A Heun differential equation derived from the Gauss hypergeometric differential equation

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Abstract

We study a Heun differential equation derived from the Gauss hypergeometric differential equation. We show that the periods for the family of cubic curves of the Hesse normal form satisfy this differential equation for some parameters. We give a monodromy representation of this differential equation; we find parameters such that the monodromy group is isomorphic to the fundamental group of the complement of the Borromean rings.

Keywords: Heun differential equation, Monodromy representation, Borromean rings,
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1 Introduction

In this paper, we study the differential equation $H(\alpha,\beta)$ for the function $f(x^3)$ under the condition $\gamma = 2/3$, where $f(y)$ is a solution of the Gauss hypergeometric differential equation

$$E(\alpha,\beta,\gamma):\ y(1-y)\frac{d^2}{dy^2}f(y)+\{\gamma-(\alpha+\beta+1)y\}\frac{d}{dy}f(y)-\alpha\beta f(y)=0.$$

This differential equation $H(\alpha,\beta)$ has four regular singular points $x=1,\omega,\omega^2$ and ∞, where ω is the third root of unity; this is a Heun differential equation.

We first show that the periods for the family $\{C(x) \mid x\in\mathbb{C}-\{1,\omega,\omega^2\}\}$ of cubic curves of the Hesse normal form in the projective plane $\mathbb{P}^2$ satisfy the differential equation $H(1/3, 1/3)$. We next give a monodromy representation of $H(\alpha,\beta)$. Finally, we find parameters α,β and a system of fundamental solutions of $H(\alpha,\beta)$ such that the monodromy group of this system coincides with the representation of the fundamental group of the Borromean-rings-complement studied in [M] and [W].

2 The Heun differential equation derived from the Gauss hypergeometric differential equation.

Let f be a solution of the Gauss hypergeometric differential equation $E(\alpha,\beta,\gamma)$ and $\imath$ be the map $\mathbb{C} \ni x \mapsto y = x^3 \in \mathbb{C}$. We study the differential equation for the function $h(x) = f(x^3) = \imath^*(f)$. Since we have

$$\frac{d}{dx}h(x) = 3x^2\frac{d}{dy}f(y), \quad \frac{d^2}{dx^2}h(x) = 6x\frac{d}{dy}f(y) + 9x^4\frac{d^2}{dy^2}f(y),$$

$\frac{d}{dy}f(y)$ and $\frac{d^2}{dy^2}f(y)$ are expressed as

$$\frac{1}{3x^2}\frac{d}{dx}h(x), \quad \frac{1}{9x^4}\frac{d^2}{dx^2}h(x) - \frac{2}{9x^5}\frac{d}{dx}h(x),$$

respectively. Thus $h(x)$ satisfies the differential equation

$$\begin{aligned} &x^3(1-x^3)[\frac{1}{9x^4}\frac{d^2}{dx^2}h(x) - \frac{2}{9x^5}\frac{d}{dx}h(x)] \\ &+\{\gamma - (\alpha+\beta+1)x^3\}[\frac{1}{3x^2}\frac{d}{dx}h(x)] - \alpha\beta h(x) = 0, \end{aligned}$$

which is equivalent to

$$x(1-x^3)\frac{d^2}{dx^2}h(x) + \{(3\gamma-2) - (3\alpha+3\beta+1)x^3\}\frac{d}{dx}h(x) - 9\alpha\beta x^2 h(x) = 0.$$

When $\gamma = 2/3$, this equation reduces to

$$H(\alpha,\beta): \ (1-x^3)\frac{d^2}{dx^2}h(x) - (3\alpha+3\beta+1)x^2\frac{d}{dx}h(x) - 9\alpha\beta x h(x) = 0,$$

which has four regular singular points $x = 1, \omega, \omega^2$ and ∞. Hence, $H(\alpha,\beta)$ is a Heun differential equation.

