}F_D(\frac{c}{4},\frac{c+2}{12},\frac{c+2}{12},\frac{c+2}{12},\frac{c+2}{3};1-z_1^2,1-z_2^2,1-z_3^2)\\ =\ & F_D(\frac{c}{4},\frac{c+2}{12},\frac{c+2}{12},\frac{c+2}{12},\frac{c+5}{6};z_1',z_2',z_3'),\end{aligned}\tag{5}$$

$$\begin{aligned}\xi:(z_1,z_2,z_3)&\mapsto(z_1',z_2',z_3'),\\ z_1'&=(\frac{1-z_1-z_2+z_3}{1+z_1+z_2+z_3})^2,\\ z_2'&=(\frac{1-z_1+z_2-z_3}{1+z_1+z_2+z_3})^2,\\ z_3'&=(\frac{1+z_1-z_2-z_3}{1+z_1+z_2+z_3})^2,\end{aligned}$$

where $z=(z_1,z_2,z_3)$ is in a small neighborhood U of $(1,1,1)$, and the value of $(\frac{1+z_1+z_2+z_3}{4})^{c/2}$ at $(z_1,z_2,z_3)=(1,1,1)$ is 1. This functional equation admits no other parameters when we specify the transformations of variables of the both sides of F_D in (3) and an admissible factor to $\Delta(z)=a\prod_{i=1}^k p_i(z)^{c_i}$, where $p_1(z),\dots,p_k(z)$ are the irreducible factors of the product $\prod_{1\le i<j\le m+2}\xi^*\ell_{ij}$ of the pull-back of ℓ_{ij} in (2) for $m=3$ under the map ξ.

Remark 1 The transformation formulas (3) and (4) for $c=1$ are utilized for the study of arithmetic-geometric means of three terms in [KS1] and [KS2], respectively. The transformation formula (5) for $c=1$ appears in [KM] as Proposition 1, which is a key to express the common limit of a quadruple sequence by Lauricella's hypergeometric function F_D of 3 variables.

4 Proof

In this section, we prove Theorem 1. Since we can show the others similarly, we omit their proofs.

Let $\Omega_{\hat{f}}(z)$ and $\Omega_{\hat{g}}(z)$ be the connection 1-forms in Fact 1 for

$$\begin{aligned} \hat{f}(z) &= {}^t(f_0(z_1,z_2), z_1\frac{\partial f_0}{\partial z_1}(z_1,z_2), z_2\frac{\partial f_0}{\partial z_2}(z_1,z_2)), \\ f_0(z_1,z_2) &= F_D(\alpha,\beta_1,\beta_2,\gamma;z_1,z_2), \end{aligned}$$

and

$$\begin{aligned} \hat{g}(z) &= {}^t(g_0(z_1,z_2), z_1\frac{\partial g_0}{\partial z_1}(z_1,z_2), z_2\frac{\partial g_0}{\partial z_2}(z_1,z_2)), \\ g_0(z_1,z_2) &= F_D(\alpha',\beta_1',\beta_2',\gamma';z_1,z_2), \end{aligned}$$

respectively. It is easy to see that the vector-valued functions

$$f(z) = {}^t(f_0, \frac{\partial f_0}{\partial z_1}, \frac{\partial f_0}{\partial z_2}) \text{ and } g(z) = {}^t(g_0, \frac{\partial g_0}{\partial z_1}, \frac{\partial g_0}{\partial z_2})$$

satisfy the Pfaffian systems

$$df = \Omega_f(z)f, \quad dg = \Omega_g(z)g,$$

respectively, where

$$\Omega_f(z) = P\Omega_{\hat{f}}(z)P^{-1} + dPP^{-1}, \quad \Omega_g(z) = P\Omega_{\hat{g}}(z)P^{-1} + dPP^{-1},$$

$$P = \mathrm{diag}(1, \frac{1}{z_1}, \frac{1}{z_2}) = \begin{pmatrix} 1 & & \\ & \frac{1}{z_1} & \\ & & \frac{1}{z_2} \end{pmatrix}.$$

Consider the vector-valued function

$$G(x) = {}^t(G_0, \frac{\partial G_0}{\partial x_1}, \frac{\partial G_0}{\partial x_2})$$

for the pull-back $G_0(x_1,x_2)$ of $g_0(z_1,z_2)$ under the map

$$\xi : (x_1,x_2) \mapsto (z_1,z_2) = \left((\frac{1+\omega x_1+\omega^2 x_2}{1+x_1+x_2})^3, (\frac{1+\omega^2 x_1+\omega x_2}{1+x_1+x_2})^3 \right).$$

It satisfies

$$G(1,1) = {}^t(1,0,0) \tag{6}$$

and the Pfaffian system $dG = \Omega_G(x)G$, where

$$\Omega_G(x) = J_2\Omega_g(x)J_2^{-1} + dJ_2J_2^{-1}, \quad J_2 = \begin{pmatrix} 1 & \\ & {}^tJ \end{pmatrix},$$

