The common limit of a quadruple sequence and the hypergeometric function F_D of three variables

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Abstract

We study a quadruple sequence and express its common limit by Lauricella's hypergeometric function $F_D(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},1;z_1,z_2,z_3)$ of three variables. We give a functional equation of F_D, which is the key to get our expression of the common limit.

1 Introduction

For two positive real numbers a_0 and b_0 with $a_0 \geq b_0$, the double sequence $\{a_n\}$ and $\{b_n\}$ given as

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n},$$

has a common limit, which is called the arithmetic-geometric mean $M(a_0, b_0)$ of a_0 and b_0. It is shown by C.F. Gauss that $M(a_0, b_0)$ can be expressed by the hypergeometric function:

$$\frac{a_0}{M(a_0,b_0)} = \frac{1}{M(1,x)} = F(\frac{1}{2},\frac{1}{2},1;1-x^2),$$

where $x = b_0/a_0$.

[1]**MSC2000:** primary 26A18; secondary 33C65.
[2]**Keywords:** arithmetic-geometric mean, hypergeometric function.

In this paper, we study a quadruple sequence $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ given by

$$(a_0, b_0, c_0, d_0) = (a, b, c, d), \quad a \geq b \geq c \geq d \geq 0,$$

$$(1) \quad \begin{aligned} a_{n+1} &= \frac{a_n + b_n + c_n + d_n}{4}, & b_{n+1} &= \frac{\sqrt{(a_n + d_n)(b_n + c_n)}}{2}, \\ c_{n+1} &= \frac{\sqrt{(a_n + c_n)(b_n + d_n)}}{2}, & d_{n+1} &= \frac{\sqrt{(a_n + b_n)(c_n + d_n)}}{2}. \end{aligned}$$

We can easily see that it has a common limit $\mu(a, b, c, d)$. Our main theorem is the expression of $\mu(a, b, c, d)$ by Lauricella's hypergeometric function F_D of three variables:

$$\frac{1}{\mu(1, x_1, x_2, x_3)} = F_D(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 1; 1 - x_1^2, 1 - x_2^2, 1 - x_3^2)^2.$$

The key for our main theorem is Proposition 1, which is the functional equation of the hypergeometric function F_D corresponding to the property

$$\frac{a_0}{a_1}\mu(1, \frac{b_0}{a_0}, \frac{c_0}{a_0}, \frac{d_0}{a_0}) = \mu(1, \frac{b_1}{a_1}, \frac{c_1}{a_1}, \frac{d_1}{a_1}).$$

It turns out that

$$\mu(1, \frac{b_n}{a_n}, \frac{c_n}{a_n}, \frac{d_n}{a_n})F_D(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 1; 1 - (\frac{b_n}{a_n})^2, 1 - (\frac{c_n}{a_n})^2, 1 - (\frac{d_n}{a_n})^2)^2$$

is independent of n. This fact implies our main theorem. In order to show Proposition 1, we prepare some essential facts for integrable Pfaffian systems in §3 and give the integrable Pfaffian system with respect to $F_D(\alpha, \beta, \gamma; z)$ of three variables in Fact 4.

C.W. Borchardt considers in [B] the quadruple sequence

$$\begin{aligned} a_{n+1} &= \frac{a_n + b_n + c_n + d_n}{4}, & b_{n+1} &= \frac{\sqrt{a_n b_n} + \sqrt{c_n d_n}}{2}, \\ c_{n+1} &= \frac{\sqrt{a_n c_n} + \sqrt{b_n d_n}}{2}, & d_{n+1} &= \frac{\sqrt{a_n d_n} + \sqrt{b_n c_n}}{2}, \end{aligned}$$

with positive initial terms a_0, b_0, c_0, d_0. Its common limit $B(a_0, b_0, c_0, d_0)$ is expressed in terms of period integrals of a hyperelliptic curve C of genus 2. J.F. Mestre studies the expression of fixed points under the hyperelliptic involution on C by the initial terms and shows that $B(a_0, b_0, c_0, d_0)$ can be expressed by the arithmetic-geometric mean $M(a_0, c_0)$ when $a_0 = b_0$ and $c_0 = d_0$, refer to [M].