3 Periods of cubic curves of the Hesse normal form.

It is known that any non-singular cubic curve in the projective plane $\mathbb{P}^2$ can be transformed into the Hesse normal form

$$C(x) = \{[t_0, t_1, t_2] \in \mathbb{P}^2 \mid t_0^3 + t_1^3 + t_2^3 - 3xt_0t_1t_2 = 0\}, \quad x \in \mathbb{C} - \{1, \omega, \omega^2\},$$

by a projective transformation. Since $C(x)$ is a Riemann surface of genus 1, there exists a nowhere vanishing holomorphic 1-from

$$\varphi = \frac{t_0 dt_1 - t_1 dt_0}{t_2^2 - xt_0t_1}$$

for any $x \in \mathbb{C} - \{1, \omega, \omega^2\}$. We take an element c of $H_1(C(0), \mathbb{Z})$ for $x = 0$; we can make the continuation $c(x) \in H_1(C(x), \mathbb{Z})$ of the cycle c along a path in $\mathbb{C} - \{1, \omega, \omega^2\}$ by the local triviality of the family $\{C(x)\}$. The integral $p(x) = \int_{c(x)} \varphi$ is called a period of $C(x)$.

Proposition 1 *The period $p(x) = \int_{c(x)} \varphi$ of $C(x)$ satisfies the differential equation $H(1/3, 1/3)$.*

Proof. Set $(u, v) = (t_1/t_0, t_2/t_0)$ and $q = q(x; u, v) = u^3 + v^3 + 1 - 3xuv$; the curve $C(x)$ is expressed as $q(x; u, v) = 0$. Since $dq = q_u du + q_v dv = 0$, we have

$$dv = -\frac{q_u}{q_v} du = -\frac{u^2 - xv}{v^2 - xu} du.$$

Note that the period $p(x)$ is expressed as

$$p(x) = \int_{c(x)} \frac{du}{v^2 - xu}.$$

By the local triviality of the family $\{C(x)\}$, we have

$$\frac{d}{dx} \int_{c(x)} \psi(x; u, v) du = \int_{c(x)} \{\frac{\partial}{\partial x}\psi + \frac{\partial}{\partial v}\psi \frac{\partial v(x, u)}{\partial x}\} du,$$

where $\psi du = \psi(x; u, v) du$ is a meromorphic 1-form on $C(x)$, and we regard the variable v as the implicit function of x and u by the equality $q(x; u, v) = 0$.

Differentiating the equality $q(x;u,v) = u^3 + v(x,u)^3 - 3xuv(x,u) = 0$ with respect to x, we have

$$3v(x,u)^2\frac{\partial v(x,u)}{\partial x} - 3uv(x,u) - 3xu\frac{\partial v(x,u)}{\partial x} = 0,$$

which is equivalent to

$$\frac{\partial v(x,u)}{\partial x} = \frac{uv(x,u)}{v^2(x,u) - xu}.$$

Thus $\frac{d}{dx}\int_{c(x)}\psi(x;u,v)du$ is given as

$$\int_{c(x)}\{(\frac{\partial}{\partial x} + \frac{uv}{v^2 - xu}\frac{\partial}{\partial v})\psi\}du.$$

Hence we have

$$\begin{aligned}\frac{d}{dx}p(x) &= \int_{c(x)}\frac{-u(v^2+xu)}{(v^2-xu)^3}du,\\ \frac{d^2}{dx^2}p(x) &= \int_{c(x)}\frac{2xu^3(5v^2+xu)}{(v^2-xu)^5}du.\end{aligned}$$

We show that the 1-from $\eta(x;u,v)du$ is exact, where

$$[(1-x^3)\frac{d^2}{dx^2} - 3x^2\frac{d}{dx} - x]p(x) = x\int_{c(x)}\eta(x;u,v)du,$$

$$\eta(x;u,v) = \frac{2xu^4 - (9x^3-10)u^3v^2 - 9x^2u^2v^4 + 7xuv^6 - v^8}{(v^2-xu)^5}.$$

In fact, for a meromorphic function $F = \frac{(u^3-1)uv}{(v^2-xu)^3}$ on $C(x)$, dF is

$$\frac{\partial}{\partial u}Fdu + \frac{\partial}{\partial v}Fdv = \{(\frac{\partial}{\partial u} - \frac{u^2-xv}{v^2-xu}\frac{\partial}{\partial v})F\}du,$$

and $\eta + dF$ is

$$\frac{xu^4 + 5u^3v^2 + 3x^2u^2v + 4xuv^3 - v^5}{(v^2-xu)^5}q(x;u,v)du,$$

which vanishes on $C(x)$. □

4 Monodromy representation.

We use the monodromy representation of the Gauss hypergeometric differential equation given in [K].

