GEOMETRIC APPROACH TO GOURSAT FLAGS *

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Abstract. A Goursat flag is a chain $D_s \subset D_{s-1} \subset D_1 \subset D_0 = TM$ of subbundles of the tangent bundle TM such that *corank* $D_i = i$ and D_{i-1} is generated by the vector fields in D_i and their Lie brackets. Engel, Goursat, and Cartan studied these flags and established a normal form for them, valid at generic points of M. Recently Kumpera, Ruiz and Mormul discovered that Goursat flags can have singularities, and that the number of these grows exponentially with the corank s. Our theorem 1 says that every corank s Goursat germ, including those yet to be discovered, can be found within the s-fold Cartan prolongation of the tangent bundle of a surface. Theorem 2 says that every Goursat singularity is structurally stable, or irremovable, under Goursat perturbations. Theorem 3 establishes the global structural stability of Goursat flags, subject to perturbations which fix a certain canonical foliation. It relies on a generalization of Gray's theorem for deformations of contact structures. Our results are based on a geometric approach, beginnning with the construction of an integrable subflag to a Goursat flag, and the sandwich lemma which describes inclusions between the two flags. We show that the problem of local classification of Goursat flags reduces to the problem of counting the fixed points of the circle with respect to certain groups of projective transformations. This yields new general classification results and explains previous classification results in geometric terms. In the last appendix we obtain a corollary to Theorem 1. The problems of locally classifying the distribution which models a truck pulling s trailers and classifying arbitrary Goursat distribution germs of corank $s+1$ are the same.

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1. INTRODUCTION AND MAIN RESULTS

This paper is devoted to Goursat distributions and Goursat flags. A *Goursat flag* of length s on a manifold M^n of dimension $n \geq 4$ is a chain

$$D_s \subset D_{s-1} \subset \cdots \subset D_3 \subset D_2 \subset D_1 \subset D_0 = TM, \quad s \geq 2 \tag{F}$$

of distributions on M^n (subbundles of the tangent bundle TM^n of constant rank) satisfying the following (Goursat) conditions:

$$corank\ D_i = i, \quad i = 1, 2, \ldots, s$$

$$D_{i-1} = D_i^2 \text{ where} D_i^2 := [D_i, D_i], \quad i = 1, 2, \cdots, s. \tag{G}$$

The first condition means that $D_i(p)$ is a subspace of T_pM^n of codimension i, for any point $p \in M^n$. It follows that $D_{i+1}(p)$ is a hyperplane in $D_i(p)$, for any $i = 0, 1, 2, \ldots, s-1$ and $p \in M^n$. In condition (G) we use the standard notation D^2 or $[D, D]$ for the sheaf of vector fields generated by D and the Lie brackets $[X, Y]$, $X, Y \in D$, of vector fields in D.

By a Goursat distribution we mean any distribution of any corank $s \geq 2$ of any Goursat flag (F).

An equivalent definition is as follows. A distribution D of corank $s \geq 2$ is Goursat if the subsheaves D^i of the tangent bundle defined inductively by $D^{i+1} = [D^i, D^i]$ ($i = 1, 2, \ldots, s;\ D^1 = D$) correspond to distributions, i.e. they have constant rank, and this rank is $rank\ D^{i+1} = rank\ D^i + 1,\ i = 1, \ldots, s$.

Since the whole flag (F) is uniquely determined by the distribution $D = D_s$ of the largest corank, we will say that $D = D_s$ *generates* (F). The study of Goursat flags and Goursat distributions is the same problem.

The name "Goursat distributions" is related to the work [Goursat, 1923] in which Goursat popularized these distributions. Goursat's predecessors were Engel and Cartan.

Engel studied the case $n = 4, s = 2$. *This is the only case where the Goursat condition holds for generic germs.* He proved [Engel, 1889] that the germ of such a distribution is equivalent to a single normal form without parameters. (See (C) below.)