J.M. Borwein and P.B. Borwein consider in [BB] two double sequences

$$a_{n+1} = \frac{a_n + 2b_n}{3}, \quad b_{n+1} = \sqrt[3]{b_n \frac{a_n^2 + a_n b_n + b_n^2}{3}},$$

and

$$a_{n+1} = \frac{a_n + 3b_n}{4}, \quad b_{n+1} = \sqrt{b_n \frac{a_n + b_n}{2}};$$

they express their common limits $M_3(a_0, b_0)$ and $M_4(a_0, b_0)$ by $F(\frac{1}{3}, \frac{2}{3}, 1; z)$ and $F(\frac{1}{4}, \frac{3}{4}, 1; z)$, respectively. We remark that the expression of $M_4(a_0, b_0)$ can be obtained by our main theorem as a special case $b_0 = c_0 = d_0$.

As a generalization of $M_3(a_0, b_0)$, K. Koike and H. Shiga give a triple sequence and express its common limit by Appell's hypergeometric function $F_1(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1; z_1, z_2)$ of two variables z_1, z_2, refer to [KS1]. They study an extension of the arithmetic-geometric mean and give its expression by Appell's hypergeometric function F_1 with different parameters in [KS2].

For other studies related to the arithmetic-geometric mean, refer to [MM].

Acknowledgment. The authors express their gratitude to Professor K. Ohara for informing of the correct Pfaffian system with respect to the hypergeometric function F_D obtained by the system [O].

2 The quadruple sequence

Lemma 1 *The quadruple sequence $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ given as (1) satisfies*

$$a \geq a_{n-1} \geq a_n \geq b_n \geq c_n \geq d_n \geq d_{n-1} \geq d$$

for any $n \in \mathbb{N}$. It has a common limit, which is denoted by $\mu(a, b, c, d)$.

Proof. We assume $a_n \geq b_n \geq c_n \geq d_n \geq 0$ for $n \in \mathbb{N}$. Then we have

$$\begin{aligned}
(4a_{n+1})^2 - (4b_{n+1})^2 &= (a_n - b_n - c_n + d_n)^2 \geq 0, \\
(2b_{n+1})^2 - (2c_{n+1})^2 &= (a_n - b_n)(c_n - d_n) \geq 0, \\
(2c_{n+1})^2 - (2d_{n+1})^2 &= (a_n - d_n)(b_n - c_n) \geq 0,
\end{aligned}$$

$$\begin{aligned} a_{n+1} = \frac{a_n+b_n+c_n+d_n}{4} &\leq \frac{a_n+a_n+a_n+a_n}{4} = a_n, \\ d_{n+1} = \frac{\sqrt{(a_n+b_n)(c_n+d_n)}}{2} &\geq \frac{\sqrt{(d_n+d_n)(d_n+d_n)}}{2} = d_n, \end{aligned}$$

which imply $a_n \geq a_{n+1} \geq b_{n+1} \geq c_{n+1} \geq d_{n+1} \geq d_n \geq 0$. Since the sequences $\{a_n\}$ and $\{d_n\}$ are monotonous and bounded, they converge. Since

$$\begin{aligned} & a_{n+1} - d_{n+1} \\ = \ & \frac{1}{4}(\sqrt{a_n+b_n} - \sqrt{c_n+d_n})^2 \leq \frac{1}{4}(\sqrt{2a_n} - \sqrt{2d_n})^2 \\ = \ & \frac{1}{2}(a_n + d_n - 2\sqrt{a_n d_n}) \leq \frac{a_n - d_n}{2} \leq \frac{a-d}{2^{n+1}}, \end{aligned}$$

we have $\lim_{n\to\infty}(a_n - d_n) = 0$. Thus the quadruple sequence (1) has a common limit. □

Remark 1 *1. We have*

$$\begin{aligned} (4a_{n+1})^2 - (4b_{n+1})^2 &= (a_n - b_n - c_n + d_n)^2, \\ (4a_{n+1})^2 - (4c_{n+1})^2 &= (a_n - b_n + c_n - d_n)^2, \\ (4a_{n+1})^2 - (4d_{n+1})^2 &= (a_n + b_n - c_n - d_n)^2. \end{aligned}$$

