Amoebas & Tropical Geometry

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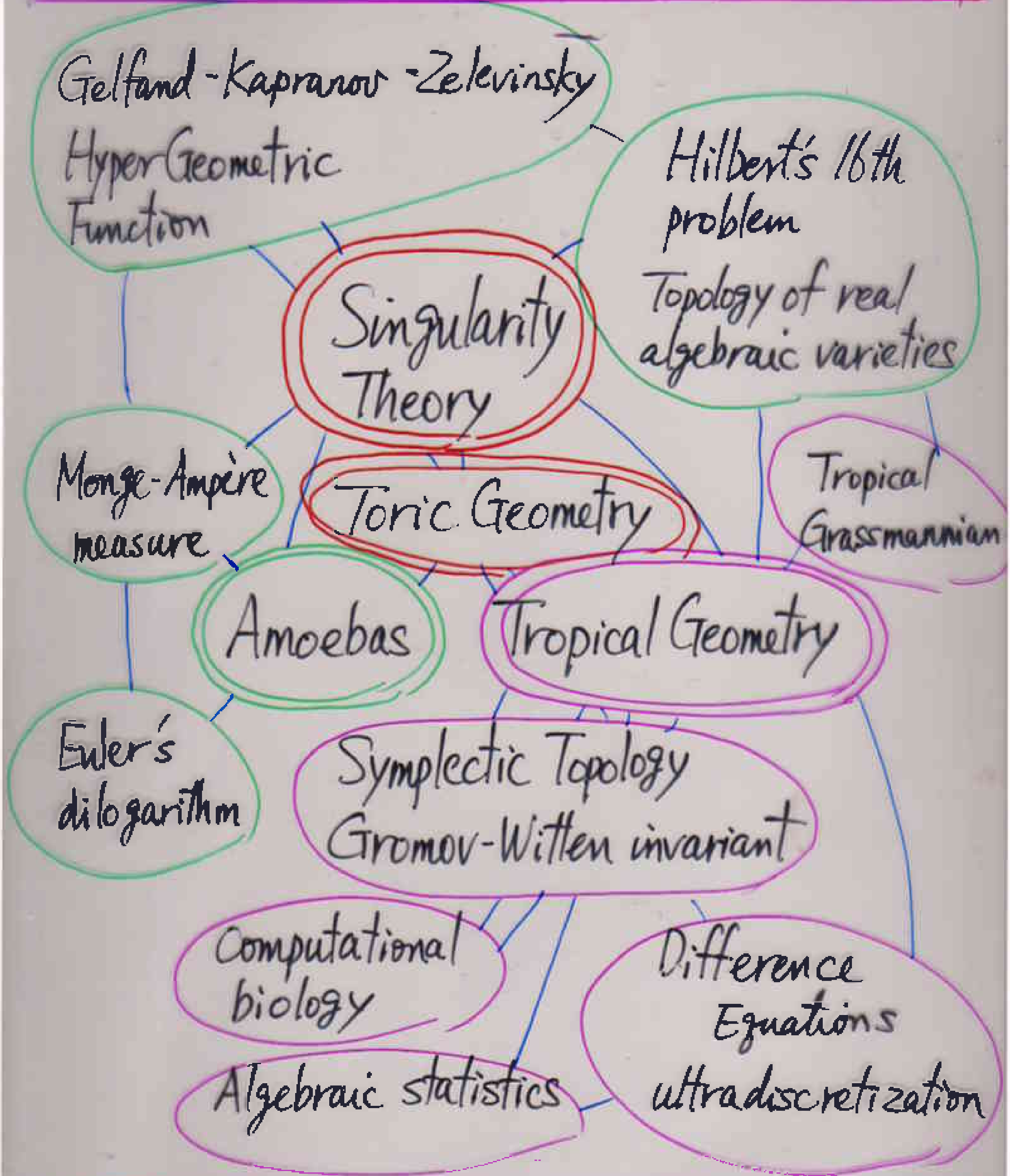

Gelfand-Kapranov-Zelevinsky
HyperGeometric Function
Hilbert's 16th problem
Topology of real algebraic varieties
Singularity Theory
Monge-Ampère measure
Toric Geometry
Tropical Grassmannian
Amoebas
Tropical Geometry
Euler's dilogarithm
Symplectic Topology
Gromov-Witten invariant
Computational biology
Difference Equations
ultradiscretization
Algebraic statistics

$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$

$$f(z) = \sum_{j \in A} a_j z^j \in \mathbb{C}[z_1^{\pm 1}, \cdots, z_n^{\pm 1}]$$