If $(n, s) \neq (4, 2)$ then the set of germs of Goursat distributions of corank s on M^n is a subset of infinite codimension in the space of all germs. Nevertheless, Goursat distributions appear naturally through Cartan's prolongation procedure. See, for example, [Bryant, 1991] and section 5 of the present paper. The simplest realization of prolongation leads to a canonical Goursat 2-distribution (i.e., distribution of rank 2) on the $(2+s)$-dimensional space of s-jets of functions $f(x)$ in one variable. This distribution can be described by s differential 1-forms

$$\omega_1 = dy - z_1 dx,\ \omega_2 = dz_1 - z_2 dx,\ \ldots,\ \omega_s = dz_{s-1} - z_s dx, \tag{C}$$

where y represents the value of f at x and z_i represents the value at x of the i-th derivative of f. Cartan proved that **a generic germ** of a Goursat 2-distribution can always be described by the 1-forms (C). Indeed he proved the stronger statement:

[Cartan, 1914]: *The germ at a generic point of* **any** *Goursat distribution of corank* $s \geq 2$ *on a manifold* M *of any dimension* $n \geq s+2$ *is equivalent to the germ at the origin of the distribution described by the 1-forms (C).*

This theorem together with all the assertions in the present paper hold in both the smooth (C^∞) and real-analytic categories. Two global distributions on M are called *equivalent* if there exists a global diffeomorphism of M sending one of them to the other. Local equivalence is defined in a usual way: the germ of D at a point p is equivalent to the germ of $\tilde{D}$ at a point $\tilde{p}$ if there exists neighbourhoods U of p and $\tilde{U}$ of $\tilde{p}$ and a diffeomorphism $\Phi : U \to \tilde{U}$, $\Phi(p) = \tilde{p}$ which sends the restriction of D to U onto the restriction of $\tilde{D}$ to $\tilde{U}$.

We will say that a point $p \in M$ is a *singularity* for a Goursat distribution if the distribution is **not** locally equivalent at p to the model distribution described by 1-forms (C). An equivalent definition in invariant terms is given in Setion 2.

Some researchers believe that Cartan missed the singularities in the problem of classifying Goursat distributions. It would be more accurate to say that he was not interested in them. Recently there has been interest. Researchers have realized that the number of different singularities grows very fast, indeed exponentially, with the corank s. Recent results on the number of singularities are given in the following table. Here $or(s)$ denotes the number of orbits (inequivalent germs) within the space of all Goursat germs of corank s at the origin of R^n (results are the same for all $n \geq s+2$):

s	2	3	4	5	6	7	≥ 8
$or(s)$	1	2	5	13	34	93	∞
$Author:$	$Engel$	$Giaro$ $Kumpera$ $Ruiz$	$Kumpera$ $Ruiz$	$Gaspar$	$Mormul$	$Mormul$	$Mormul$
$reference:$	1889	1978	1982	1985	1997	1998	1998

Although the entries of this table were obtained originally just for rank two Goursat distributions on R^{2+s} they hold for Goursat distributions of arbitrary rank k and corank s distributions on R^{k+s}, with $k, s \geq 2$. Indeed, a reduction theorem due to one of us [Zhitomirskii, 1990] implies that *any Goursat distribution of corank* s *is locally equivalent to one of the form* $D = W \oplus R^{k-2}$ *on* $R^{k+s} = R^{2+s} \times R^{k-2}$*, where* W *is a rank two Goursat distribution on* R^{2+s}*.*

The theorems summarized by the above table are in marked contrast with the spirit of Cartan's result . This contrast inspired our two main theorems. Theorem 1 says that the Cartan prolongation procedure accounts not only for the Cartan normal form (C), but for **all possible singularities**. This includes any singularities yet to be discovered, in addition to the list above. Theorem 2 asserts that every Goursat singularity, however complicated, cannot be perturbed away while keeping the distribution Goursat. In other words, theorem 2 asserts that Goursat singularities are "irremovable".

Theorem 1. *Apply the Cartan prolongation procedure (see section 5) s times, starting with a two-dimensional surface. The resulting " monster Goursat manifold" Q of dimension $2+s$ is endowed with a Goursat distribution H which is universal in the following sense. The germ at any point of any rank two Goursat distribution on a $(2+s)$-dimensional manifold is equivalent to the germ of H at some point of Q.*

In section 5 the Cartan prolongation procedure is described, the monster manifold constructed, and the theorem proved.