2. The quadruple sequence (1) quadratically converges, since

$$a_{n+1} - d_{n+1} \leq \frac{1}{4}(\sqrt{2a_n} - \sqrt{2d_n})^2 = \frac{1}{2}\frac{(a_n - d_n)^2}{(\sqrt{a_n} + \sqrt{d_n})^2}.$$

It is easy to see that

$$\mu(a,b,c,d) = a\mu(1, \frac{b}{a}, \frac{c}{a}, \frac{d}{a}),$$

$$\begin{aligned} & \mu(a,b,c,d) \\ = \ & \mu\left(\frac{a+b+c+d}{4}, \frac{\sqrt{(a+d)(b+c)}}{2}, \frac{\sqrt{(a+c)(b+d)}}{2}, \frac{\sqrt{(a+b)(c+d)}}{2}\right). \end{aligned}$$

By putting $x_1 = b/a$, $x_2 = c/a$, $x_3 = d/a$ for these equalities, we have the following lemma.

Lemma 2 *Let (y_1, y_2, y_3) be the image of (x_1, x_2, x_3) by the map φ*

$$\varphi(x_1, x_2, x_3) = \left(\frac{2\sqrt{(1+x_3)(x_1+x_2)}}{1+x_1+x_2+x_3}, \frac{2\sqrt{(1+x_2)(x_1+x_3)}}{1+x_1+x_2+x_3}, \frac{2\sqrt{(1+x_1)(x_2+x_3)}}{1+x_1+x_2+x_3}\right).$$

Then μ satisfies the relation

$$\frac{4}{1+x_1+x_2+x_3}\mu(1, x_1, x_2, x_3) = \mu(1, y_1, y_2, y_3) \tag{2}$$

for $0 < x_3 \leq x_2 \leq x_1 \leq 1$.

3 Integrable Pfaffian systems

In this section, we prepare some facts of integrable Pfaffian systems. We consider a system of first-order partial differential equations with r unknowns $f_1, \ldots, f_r$ and n variables $x_1, \ldots, x_n$ in the following form

$$df(x) = \Omega(x)f(x), \tag{3}$$

where $x = (x_1, \ldots, x_n)$ is in an open set U, $f(x) = {}^t(f_1(x), \ldots, f_r(x))$ and $\Omega(x)$ is an $r \times r$ matrix whose entries are 1-forms on U. The system (3) is called a Pfaffian system on U and $\Omega(x)$ is called the connection matrix of (3). If $\Omega(x)$ satisfies the integrability condition

$$d\Omega(x) = \Omega(x) \wedge \Omega(x),$$

then the system (3) is integrable.

Fact 1
1. *The system (3) has exactly r linearly independent vector valued solutions if and only if it is integrable.*
2. *If the system (3) is integrable, then there exists a unique solution f around $u \in U$ such that $f(u) = p$ for a given initial vector $p \in \mathbb{C}^r$.*

Fact 2 *For an integrable Pfaffian system (3) and an invertible $r \times r$ functional matrix $P(x)$ on U, the vector valued function $g(x) = P(x)f(x)$ satisfies the Pfaffian system*

$$dg(x) = [P(x)\Omega(x)P(x)^{-1} + dP(x)P(x)^{-1}]g(x).$$

Let f_0 be a function of n-variables $(y_1, \dots, y_n)$ on an open set V. We assume that the vector valued function

$$f(y) = {}^t(f_0(y), \frac{\partial f_0}{\partial y_1}(y), \dots, \frac{\partial f_0}{\partial y_n}(y))$$

satisfies an integrable Pfaffian system

$$df(y) = \Omega(y) f(y)$$

on V. Let η be a map from an open set U to V given as

$$\eta : U \ni x = (x_1, \dots, x_n) \mapsto y = (\eta_1(x), \dots, \eta_n(x)) \in V,$$

and let J be the Jacobi matrix of η:

$$J = \left(\frac{\partial \eta_i}{\partial x_j}\right)_{ij} = \begin{pmatrix} \frac{\partial \eta_1}{\partial x_1} & \cdots & \frac{\partial \eta_1}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \frac{\partial \eta_n}{\partial x_1} & \cdots & \frac{\partial \eta_n}{\partial x_n} \end{pmatrix}.$$

Fact 3 *If $\det(J) \neq 0$ on U then the function $h_0(x) = f_0(\eta(x))$ satisfies*

$$dh(x) = [\tilde{J}\Omega(x)\tilde{J}^{-1} + d\tilde{J}\tilde{J}^{-1}]h(x),$$

where

$$h(x) = {}^t(h_0(x), \frac{\partial h_0}{\partial x_1}(x), \dots, \frac{\partial h_0}{\partial x_n}(x)), \quad \tilde{J} = \begin{pmatrix} 1 & \\ & {}^tJ \end{pmatrix},$$

and $\Omega(x)$ is the pull-back of $\Omega(y)$ under the map η.