$\Omega_g(x)$ is the pull-back of $\Omega_g(z)$ under the map ξ, and J is the Jacobi matrix of the map ξ. The singular locus of $\Omega_G(x)$ consists of 12 lines $p_i(x) = 0$ $(i = 1, \dots, 12)$, where

$$\begin{aligned}
&p_1(x) = 1 + x_1 + x_2, \quad p_2(x) = x_1 - \omega, \quad p_3(x) = x_1 - \omega^2, \\
&p_4(x) = x_2 - \omega, \quad p_5(x) = x_2 - \omega^2, \quad p_6(x) = x_1 - \omega x_2, \\
&p_7(x) = x_1 - \omega^2 x_2, \quad p_8(x) = x_1 - x_2, \quad p_9(x) = x_1 - 1, \\
&p_{10}(x) = x_2 - 1, \quad p_{11}(x) = 1 + \omega x_1 + \omega^2 x_2, \quad p_{12}(x) = 1 + \omega^2 x_1 + \omega x_2.
\end{aligned}$$

Put $a_i = p_i(1,1)$; note that $a_i \neq 0$ for $i = 1, \dots, 7$, $a_i = 0$ for $i = 8, \dots, 12$. Since $f_0(1,1) = g_0(1,1) = 1$, we have $\Delta(1,1) = 1$. Thus $\Delta(x)$ should be

$$\Delta(x) = \prod_{i=1}^{7} \left(\frac{p_i(x)}{a_i}\right)^{c_i}.$$

Consider the vector-valued function

$$F(x) = {}^t(F_0, \frac{\partial F_0}{\partial x_1}, \frac{\partial F_0}{\partial x_2})$$

for

$$F_0(x_1, x_2) = \Delta(x) f_0(1 - x_1^3, 1 - x_2^3).$$

It satisfies

$$F(1,1) = {}^t(1, \sum_{i=1}^{7} \frac{c_i}{a_i}\frac{\partial p_i}{\partial x_1} - \frac{3\alpha\beta_1}{\gamma}, \sum_{i=1}^{7} \frac{c_i}{a_i}\frac{\partial p_i}{\partial x_2} - \frac{3\alpha\beta_2}{\gamma}) \tag{7}$$

and the Pfaffian system $dF = \Omega_F(x)F$, where

$$\Omega_F(x) = Q[J_1\Omega_f(x)J_1^{-1} + dJ_1J_1^{-1}]Q^{-1} + dQQ^{-1}, \quad J_1 = \mathrm{diag}(1, -3x_1^2, -3x_2^2),$$

$$Q = \begin{pmatrix} \Delta(x) & & \\ \frac{\partial \Delta(x)}{\partial x_1} & \Delta(x) & \\ \frac{\partial \Delta(x)}{\partial x_2} & & \Delta(x) \end{pmatrix} = \Delta(x) \begin{pmatrix} 1 & & \\ \frac{\partial \log \Delta(x)}{\partial x_1} & 1 & \\ \frac{\partial \log \Delta(x)}{\partial x_1} & & 1 \end{pmatrix},$$

and $\Omega_f(x)$ is the pull-back of $\Omega_f(z)$ under the map

$$(x_1, x_2) \mapsto (z_1, z_2) = (1 - x_1^3, 1 - x_2^3).$$

The singular locus of $\Omega_F(x)$ consists of 12 lines $x_1 = 0$, $x_2 = 0$ and $p_i(x) = 0$ $(i = 1, \dots, 10)$.

Note that $F_0(x) = G_0(x)$ on U if and only if $F(1,1) = G(1,1)$ and $\Omega_F(x) = \Omega_G(x)$. By (6) and (7), we have

$$\frac{c_1}{3} + \frac{c_2 + c_6}{1-\omega} + \frac{c_3 + c_7}{1-\omega^2} - \frac{3\alpha\beta_1}{\gamma} = \frac{c_1}{3} + \frac{c_4 - c_6\omega}{1-\omega} + \frac{c_5 - c_7\omega^2}{1-\omega^2} - \frac{3\alpha\beta_2}{\gamma} = 0. \quad (8)$$