Theorem 2. *Every Goursat singularity is irremovable. Namely, within the space of all germs of Goursat distributions of corank s, any germ is structurally stable in the C^{s+1}-topology on the space of Goursat germs.*

Any such germ is s-determined.

Structural stability of the germ of a Goursat distribution D at a point p means the following. Let D_N be any sequence of Goursat distributions defined in a (fixed) neighborhood of p, and such that $j_p^{s+1}D_N \to j_p^{s+1}D$ as $N \to \infty$. Then there exists a sequence of points p_N tending to p such that for all sufficiently big N the germ of D_N at p_N is equivalent to the germ at p of D. In other words, if we perturb D within the space of Goursat distributions, then nearby to p there will be points p_N at which the germ of the perturbed distribution D_N is equivalent to that of the original distribution at p.

To say that D is *s-determined* (at p) means that if $\tilde{D}$ is another Goursat distribution defined near p, and if $j_p^s D = j_p^s \tilde{D}$ then the germs at p of D and of $\tilde{D}$ are equivalent.

We also have a result on global structural stability, one inspired by works [Golubev, 1997] and [Montgomery, 1997] on deformations of global Engel distributions.

Theorem 3. *Any cooriented Goursat flag (F) of length s on a manifold M is structurally stable with respect to sufficiently Whitney C^{s+1}-small perturbations within the space of global Goursat flags, provided these peturbations do not change the characteristic codimension 3 foliation $L(D_1)$.*

The *characteristic codimension 3 foliation* $L(D_1)$ is defined in section 2. It is invariantly related to the corank one distribution D_1 and generalizes the characteristic vector field of an Engel distribution. To say that the flag (F) is *cooriented* means that there exist s global 1-forms $\omega_1, \ldots, \omega_s$ such that the distribution D_i can be described as the vanishing of $\omega_1, \ldots, \omega_i$, $i = 1, \ldots, s$. Theorem 3 says that if two global Goursat flags F and $\tilde{F}$ is sufficiently close in the Whitney C^{s+1}-topology and if $L(\tilde{D}_1) = L(D_1)$ then there exists a global diffeomorphism of M sending $\tilde{F}$ to F. The condition $L(\tilde{D}_1) = L(D_1)$ can be, of course, replaced by the condition that the foliations $L(\tilde{D}_1)$ and $L(D_1)$ are equivalent via a diffeomorphism close to the identity. This condition is essential even for the case $s = 2$ of Engel distributions, see [Gershkovich, 1995]. The foliation $L(D_1)$, viewed as a global object, is a complicated, poorly understood topological invariant of D. In particular it is not known what types of foliations are realizable, even in the simplest case of Engel distributions.

Outline.

To prove Theorems 1-3 we develop a geometric approach to Goursat flags in sections 2 and 3. The starting point is the flag of foliations associated to a Goursat flag. The relations between the two flags is described by the sandwich lemma. This allows us to formulate the Cartan theorem in pure geometric terms, and to define singular points.

In section 3 we develop the geometric approach in order to show that : *the problem of classifying Goursat flags reduces to the problem of finding fixed points of the circle with respect to certain subgroups of the group of projective transformations.* Using this reduction we obtain some general classification results. In section 4 we use our methods to explain the recent results, as summarized in table 1, by purely geometric geometric reasoning.

In section 5 we present Cartan's prolongation and deprolongation constructions and prove
Theorem 1.

Theorems 2 and 3 are proved in section 6. One tool in the proof is a generalization of Gray's theorem [Gray, 1959] on deformations of global contact structures. We prove that any two global C^{l+1}-close corank one distributions of the same constant class (in Cartan's sense) are equivalent via a C^l-close to identity global diffeomorphism. This result is of independent significance, therefore we put it to Appendix A.

In Appendix B we prove one of the lemmata used in section 3.

In Appendix C we explain the canonical meaning of the Kumpera-Ruiz normal forms and we explain P.Mormul's codes for symbolizing finer normal forms. We also summarize what is known about when and how the growth vector distinguishes singularities.