Fact 1 (Theorem 6.1 in [K]) *If none of $\alpha, \beta, \gamma-\alpha$ and $\gamma-\beta$ is an integer, then there exists a fundamental system $\mathbf{f}(y) = \begin{pmatrix} f_0(y) \\ f_1(y) \end{pmatrix}$ of $E(\alpha,\beta,\gamma)$ such that the monodromy group with respect to this system is generated by*

$$\begin{pmatrix} 1 & 0 \\ -(1-e^{-2\pi i\beta}) & e^{-2\pi i\gamma} \end{pmatrix}, \quad \begin{pmatrix} 1 & 1-e^{-2\pi i\alpha} \\ 0 & e^{-2\pi i(\alpha+\beta-\gamma)} \end{pmatrix}.$$

These matrices are given by the continuation of $\mathbf{f}(y)$ along a loop encircling the point $x=0$ once in the positive sence and along a loop encircling the point $x=1$ once in the positive sence, respectively.

By putting $\gamma = 2/3$ for the matrices in Fact 1, we set

$$\rho_0 = \begin{pmatrix} 1 & 0 \\ -(1-e^{-2\pi i\beta}) & \omega \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} 1 & 1-e^{-2\pi i\alpha} \\ 0 & \omega^2 e^{-2\pi i(\alpha+\beta)} \end{pmatrix}.$$

Note that the eigenvalues of ρ_0 are 1 and ω and that

$$\rho_0^3 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Proposition 2 *If none of $\alpha, \beta, 2/3-\alpha$ and $2/3-\beta$ is an integer, then there exists a fundamental system of $H(\alpha,\beta)$ such that the monodromy group with respect to this system is generated by*

$$\rho_1, \quad \rho_0\rho_1\rho_0^{-1}, \quad \rho_0^2\rho_1\rho_0^{-2}.$$

Proof. Under the condition for parameters in this proposition, $\mathbf{h}(x) = \begin{pmatrix} f_0(x^3) \\ f_1(x^3) \end{pmatrix}$ is a fundamental system of solutions of $H(\alpha,\beta)$. We take a base point x_0 as a small positive real number ε.

Let ℓ_1 be a loop starting at x_0, going to $x = 1-\varepsilon$ along the real axis, encircling the point $x=1$ once in the positive sence and going back along the real axis. When x varies along ℓ_1, $y = x^3$ turns the point $y=1$ once in the positive sence. Thus $\mathbf{h}(x)$ changes into $g_1\mathbf{h}(x)$ by the continuation along the loop ℓ_1.

Let ℓ_ω be the loop $(r_1^\omega(\varepsilon))\cdot(\omega\ell_1)\cdot(r_1^\omega(\varepsilon))^{-1}$, where $r_1^\omega(\varepsilon)$ is the arc from ε to $\omega\varepsilon$ with center at 0, and $\omega\ell_\omega$ is the image of ℓ_1 under the map $\mathbb{C}\ni x\mapsto\omega x\in\mathbb{C}$. Since $y=x^3$ turns the point $y=0$ once in the positive sence when x varies along the arc $r_1^\omega(\varepsilon)$, $\mathbf{h}(x)$ changes into $g_0\mathbf{h}(x)$ by the continuation along $r_1^\omega(\varepsilon)$. Thus $\mathbf{h}(x)$ changes into $g_0g_1g_0^{-1}\mathbf{h}(x)$ by the continuation along the loop ℓ_ω.

Similarly, $\mathbf{h}(x)$ changes into $g_0^2g_1g_0^{-2}\mathbf{h}(x)$ by the continuation along a certain loop ℓ_{ω^2} starting at x_0 and turning the point $x=\omega^2$.