4 Lauricella's hypergeometric function F_D

Lauricella's hypergeometric function F_D of m-variables $z_1, \dots, z_m$ with parameters $\alpha, \beta_1, \dots, \beta_m, \gamma$ is defined as

$$F_D(\alpha, \beta, \gamma; z) = \sum_{n_1, \dots, n_m \geq 0}^{\infty} \frac{(\alpha, \sum_{j=1}^m n_j) \prod_{j=1}^m (\beta_j, n_j)}{(\gamma, \sum_{j=1}^m n_j) \prod_{j=1}^m (1, n_j)} \prod_{j=1}^m z_j^{n_j},$$

where $z = (z_1, \dots, z_m)$ satisfies $|z_j| < 1$ $(j = 1, \dots, m)$, $\beta = (\beta_1, \dots, \beta_m)$, $\gamma \neq 0, -1, -2, \dots$ and $(\alpha, n) = \alpha(\alpha+1)\cdots(\alpha+n-1) = \Gamma(\alpha+n)/\Gamma(\alpha)$. This function admits the integral representation:

$$F_D(\alpha,\beta,\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 t^{\alpha}(1-t)^{\gamma-\alpha} \prod_{j=1}^{m} (1-z_j t)^{-\beta_j} \frac{dt}{t(1-t)}.$$

When $m = 1$, $F_D(\alpha,\beta,\gamma;z)$ coincides with the Gauss hypergeometric function $F(\alpha,\beta,\gamma;z)$. We consider $F_D(\alpha,\beta_1,\beta_2,\beta_3,\gamma;z_1,z_2,z_3)$ of three variables.

Fact 4 *The function $F_D(\alpha,\beta,\gamma;z)$ of three variables satisfies the integrable Pfaffian system given as*

$$df = \sum_{1\le i<j\le 5} A_{ij}\, d\log(z_i - z_j) f,$$

where $f = {}^t(f_0, f_1, f_2, f_3)$, $f_0 = F_D(\alpha,\beta,\gamma;z)$, $f_i = z_i \frac{\partial f_0}{\partial z_i}$ $(i = 1,2,3)$, $z_4 = 0$, $z_5 = 1$ and

$$A_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\beta_2 & \beta_1 & 0 \\ 0 & \beta_2 & -\beta_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_{14} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1+\beta_2+\beta_3-\gamma & 0 & 0 \\ 0 & -\beta_2 & 0 & 0 \\ 0 & -\beta_3 & 0 & 0 \end{pmatrix},$$

$$A_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\beta_3 & 0 & \beta_1 \\ 0 & 0 & 0 & 0 \\ 0 & \beta_3 & 0 & -\beta_1 \end{pmatrix}, \quad A_{24} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -\beta_1 & 0 \\ 0 & 0 & 1+\beta_1+\beta_3-\gamma & 0 \\ 0 & 0 & -\beta_3 & 0 \end{pmatrix},$$

$$A_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta_3 & \beta_2 \\ 0 & 0 & \beta_3 & -\beta_2 \end{pmatrix}, \quad A_{34} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\beta_1 \\ 0 & 0 & 0 & -\beta_2 \\ 0 & 0 & 0 & 1+\beta_1+\beta_2-\gamma \end{pmatrix},$$

$$A_{15} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\alpha\beta_1 & \gamma-\alpha-\beta_1-1 & -\beta_1 & -\beta_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_{25} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha\beta_2 & -\beta_2 & \gamma-\alpha-\beta_2-1 & -\beta_2 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_{35} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha\beta_3 & -\beta_3 & -\beta_3 & \gamma-\alpha-\beta_3-1 \end{pmatrix}.$$