Finally, in Appendix D we use our Theorem 1 to give a simple proof that the local classification of Goursat distributions describing a kinematic model of a truck towing s trailers and the local classification of arbitrary Goursat flags of length $s+1$ are the same problem.

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2. FLAG OF FOLIATIONS. SANDWICH LEMMA. CARTAN THEOREM.

We start the geometric approach to Goursat distributions by associating a flag of foliations

$$L(D_s) \subset L(D_{s-1}) \subset L(D_{s-2}) \subset \cdots \subset L(D_2) \subset L(D_1) \qquad (L)$$

to the Goursat flag

$$D_s \subset D_{s-1} \subset D_{s-2} \subset \cdots \subset D_2 \subset D_1 \subset D_0 = TM \qquad (F)$$

generated by the Goursat distribution $D = D_s$ of corank $s \geq 2$ on a manifold M.

Definition. Given any distribution $D \subset TM$ we denote by $L(D)$ the subsheaf of D consisting of those vector fields $X \in D$ whose flows preserve D: $[X, Y] \in D$ for all $Y \in D$. We call $L(D)$ the *characteristic foliation* of D.

The Jacobi identity implies that $L(D)$ is closed under Lie bracket. Consequently if $L(D)$ is of constant rank, then it is a foliation in the standard sense. As we will see momentarily it does have constant rank in the Goursat case, this rank being $rank(D)-2$. In other words, if we set

$$L(D)(p)=\{X(p):\ X\in L(D)\},$$

then $L(D)(p)$ has dimension $rank(D)-2$, independently of the point p.

Lemma 2.1. (Sandwich lemma). *Let D be any Goursat distribution of corank $s\geq 2$ on a manifold M. Let p be any point of M. Then*

$$L(D)(p)\subset L(D^2)(p)\subset D(p),$$

with

$$dim\ L(D)(p)=dim\ D(p)-2,\quad dim\ L(D^2)(p)=dim\ D(p)-1.$$

It follows that the relation between the Goursat flag (F) and its flag of characteristic foliations (L) is summarized by:

$$\begin{array}{ccccccccccccc}
 & & D_s & \subset & D_{s-1} & \cdots & \subset & D_3 & \subset & D_2 & \subset & D_1 \\
 & & \cup & & \cup & & & \cup & & \cup & & \\
L(D_s) & \subset & L(D_{s-1}) & \subset & L(D_{s-2}) & \cdots & \subset & L(D_2) & \subset & L(D_1) & &
\end{array}$$

Each inclusion here is a codimension one inclusion of subbundles of the tangent bundle. $L(D_i)$ has codimension 2 within D_i, which in turn has corank i within TM, so that $L(D_i)$ is a foliation of M of codimension $i+2$. In particular, $L(D_1)$ – the foliation figuring in our Theorem 3 – is a codimension 3 foliation.

The foliations $L(D_i)$ can be described using 1-forms. We will say that an ordered s-tuple $\omega_1,\ldots,\omega_s$ *describes* the flag (F) generated by a Goursat distribution $D=D_s$ of corank s if ω_1 describes the corank one distribution D_1, the forms ω_1 and ω_2 together describe the corank 2 distribution D_2, etc., the tuple $(\omega_1,\ldots,\omega_{s-1})$ describes D_{s-1} and the tuple $(\omega_1,\ldots,\omega_s)$ describes D_s. (Here "describes" means that the distribution being described consists of all vectors annihilated by the forms "describing".) Order matters. For example, consider the corank 2 Goursat distribution D defined by the vanishing of of the 1-forms $\omega_1=dy-z_1dx$ and $\omega_2=dz_1-z_2dx$. Then the pair (ω_1,ω_2) describes the flag generated by D whereas the pair (ω_2,ω_1) does not.