Since the fundamental group of $\mathbb{C}-\{1,\omega,\omega^2\}$ is generated by the three loops ℓ_1, ℓ_ω and ℓ_{ω^2}, the monodromy group with respect to $\mathbf{h}(x)$ is generated by ρ_1, $\rho_0\rho_1\rho_0^{-1}$ and $\rho_0^2\rho_1\rho_0^{-2}$. □

The monodromy group of the fundamental system $\mathbf{h}(x)$ of the differential equation $H(1/3,1/3)$ is generated by $m_{1+j}=m_0^jm_1m_0^{-j}$ $(j=0,1,2)$, where

$$m_0=\begin{pmatrix}1 & 0\\ -1+\omega^2 & \omega\end{pmatrix},\quad m_1=\begin{pmatrix}1 & 1-\omega^2\\ 0 & 1\end{pmatrix}.$$

For the matrix $P=\begin{pmatrix}0 & \omega^2\\ -1+\omega^2 & -1\end{pmatrix}$, Pm_jP^{-1} $(j=0,1,2,3)$ are

$$\omega^2\begin{pmatrix}-1 & 1\\ -1 & 0\end{pmatrix},\quad \begin{pmatrix}1 & 0\\ 3 & 1\end{pmatrix},\quad \begin{pmatrix}1 & -3\\ 0 & 1\end{pmatrix},\quad \begin{pmatrix}4 & -3\\ 3 & -2\end{pmatrix},$$

respectively. It is known that the group generated by Pm_jP^{-1} $(j=1,2,3)$ coincides with the level 3 principal congruence subgroup

$$\Gamma(3)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})\middle|\, a-1,b,c,d-1\in 3\mathbb{Z}\right\}.$$

The group generated by $\begin{pmatrix}-1 & 1\\ -1 & 0\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\ 3 & 1\end{pmatrix}$ is conjugate to the congruence subgroup

$$\Gamma_0(3)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})\middle|\, a-1,c,d-1\in 3\mathbb{Z}\right\},$$

since $\Gamma(3)$ is a normal subgroup of $GL_2(\mathbb{Z})$, $\Gamma_0(3)/\Gamma(3) \simeq \mathbb{Z}/(3\mathbb{Z})$, and $(QP)m_0(QP)^{-1} = \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix}$ belongs to $\Gamma_0(3)$, where $Q = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \in GL_2(\mathbb{Z})$. We have the commutative diagram:

$$\begin{array}{ccc} \mathbb{C} - \{0,1,\omega,\omega^2\} & \overset{\tilde{h}}{\longrightarrow} & \mathbb{H}/\Gamma(3) \\ \imath \downarrow & & pr \downarrow \\ \mathbb{C} - \{0,1\} & \overset{\tilde{f}}{\longrightarrow} & \mathbb{H}/\Gamma_0(3), \end{array}$$

where $\mathbb{H}$ is the upper half space, the map $\imath$ is $x \mapsto y = x^3$, the map pr is the natural projection, the maps $\tilde{h}$ and $\tilde{f}$ are given by the ratio of the fundamental solutions of $(QP)\mathbf{h}(x)$ and $(QP)\mathbf{f}(y)$, respectively.

5 A representation of the fundamental group of the Borromean-rings-complement

It is shown in [W] that the fundamental group of the Borromean-rings-complement is isomorphic to the subgroup B of $SL_2(\mathbb{Z}[i])$ generated by three elements

$$g_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 2i \\ 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 2+i & 2i \\ -1 & -i \end{pmatrix}.$$

Lemma 1 *We have*

$$g_0^3 = I, \quad g_2 = g_0 g_1 g_0^{-1}, \quad g_3 = g_0^2 g_1 g_0^{-2},$$

where

$$g_0 = \begin{pmatrix} -1 & -1-i \\ \frac{1-i}{2} & 0 \end{pmatrix} \in SL_2(\mathbb{C}).$$

Proof. We can easily show this lemma by direct computations. We here explain how to find the matrix g_0. The matrices g_1, g_2 and g_3 can be expressed as

$$g_j = I - v_j\, {}^t v_j J \quad (j = 1, 2, 3),$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1+i \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1+i \\ -1 \end{pmatrix}.$$