Remark 2 *The A_{ij} and $A_{i,n+1}$ in the proof of Proposition 9.1.4 in [IKSY] are wrong. Professor K. Ohara informed us of the correct Pfaffian system given by the system [O].*

5 Main theorem

Theorem 1 *For any numbers x_1, x_2, x_3 satisfying $0 < x_3 \le x_2 \le x_1 \le 1$, we have*

$$\frac{1}{\mu(1, x_1, x_2, x_3)} = F_D(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 1; 1-x_1^2, 1-x_2^2, 1-x_3^2)^2,$$

where $\mu(1, x_1, x_2, x_3)$ is the common limit of the quadruple sequence (1) with initial $(1, x_1, x_2, x_3)$ and F_D is Lauricella's hypergeometric function.

We put

$$F(z_1, z_2, z_3) = F_D(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 1; z_1, z_2, z_3).$$

Proposition 1 *The function F satisfies*

$$\frac{1+x_1+x_2+x_3}{4} F(1-x_1^2, 1-x_2^2, 1-x_3^2)^2 = F(1-y_1^2, 1-y_2^2, 1-y_3^2)^2$$
$$= F\left(\left(\frac{1-x_1-x_2+x_3}{1+x_1+x_2+x_3}\right)^2, \left(\frac{1-x_1+x_2-x_3}{1+x_1+x_2+x_3}\right)^2, \left(\frac{1+x_1-x_2-x_3}{1+x_1+x_2+x_3}\right)^2\right)^2,$$

where $(y_1, y_2, y_3) = \varphi(x_1, x_2, x_3)$ is defined in Lemma 2.

Proof. Put

$$(\xi_1, \xi_2, \xi_3) = \left(\frac{1-x_1-x_2+x_3}{1+x_1+x_2+x_3}, \frac{1-x_1+x_2-x_3}{1+x_1+x_2+x_3}, \frac{1+x_1-x_2-x_3}{1+x_1+x_2+x_3}\right).$$

Then we have

$$
\begin{aligned}
&(x_1,x_2,x_3)=\left(\frac{1-\xi_1-\xi_2+\xi_3}{1+\xi_1+\xi_2+\xi_3},\frac{1-\xi_1+\xi_2-\xi_3}{1+\xi_1+\xi_2+\xi_3},\frac{1+\xi_1-\xi_2-\xi_3}{1+\xi_1+\xi_2+\xi_3}\right),\\
&\frac{1+x_1+x_2+x_3}{4}=\frac{1}{1+\xi_1+\xi_2+\xi_3},\\
&(1-x_1^2,1-x_2^2,1-x_3^2)\\
=\ &\left(\frac{4(1+\xi_3)(\xi_1+\xi_2)}{(1+\xi_1+\xi_2+\xi_3)^2},\frac{4(1+\xi_2)(\xi_1+\xi_3)}{(1+\xi_1+\xi_2+\xi_3)^2},\frac{4(1+\xi_1)(\xi_2+\xi_3)}{(1+\xi_1+\xi_2+\xi_3)^2}\right).
\end{aligned}
$$

Thus the equality in Proposition 1 is equivalent to

$$
\begin{aligned}
&\sqrt{1+\xi_1+\xi_2+\xi_3}F(\xi_1^2,\xi_2^2,\xi_3^2)\\
=\ &F\left(\frac{4(1+\xi_3)(\xi_1+\xi_2)}{(1+\xi_1+\xi_2+\xi_3)^2},\frac{4(1+\xi_2)(\xi_1+\xi_3)}{(1+\xi_1+\xi_2+\xi_3)^2},\frac{4(1+\xi_1)(\xi_2+\xi_3)}{(1+\xi_1+\xi_2+\xi_3)^2}\right)
\end{aligned}
$$

for $0\le\xi_1\le\xi_2\le\xi_3<1$. We show that the Pfaffian systems obtained by the functions in the both sides of the above equality coincide.