Given a tuple of 1-forms $\omega_1,\ldots,\omega_s$ describing the Goursat flag (F), denote by

$$\theta_i(p)=d\omega_i(p)|_{D_i(p)} \tag{2.1}$$

the restriction of the 2-form $d\omega_i(p)$ to the space $D_i(p)$, $p\in M$. By the kernel of a 2-form θ on a vector space V we mean the space of vectors v such that $\theta(v,Y)=0$ for any $Y\in V$. The proof of Lemma 1 is base on the following statement.

Lemma 2.2. *Let (F) be the Goursat flag generated by a distribution $D = D_s$ and described by the tuple $\omega_1, \dots, \omega_s$ of 1-forms. Define the 2-forms $\theta_i(p)$ by (2.1). Then for any point p of the manifold and for any $i = 1, 2, \dots, s$ we have:*

$$rank\ \theta_i(p) = 2; \quad L(D_i)(p) = ker\ \theta_i(p).$$

Example. Let D be the corank s Goursat distribution described by the 1-forms (C). Then the tuple $(\omega_1, \dots, \omega_s)$ describes the flag (F) generated by $D = D_s$, and the foliation $L(D_i)$ is described by the 1-forms $dx, dy, dz_1, \dots, dz_i$.

Proof of lemmata 2.1 and 2.2. We first show that the rank of $\theta_i(p)$ is two, for $i < s$. Recall that ω_i vanishes on D_i but not on D_{i-1}, and that its vanishing **defines** D_i **within** D_{i-1}. The identity

$$d\omega_i(X, Y) = -\omega_i([X, Y]), \quad X, Y \in D_i \tag{2.2}$$

and the fact that $[D_{i+1}, D_{i+1}]$ is a subset of D_i imply that $\theta_i(p)$ vanishes upon restriction to the hyperplane $D_{i+1}(p)$ of $D_i(p)$ (provided that $i < s$ so that D_{i+1} is defined). The fact that $[D_i, D_i] = D_{i-1}$ implies that $\theta_i(p) \neq 0$. In other words, $\theta_i(p)$ is a non-zero skew-symmetric form which admits $D_{i+1}(p)$ as an isotropic subspace of codimension 1. Basic linear algebra now implies that $rank\ \theta_i(p) = 2$, $dim\ ker\ \theta_i(p) = 2$, and that $ker\ \theta_i(p) \subset D_{i+1}(p)$. This is valid for all points p and all $i = 1, 2, \dots, s-1$.

It follows directly from the identity (2.2) that $L(D_i)(p) \subset ker\ \theta_i(p)$. To prove that $L(D_i)(p) = ker\ \theta_i(p)$ we use the constancy of rank of these kernels. Suppose $X_p \in ker\ \theta_i(p)$. Since the field of kernels of θ_i has constant rank we may extend X_p to a vector field X tangent to this field of kernels. Now (2.2), together with the fact that the vanishing of ω_i defines D_i within D_{i-1}, implies that $X \in L(D_i)$ so that $X_p \in L(D_i)(p)$. This completes the proof of Lemma 2.2 for $i = 1, 2, \dots, s-1$.

The case $i = s$ remains. We know that $L(D_i)$ is involutive for all i and that $L(D_i)(p) = ker\ d\theta_i(p)$ is a hyperplane in $D_{i+1}(p)$, for $i < s$. Identity (2.2) now implies that θ_{i+1} vanishes upon restriction to the hyperplane $L(D_i)(p)$ of $D_{i+1}(p)$, again for $i < s$. Therefore for $1 < i < s$ the form $\theta_i(p)$ has **two** (possibly equal) isotropic subspaces: $D_{i+1}(p)$ and $L(D_{i-1})(p)$, whereas the "end" forms $\theta_1(p)$ and $\theta_s(p)$ have only one isotropic subspace each: $D_2(p)$ and $L(D_{s-1})(p)$ respectively. The fact that $\theta_s(p)$ has $L(D_{s-1})(p) \subset D_s(p)$ as an isotropic hyperplane implies that $rank\ \theta_s(p) \leq 2$. The condition $rank\ [D_s, D_s](p) = s - 1$ implies that $rank\ \theta_s(p) \geq 2$. Therefore $rank\ \theta_s(p) = 2$. Repeating the above arguments, we see that $L(D_s)(p) = ker\ \theta_s(p)$, and therefore $L(D_s)(p)$ is a subspace of $D(p)$ of codimension 2. This completes the proof of Lemma 2.2.