We compare the entries of $\Omega_F(x)$ with those of $\Omega_G(x)$ by utilizing yang [O1], which is a package of computer algebra system Risa/Asir for the ring of differential-difference operators. We have a necessary and sufficient condition for the identity $\Omega_F(x) = \Omega_G(x)$ expressed as a system of 802 linear and 399 quadratic equations of 15 variables ${}^t v = (c_1, \dots, c_7, \alpha, \beta_1, \beta_2, \gamma, \alpha', \beta_1', \beta_2', \gamma')$. The 802 linear equations include the followings 14 linear equations:

$$\begin{aligned}
-c_1 - c_4 - c_5 - c_6 - c_7 + 3\beta_2 + \alpha' + \beta_1' + \beta_2' - \gamma' &= 0,\\
-2c_4 - 2c_5 + 3\alpha + 3\beta_2 - 3\gamma - 2\alpha' + \beta_1' + \beta_2' + 2\gamma' &= 1,\\
-2c_6 - 2c_7 + 3\alpha + 3\beta_1 + 3\beta_2 - 3\gamma - \alpha' - \beta_1' - \beta_2' + \gamma' &= 0,\\
-c_6 - c_7 + 3\beta_1 - \alpha' - \beta_1' - \beta_2' + \gamma' &= 1,\\
c_6 + c_7 + \alpha' + \omega\beta_1' - (\omega+1)\beta_2' - \gamma' &= -1,\\
3\alpha - 3\beta_1 + 3\beta_2 - 3\beta_1' - 3\beta_2' &= -1,\\
\alpha' - 2\beta_1' - 2\beta_2' + 2\gamma' &= 1,\\
3\alpha + 3\beta_1 - 3\gamma &= -1,\\
3\alpha + 3\beta_2 - 3\gamma &= -1,\\
(\omega+1)c_2 - \omega c_3 &= 0,\\
c_2 + c_3 &= 0,\\
(\omega+1)c_4 - \omega c_5 &= 0,\\
c_4 + c_5 &= 0,\\
\omega c_6 - (\omega+1)c_7 &= 0.
\end{aligned}$$

Thus we have a 1-dimensional solution space of these 14 linear equations, which can be expressed as

$$\begin{aligned} &\alpha=\alpha'=\frac{c_1}{3}, \quad \beta_1=\beta_2=\beta_1'=\beta_2'=\frac{c_1+1}{6}, \\ &\gamma=\frac{c_1+1}{2}, \quad \gamma'=\frac{c_1+5}{6}, \quad c_2=c_3=c_4=c_5=c_6=c_7=0, \end{aligned} \tag{9}$$

where we regard c_1 as a free parameter in $\mathbb{C}$.

We can see that the solution (9) satisfies the 802 linear and 399 quadratic equations and (8) by Risa/Asir. Hence $F_0(x)=G_0(x)$ on U if and only if the condition (9) holds. Refer to [O2] for our computation by yang and Risa/Asir. □

5 Restrictions

In this section, we derive some corollaries by considering restrictions of variables for our transformation formulas.

Corollary 1 (Theorem 2.3 in [BBG]) *We have*

$$\begin{aligned} &\left(\frac{1+2z}{3}\right)^{c} F(\frac{c}{3},\frac{c+1}{3},\frac{c+1}{2};1-z^3) \\ =\ &F(\frac{c}{3},\frac{c+1}{3},\frac{c+5}{6};(\frac{1-z}{1+2z})^3) \end{aligned}$$

for z sufficiently near to 1, where the value of $(\frac{1+2z}{3})^c$ at $z=1$ is 1.

Proof. Put $z=z_1=z_2$ for the transformation formula (3) and use the integral representation (1). □

Corollary 2 *We have*

$$\begin{aligned} &\left(\frac{1+z}{2}\right)^{2c-1} F(\frac{c}{2},\frac{2c-1}{4},c;1-z^4) \\ =\ &F(\frac{c}{2},\frac{2c-1}{2},\frac{c+1}{2};-(\frac{1-z}{1+z})^2). \end{aligned}$$

for z sufficiently near to 1, where the value of $(\frac{1+z}{2})^{2c-1}$ at $z=1$ is 1.

Proof. Put $z=\sqrt{z_1}$, $z_2=1$ for the transformation formula (4) and use the integral representation (1). □

Corollary 3 *We have*

$$\begin{aligned} &\left(\frac{1+3z}{4}\right)^{\frac{c}{2}} F\left(\frac{c}{4}, \frac{c+2}{4}, \frac{c+2}{3}; 1-z^2\right) \\ = \ & F\left(\frac{c}{4}, \frac{c+2}{4}, \frac{c+5}{6}; \left(\frac{1-z}{1+3z}\right)^2\right). \end{aligned}$$

for z sufficiently near to 1, where the value of $(\frac{1+3z}{4})^{c/2}$ at $z=1$ is 1.

Proof. Put $z = z_1 = z_2 = z_3$ for the transformation formula (3) and use the integral representation (1). $\square$

Remark 2 The equalities in Corollaries 1 and 3 for $c = 1$ are used in [BB] to study modified arithmetic-geometric means.

Remark 3 It is written in [BBG] that Corollary 1 can not be deduced from the cubic transformation formulas in [G]. The authors remark that our corollaries are not listed in [E] and [G], and think that Corollaries 2 and 3 can not be obtained by classical results.

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