Laurent polynomial $\quad A \underset{\text{finite}}{\subset} \mathbb{Z}^n$

$$V_f := \{f(z) = 0\} \subset (\mathbb{C}^*)^n$$

$$\mathrm{Log} : (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n$$

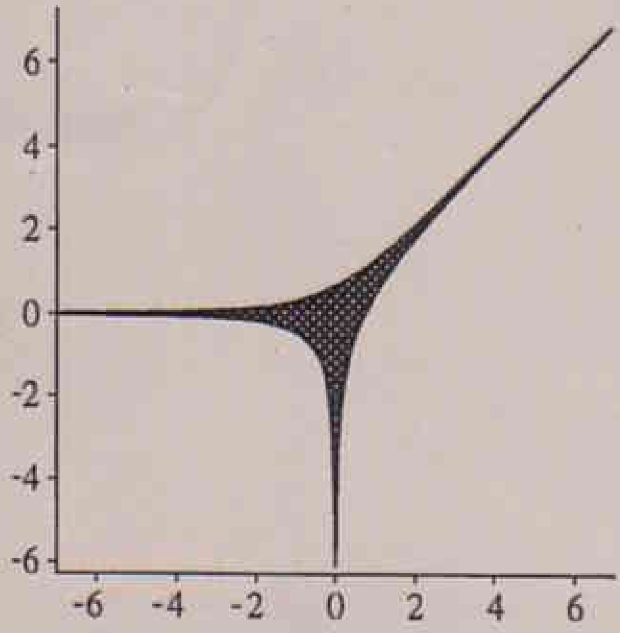
$$\mathrm{Log}(z_1, \cdots, z_n) := (\log|z_1|, \cdots, \log|z_n|)$$

$$\left\{ \begin{array}{l} \mathcal{A} := \mathrm{Log}(V_f) \subset \mathbb{R}^n \\ \text{Amoeba of } V_f \end{array} \right.$$

(Gelfand-Kapranov-Zelevinski)

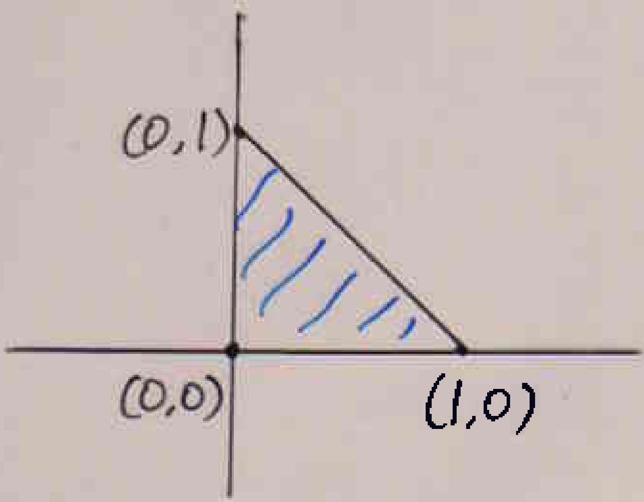
Example $f(z_1, z_2) = z_1 + z_2 + 1$

Amoeba



$$V = \{ z_1 + z_2 + 1 = 0 \} \subset (\mathbb{C}^*)^2$$

Newton polytope



Newton polytope (polyhedron)

$$f(z) = \sum_{j \in A} a_j z^j \quad a_j \in \mathbb{C},\ A \subset \mathbb{Z}^n$$

$$\Delta = \Delta(f) =$$

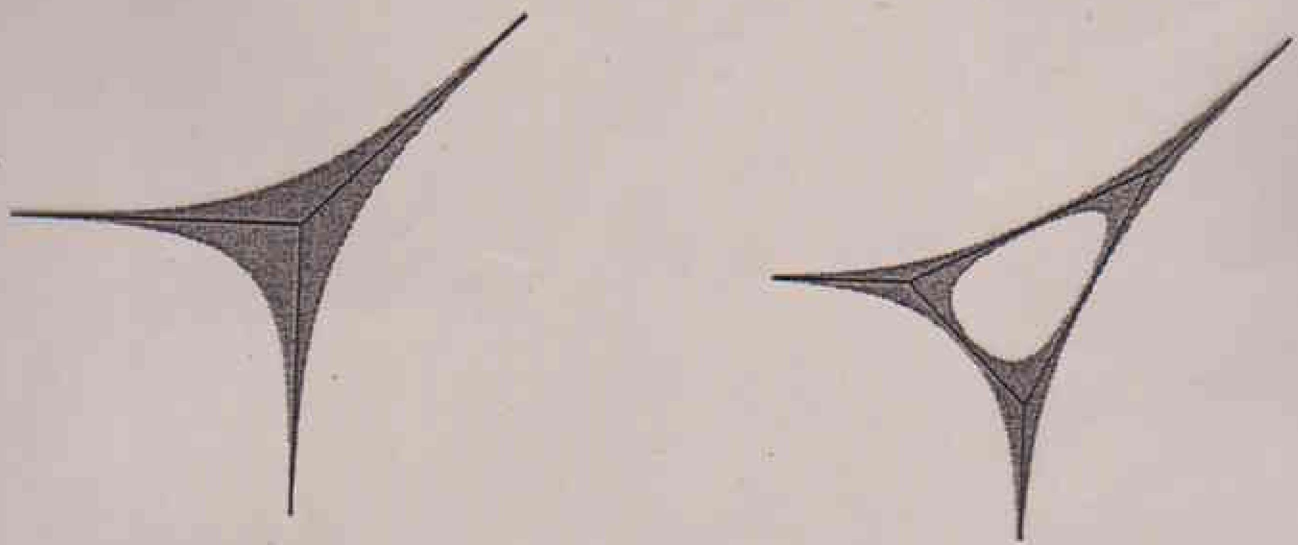
convex hull of $\{j \mid a_j \neq 0\} \subset \mathbb{R}^n$