To prove Lemma 2.1 , it only remains to show that $L(D_i) \subset L(D_{i-1})$. Again use the fact that if a skew-symmetric nonzero 2-form has an isotropic hyperplane then its kernel belongs to this hyperplane. We have proved that $L(D_{i-1})(p)$ is an isotropic hyperplane for $\theta_i(p)$ and $L(D_i)(p)$ is the kernel of $\theta_i(p)$. Therefore $L(D_i) \subset L(D_{i-1})$. Q.E.D.

By Lemma 2.1 for each $i = 3, 4, \dots, s$ the space $D_{i-1}(p)$ has two invariantly defined hyperplanes: $D_i(p)$ and $L(D_{i-2})(p)$. If the Goursat flag is generic then one expects that these two hyperplanes will be different. This is indeed the case, and it suggests our geometric formulation of Cartan's theorem on the normal form (C).

Proposition 2.1. (compare with [Cartan, 1914]). *The germ at a point p of a Goursat flag (F) of length s on a manifold M is equivalent to the germ at the origin of the flag described by the 1-forms (C) if and only if the condition*

$$L(D_{i-2})(p) \neq D_i(p), \quad i = 3, 4, \dots s \qquad (GEN)$$

holds. For any Goursat flag the set of points $p \in M$ satisfying (GEN) is open and dense in M.

The proof of this proposition is in section 4. Now we can give an invariant definition of a singular point of a Goursat distribution D or of its flag (F):

Definition. *A point p is nonsingular if (GEN) is satisfied. It is singular if (GEN) is violated for at least one $i \in \{3, 4, \dots, s\}$.*

We have 2^{s-2} different types of singularities, called *Kumpera-Ruiz classes* parametrized by the 2^{s-2} subsets $I \subset \{3, 4, \dots, s\}$. The class corresponding to the subset I consists of Goursat germs at a point p such that the condition (GEN) is violated for $i \in I$ and is valid for all $i \notin I$, $i \in \{3, 4, \dots, s\}$. A nonsingular point corresponds to $I = \emptyset$. Each singularity class is realized. These realizations correspond to the 2^{s-2} normal forms found by Kumpera-Ruiz [Kumpera-Ruiz, 1982], and described in Appendix C to the present paper.

As soon as $s > 3$ the Kumpera-Ruiz classification is **coarser** than the full classification of Goursat germs into equivalence classes under diffeomorphisms. In other words, for $s > 3$ there will be Kumpera-Ruiz classes which contain more than one orbit, i.e. several inequivalent Goursat germs. See the table in section 1. For example, when $s = 4$, we see that $or(s) = 5 \geq 2^{s-2} = 4$.

In the next two section we further develop the geometric approach to Goursat distributions, obtain general classification results and explain in invariant terms the classification results by Mormul and his predecessors.

3. Classification of branches of $\sqrt{D}$

The classification of germs of Goursat distributions of arbitrary corank reduces to the following problem:

Given a Goursat distribution germ D of corank s, classify the Goursat distributions E of corank $s+1$ such that $[E, E] = E^2 = D$.

Notation. The set of all such distribution germs E for a given D will be denoted $\sqrt{D}$.

Imagine the tree whose vertices are equivalence classes of Goursat germs. The root of the tree is the corank 2 distribution germ, which is a single class, according to Engel's theorem. The "level" or "height" of a vertex is its corank. Thus there are $or(s)$ vertices at level s. A vertex $[E]$ at level $s+1$ is connected to a vertex $[D]$ at level s if and only if $E \in \sqrt{D}$.

If it were true that each Kumpera-Ruiz class (see the end of the previous section) consisted of a single orbit, then this tree would be a simple binary tree. One branch of the vertex D would consist of the E for which $E(p) \neq L(D^2)(p)$, and the other for which $E(p) = L(D^2)(p)$. But the table given in section 1 shows that this is false. There are D for which $|[\sqrt{D}]| > 2$. Indeed, for $s = 7$ there are D whose $\sqrt{D}$ contains infinitely many nonequivalent germs, corresponding to $or(8) = \infty$.