Let $\Omega(x)$ be the connection 1-form in Fact 4 for $\alpha=\beta_1=\beta_2=\beta_3=1/4$ and $\gamma=1$. Fact 2 implies that the vector valued function

$$g(x)={}^t(F,\frac{\partial F}{\partial x_1},\frac{\partial F}{\partial x_2},\frac{\partial F}{\partial x_3})$$

satisfies the Pfaffian system $dg=\Omega_1(x)g$, where

$$\Omega_1(x)=P\Omega(x)P^{-1}+dPP^{-1},\ P=\mathrm{diag}(1,\frac{1}{x_1},\frac{1}{x_2},\frac{1}{x_3})=\begin{pmatrix}1&&&\\&\frac{1}{x_1}&&\\&&\frac{1}{x_2}&\\&&&\frac{1}{x_3}\end{pmatrix}.$$

The vector valued function

$$h(\xi)={}^t(h_0,\frac{\partial h_0}{\partial\xi_1},\frac{\partial h_0}{\partial\xi_2},\frac{\partial h_0}{\partial\xi_3})$$

for

$$h_0(\xi_1,\xi_2,\xi_3)=\sqrt{1+\xi_1+\xi_2+\xi_3}F(\xi_1^2,\xi_2^2,\xi_3^2)$$

satisfies $h(0,0,0)={}^t(1,1/2,1/2,1/2)$ and the Pfaffian system $dh=\Omega_2(\xi)h$, where

$$\Omega_2(\xi)=Q[J_1\Omega_1(\xi)J_1^{-1}+dJ_1J_1^{-1}]Q^{-1}+dQQ^{-1},\quad J_1=\mathrm{diag}(1,2\xi_1,2\xi_2,2\xi_3),$$

$$Q = \begin{pmatrix} \zeta & & & \\ \frac{1}{2\zeta} & \zeta & & \\ \frac{1}{2\zeta} & & \zeta & \\ \frac{1}{2\zeta} & & & \zeta \end{pmatrix}, \quad \zeta = \sqrt{1+\xi_1+\xi_2+\xi_3},$$

and $\Omega_1(\xi)$ is the pull-back of $\Omega_1(x)$ under the map

$$(\xi_1, \xi_2, \xi_3) \mapsto (x_1, x_2, x_3) = (\xi_1^2, \xi_2^2, \xi_3^2).$$

On the other hand, the vector valued function

$$h(x) = {}^t(h_0, \frac{\partial h_0}{\partial \xi_1}, \frac{\partial h_0}{\partial \xi_2}, \frac{\partial h_0}{\partial \xi_3})$$

for

$$h_0(\xi_1, \xi_2, \xi_3) = F\left(\frac{4(1+\xi_3)(\xi_1+\xi_2)}{(1+\xi_1+\xi_2+\xi_3)^2}, \frac{4(1+\xi_2)(\xi_1+\xi_3)}{(1+\xi_1+\xi_2+\xi_3)^2}, \frac{4(1+\xi_1)(\xi_2+\xi_3)}{(1+\xi_1+\xi_2+\xi_3)^2}\right)$$

satisfies $h(0,0,0) = {}^t(1, 1/2, 1/2, 1/2)$ and the Pfaffian system $dh = \Omega_3(\xi)h$, where

$$\Omega_3(\xi) = J_2 \Omega_1'(\xi) J_2^{-1} + dJ_2 J_2^{-1}, \quad J_2 = \begin{pmatrix} 1 & \\ & {}^tJ \end{pmatrix},$$

$\Omega_1'(\xi)$ is the pull-back of $\Omega_1(x)$ under the map

$$\varphi' : (\xi_1, \xi_2, \xi_3) \mapsto \left(\frac{4(1+\xi_3)(\xi_1+\xi_2)}{(1+\xi_1+\xi_2+\xi_3)^2}, \frac{4(1+\xi_2)(\xi_1+\xi_3)}{(1+\xi_1+\xi_2+\xi_3)^2}, \frac{4(1+\xi_1)(\xi_2+\xi_3)}{(1+\xi_1+\xi_2+\xi_3)^2}\right),$$

and J is the Jacobi matrix of the map φ'. By a straight forward calculation, we can show that $\Omega_2(\xi) = \Omega_3(\xi)$. Thus we have the required equality around $\xi = (0,0,0)$. □