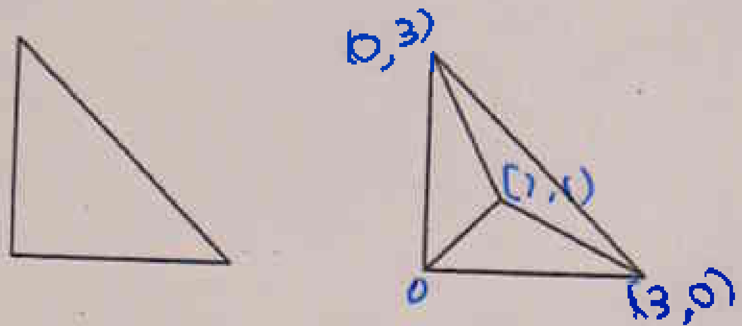
$\#$ conn. comp. of $(\mathbb{R}^n \setminus \mathcal{A})$

$$\leqq \#(\Delta \cap \mathbb{Z}^n)$$

(Forsberg-Passare-Tsikh)

Amoeba and Newton polytope





$f(z_1, z_2)$
$= z_1^3 + z_2^3 + 1$

$g(z_1, z_2)$
$= z_1^3 + z_2^3 - 6z_1z_2 + 1$

Tropical Geometry

tropical semiring $\mathbb{R}_{trop}$

$\mathbb{R}$ $\quad$ "$x+y$" $:= \max\{x, y\}$

"xy" $:= x+y$

Imre Simon (São Paolo)

tropical polynomial $\quad a_j \in \mathbb{R} \quad A \subset (\mathbb{Z}_{\geq 0})^n$ finite

$$F(x) = \text{"}\sum_{j\in A} a_j x^j\text{"} = \max_{j\in A}\left(j_1x_1 + \cdots + j_nx_n + a_j\right)$$