In this section we reduce the problem of classification of $\sqrt{D}$ to classification of points of the circle $S^1 = RP^1$ with respect to the action of a certain group $\Gamma = \Gamma(D) \subset PGL(2)$ of projective transformations of the circle. The orbits in $\sqrt{D}$ correspond to the Γ-orbits in S^1. We will show that the number of orbits is either $2, 3, 4$ or ∞, according to the number of fixed points of Γ.

The first step in such reduction is the following proposition (proved in section 6).

Proposition 3.1. *Let E and $\tilde{E}$ be the germs at a point p of Goursat distributions of corank $s+1$ such that $E^2 = \tilde{E}^2$ and $E(p) = \tilde{E}(p)$. Then the germs E and $\tilde{E}$ are equivalent.*

Set

$$(\sqrt{D})(p) = \{E(p): \quad E \in \sqrt{D}\}.$$

Recall that the sandwich lemma asserts that $L(D)(p) \subset E(p) \subset D(p)$ for any $E \in \sqrt{D}$. Also recall that $codim\ L(D)(p) = 2$ in $D(p)$. In other words

$$(\sqrt{D})(p) \subset S^1_D(p) = \{subspaces\ V \subset T_pM: \quad codim\ V = s+1, \quad L(D)(p) \subset V \subset D(p)\}.$$

We use the notation $S^1_D(p)$ because this set is topologically a circle. Indeed it can be canonically identified with the set of all one-dimensional subspaces of the 2-dimensional factor space $D(p)/L(D)(p)$, which is to say with the real projective line. The real projective line is topologically a circle:

$$S^1_D(p) \cong P[D(p)/L(D)(p)] \cong RP^1 \cong S^1.$$

Lemma 3.1. $(\sqrt{D})(p) = S^1_D(p)$ *for any Goursat distribution germ D such that $rank(D) > 2$.*

Proof. We must show that every $V \in S^1_D(p)$ can be realized as $V = E(p)$ for some $E \in \sqrt{D}$. Since $rank(D) > 2$ and consequently $dim\ V > 1$ we can fix a nonvanishing 1-form ω which annihilates the involutive distribution $L(D)$, and for which $\omega(p)$ annihilates V, and for which $d\omega(p)$ restricted to V is nonzero. Define E to be the subdistribution of D annihilated by ω. We claim that $E^2 = D$, and consequently $V \in \sqrt{D}(p)$.

We first show that $E^2 \subset D$. Take two vector fields $X, Y \in E$, and any 1-form μ annihilating D. We must show that μ annihilates $[X, Y]$ or, equivalently that $d\mu(X, Y) = 0$. $L(D^2)$ is a corank one subdistribution of E. Pick any nonvanishing vector field Z tangent to E such that Z mod $L(D)$ spans $E/L(D)$ (near p). Then there are functions k_1, k_2 such that $X = k_1 Z$ modulo $L(D)$ and $Y = k_2 Z$ modulo $L(D)$. Since $L(D) \subset D$ and $L(D)$ is involutive any vector field in $L(D)$ belongs to the kernel of $d\mu$. Therefore $d\mu(X, Y) = d\mu(k_1 Z, k_2 Z) = 0$.

The fact that $d\omega|_V \neq 0$ implies that the rank of E^2 is greater than that of E. But $rank(E) = rank(D) - 1$ and $E \subset E^2 \subset D$. Consequently $E^2 = D$. Q.E.D.