Proof of Theorem 1.
Consider the quadruple sequence (1) with initial $(a_0, b_0, c_0, d_0) = (1, x_1, x_2, x_3)$. Lemma 2 and Proposition 1 imply that

$$\mu(1, x_1, x_2, x_3)F(1-x_1^2, 1-x_2^2, 1-x_3^2)^2 = \mu(1, y_1, y_2, y_3)F(1-y_1^2, 1-y_2^2, 1-y_3^2)^2.$$

Thus we have

$$\begin{aligned} & \mu(1, x_1, x_2, x_3)F(1-x_1^2, 1-x_2^2, 1-x_3^2)^2 \\ = \ & \mu(1, \frac{b_n}{a_n}, \frac{c_n}{a_n}, \frac{d_n}{a_n})F(1-(\frac{b_n}{a_n})^2, 1-(\frac{c_n}{a_n})^2, 1-(\frac{d_n}{a_n})^2)^2 \end{aligned}$$

for any $n \in \mathbb{N}$. Since

$$\lim_{n\to\infty}\frac{b_n}{a_n}=\lim_{n\to\infty}\frac{c_n}{a_n}=\lim_{n\to\infty}\frac{d_n}{a_n}=1,$$

and $\mu(1,1,1,1)=F(0,0,0)=1$, we have

$$\begin{aligned}
&\mu(1,x_1,x_2,x_3)F(1-x_1^2,1-x_2^2,1-x_3^2)^2\\
=&\lim_{n\to\infty}\mu(1,\frac{b_n}{a_n},\frac{c_n}{a_n},\frac{d_n}{a_n})F(1-(\frac{b_n}{a_n})^2,1-(\frac{c_n}{a_n})^2,1-(\frac{d_n}{a_n})^2)^2\\
=&\mu(1,1,1,1)F(0,0,0)^2=1,
\end{aligned}$$

which is the desired equality. □

Corollary 1 *For $1>x_1\geq x_2\geq x_3\geq 0$, we have*

$$F(1-x_1^2,1-x_2^2,1-x_3^2)=\prod_{n=0}^{\infty}\sqrt{\frac{a_n}{a_{n+1}}},$$

where we set the initial of the quadruple sequence (1) as $(a_0,b_0,c_0,d_0)=(1,x_1,x_2,x_3)$.

Proof. By Proposition 1, we have

$$\begin{aligned}
&F(1-x_1^2,1-x_2^2,1-x_3^2)\\
=&\frac{2}{\sqrt{1+x_1+x_2+x_3}}F(1-y_1^2,1-y_2^2,1-y_3^2)\\
=&\sqrt{\frac{4a_0}{a_0+b_0+c_0+d_0}}F(1-\frac{b_1^2}{a_1^2},1-\frac{c_1^2}{a_1^2},1-\frac{d_1^2}{a_1^2})\\
=&\sqrt{\frac{a_0}{a_1}}\sqrt{\frac{a_1}{a_2}}F(1-\frac{b_2^2}{a_2^2},1-\frac{c_2^2}{a_2^2},1-\frac{d_2^2}{a_2^2})\\
=&\left(\prod_{i=0}^{n-1}\sqrt{\frac{a_i}{a_{i+1}}}\right)F(1-\frac{b_n^2}{a_n^2},1-\frac{c_n^2}{a_n^2},1-\frac{d_n^2}{a_n^2}),
\end{aligned}$$

which implies this corollary. □

6 A specialization

For the case $b = c = d$, the quadruple sequence reduces to

$$a_{n+1} = \frac{a_n + 3b_n}{4}, \quad b_{n+1} = c_{n+1} = d_{n+1} = \sqrt{b_n \frac{a_n + b_n}{2}},$$

which is studied in [BB]. It is shown that the reciprocal of the common limit of the double sequences is $F(\frac{1}{4}, \frac{3}{4}, 1; 1 - x^2)^2$, where $x = b/a$ and $F(\alpha, \beta, \gamma; z)$ is the Gauss hypergeometric function. By our main theorem, we have

$$\begin{aligned} & F_D(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 1; 1 - x^2, 1 - x^2, 1 - x^2)^2 \\ = \quad & \frac{1}{\mu(1, x, x, x)} = \frac{1}{M_4(1, x)} = F(\frac{1}{4}, \frac{3}{4}, 1; 1 - x^2)^2. \end{aligned}$$

Note that the above reduction of F_D to F can be easily obtained by the integral representation of F_D.

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