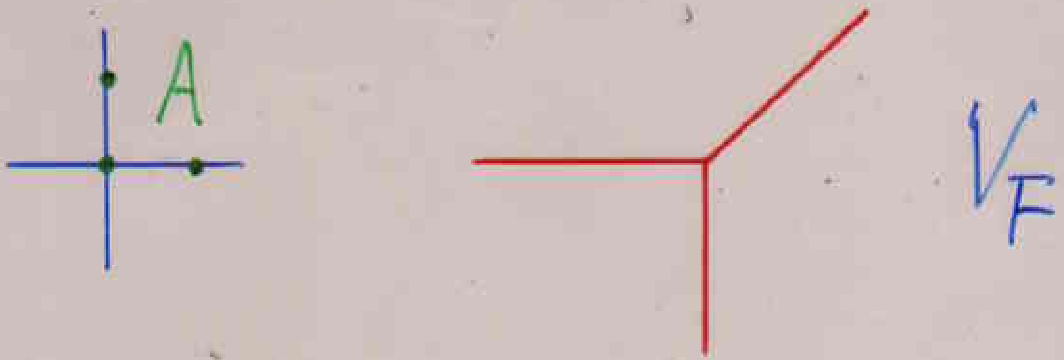
tropical hypersurface

$V_F :=$ non-smooth loci of F

$\subset \mathbb{R}^n$

Example Tropical line

$$F(x_1,x_2) = \text{"}ax_1+bx_2+c\text{"} = \text{"}0\text{"}$$

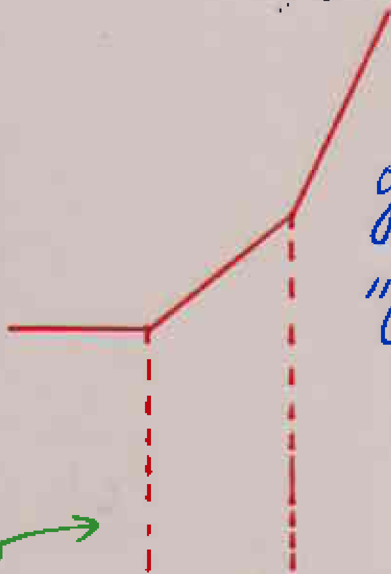


$$\text{"}ax_1+bx_2+c\text{"} = \max\{x_1+a_1,\ x_2+b,\ c\}$$

Remark "0" $= -\infty$ additive unit

$$\text{"}ax^2+bx+c\text{"} = \text{"}0\text{"}$$

$$\max\{2x+a,\ x+b,\ c\}$$
$$= -\infty$$



graph of "ax^2+bx+c"

tropical closure $\rightsquigarrow$

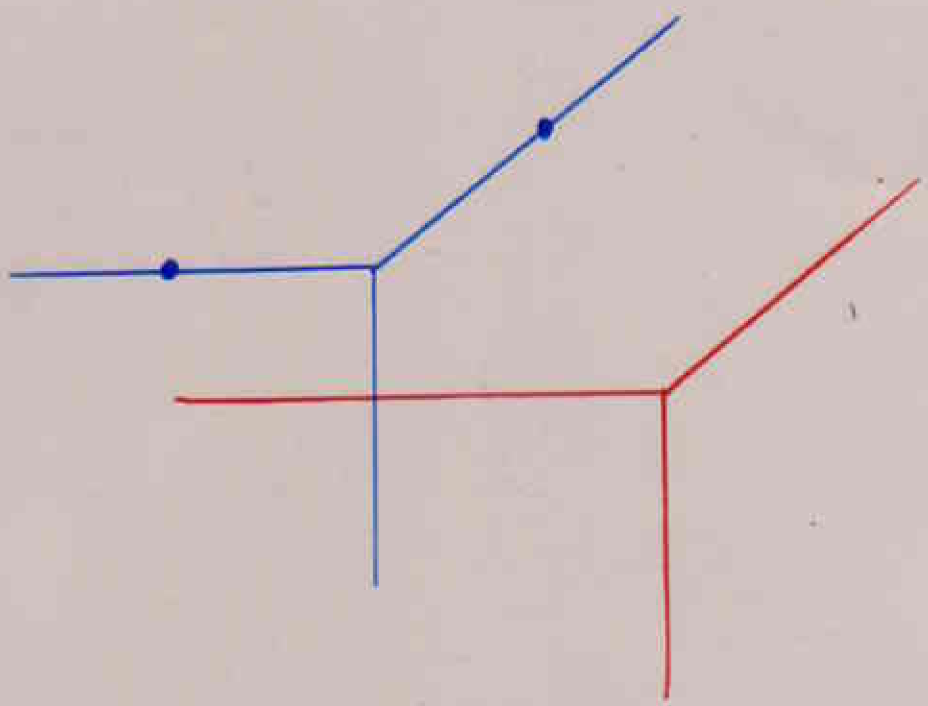
$\rightsquigarrow$ non-smooth loci.

$\forall P, P' \in \mathbb{R}^2$, $\exists$ tropical line passing through P, P'.

If P, P' generic, $\exists$ unique line.

$\forall \ell, \ell'$ trop. line, $\exists$ intersection point

If ℓ, ℓ' generic, $\exists$ unique int. pt.



Tropical Bézout-Bernstein Theorem

Generic tropical curves of degree d, degree r, intersect in dr-points.

Generic tropical curves with Newton polytopes Δ, Δ' resp. intersect in $\text{Area}(\Delta+\Delta') - \text{Area}(\Delta) - \text{Area}(\Delta')$ points.

Δ = convex hull of A in $\mathbb{R}^2$

$F(x) = \text{“} \sum_{j \in A} a_j x^j \text{”}$

$\frac{1}{2}(d+r)^2 - \frac{1}{2}d^2 - \frac{1}{2}r^2 = dr$

Example: quadratic curve

$$F(x_1,x_2) = \text{"}0x_1^2 + 1x_1x_2 + 0x_2^2 + 1x_1 + 1x_2 + 0\text{"}$$

$$G(x_1,x_2) = \text{"}2x_1^2 + \tfrac{5}{2}x_1x_2 + 1\cdot x_2^2 + 2x_1 + \tfrac{3}{2}x_2 + 0\text{"}$$