Since any element $g \in SL_2(\mathbb{C})$ satisfies ${}^t gJg = J$, we have

$$gg_jg^{-1} = I - g(v_j\,{}^tv_jJ)g^{-1} = I - (gv_j)\,{}^t(gv_j)J.$$

Thus if the matrix g satisfies $g(v_1, v_2) = (v_2, v_3)$ then $g_2 = gg_1g^{-1}$, $g_3 = g^2g_1g^{-2}$. We put $g_0 = -(v_2, v_3)(v_1, v_2)^{-1}$ so that $g_0^3 = I$. □

Theorem 1 *The monodromy group of $H(\alpha, \beta)$ for α and β satisfying*

$$e^{2\pi i\alpha} = i\omega\zeta, \qquad e^{2\pi i\beta} = i\omega\zeta'$$

is conjugate to the group B, where $\zeta = \frac{1\pm\sqrt{5}}{2}$ and $\zeta' = \frac{1\mp\sqrt{5}}{2}$.

Proof. In fact, for parameters in Theorem 1 and the matrix

$$P = \begin{pmatrix} 0 & 1+i \\ \omega - i\zeta & \omega \end{pmatrix},$$

we have

$$P\rho_0P^{-1} = g_0, \quad P\rho_1P^{-1} = g_1.$$

Proposition 2 and Lemma 1 imply this theorem.

We explain our method to find these parameters and the matrix P. If g_1 is conjugate to ρ_1 then the Jordan normal form $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of g_1 must coincide with that of ρ_1. Thus we have the condition $\omega^2e^{-2\pi i(\alpha+\beta)} = 1$. We eliminate α in ρ_1 by this condition, and put $b = e^{-2\pi i\beta}$; ρ_1 becomes

$$\begin{pmatrix} 1 & 1-\omega/b \\ 0 & 1 \end{pmatrix} = I - v\,{}^tvJ, \quad v = \begin{pmatrix} \sqrt{1-\omega/b} \\ 0 \end{pmatrix}.$$

Since

$$P_1^{-1}g_0P_1 = \omega P_2^{-1}\rho_0P_2 = \begin{pmatrix} \omega & \\ & \omega^2 \end{pmatrix},$$

we have $\omega P(z)\rho_0 P(z)^{-1} = g_0$, where z is a variable in $\mathbb{C} - \{0\}$ and

$$P(z) = \frac{1}{\sqrt{(1+i)z}} P_1 Z P_2^{-1} \in SL_2(\mathbb{C}),$$

$$P_1 = \begin{pmatrix} 1+i & 1+i \\ \omega^2 & \omega \end{pmatrix}, \quad P_2 = \begin{pmatrix} \sqrt{3}i & 0 \\ \omega(b-1) & 1 \end{pmatrix}, \quad Z = \begin{pmatrix} z & \\ & 1 \end{pmatrix}.$$

By the equality $P(z)v = v_1$, which implies $P(z)\rho_1 P(z)^{-1} = g_1$, we have two algebraic equations with variables b and z. The first equation reduces to

$$(b - \omega)(z - \omega(b-1)) = 0.$$

If $b = \omega$ then ρ_1 becomes I; thus z should be $\omega(b-1)$. By eliminating z from the second equation by this identity, we have the quadratic equation

$$b^2 - i\omega^2 b + \omega = 0,$$

of which solutions are $i\omega^2 \frac{1 \pm \sqrt{5}}{2}$. Note that their inverses are $i\omega \frac{1 \mp \sqrt{5}}{2}$. The matrix P is given by $\sqrt{(1+i)z}P(z)$ for $b = \exp(-2\pi i\beta) = i\omega^2\zeta$ and $z = \omega(i\omega^2\zeta - 1)$. □

Remark 1 *The monodromy group of the fundamental system* $P\mathbf{h}(x)$ *of the differential equation* $H(\alpha, \beta)$ *for parameters satisfying the condition in Theorem 1 coincides with the group* B.

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