Consider the group $Diff_p$ of all local diffeomorphisms with fixed point p and its subgroup $Symm_p(D)$ consisting of local symmetries of the germ at p of D:

$$Symm_p(D) = \{\Phi \in Diff_p : \quad \Phi_* D = D.\}$$

Any $\Phi \in Symm_p(D)$ automatically preserve the canonical foliation $L(D)$, and consequently it preserves $L(D)(p)$. Its derivative $d\Phi_p$ thus acts on the two-dimensional factor space $D(p)/L(D)(p)$ by a linear transformation, and consequently defines a transformation

$$g_\Phi : S^1_D(p) \to S^1_D(p); \quad g_\Phi.V = d\Phi_p(V); \quad V \in S^1_D(p).$$

This defines a group homomorphism

$$\Phi \mapsto g_\Phi; \quad Symm_p(D) \to PGl(2) = PGl(D(p)/L(D)(p)).$$

We denote the image of this homomorphism by

$$\Gamma_p(D) = \{g_\Phi, \quad \Phi \in Symm_p(D)\}.$$

Remark: $PGl(2)$ is the standard notation for the group of all invertible linear transformations of a two-dimensional vector space modulo scale. Elements of this group map lines to lines, and hence define transformations of $RP^1 = S^1$. These transformations are sometimes called *projectivities*. So $\Gamma_p(D)$ is a group of projectivities.

Proposition 3.1 and Lemma 3.1 imply:

Proposition 3.2. *Let D be the germ at a point p of a Goursat distribution of corank s. Let E and $\tilde{E}$ be the germs at p of Goursat distributions of corank $s+1$ such that $E^2 = \tilde{E}^2 = D$. The germs E and $\tilde{E}$ are equivalent if and only if the points $E(p)$ and $\tilde{E}(p)$ of the circle $S^1_D(p)$ belong to a single orbit with respect to the action of the group $\Gamma_p(D)$.*

The rest of this section is devoted to understanding the orbit structure of the action of $\Gamma_p(D)$ on the circle.

To understand the orbit structure we should first understand the fixed points of the action. By a *fixed point* $V \in S^1_D(p)$ we mean a point that is fixed by *every* transformation in the group $\Gamma_p(D)$. The set of all fixed points will be denoted $Fix_p(D)$:

$$Fix_p(D) = \{V \in S^1_D(p) : \quad g.V = V \ \textit{for any} \ g \in \Gamma_p(D)\}.$$

To reiterate $V \subset D(p)$ is a codimension 1 hyperplane which contains the codimension 2 hyperplane $L(D)(p)$, and $g.V = d\Phi_p(V)$ where $g = g_\Phi$, with $\Phi \in Symm_p(D)$.

The set $Fix_p(D)$ is never empty. Indeed, $Symm_p(D)$ preserves D^2, and hence $L(D^2)$. But $L(D^2)(p) \subset D(p)$ is a codimension 1 hyperplane, as we saw in the previous section (see the sandwich lemma). Consequently

$$L(D^2)(p) \in Fix_p(D)$$

for any Goursat distribution D.

On the other hand, *if $Fix_p(D)$ contain more than two points then $Fix_p(D) = S^1_D(p)$ – every point is a fixed point, and $\Gamma_p(D) = \{1\}$ consists of the identity transformation alone.* This follows immediately from what is sometimes called "the fundamental theorem of projective geometry": any projectivity of the projective line which fixes three or more points is the identity. At the level of linear algebra, this is the assertion that if a linear transformation of the plane R^2 has three distinct eigenspaces (the three alleged fixed points of the projective line) then that transformation is a scalar multiple of the identity.

We thus have the following possibilities.

- $\#(Fix_p(D)) = \infty$, in which case $\Gamma_p(D) = \{id\}$, and the number of inequivalent germs $E \in \sqrt{D}$ is infinite;
- $\#(Fix_p(D)) = 1$, in which case that single fixed point must be $L(D^2)(p)$;
- $\#(Fix_p(D)) = 2$, in which case the fixed points are $L(D^2)(p)$ and one other point.

The following proposition explores the middle possibilty.

Proposition 3.3. *If $\#(Fix_p(D)) = 1$ then $Fix_p(D) = \{L(D^2)(p)\}$. In this case the action of $\Gamma_p(D)$ is transitive away from the fixed point. That is to say, for any two points $V, \tilde{V} \in S^1_D(p)$ different from $L(D^2)(p)$ there exists a $g \in \Gamma_p(D)$ such that $g.V = \tilde{V}$. Consequently, the circle $S^1_D(p)$ consists of two orbits with respect to the group $\Gamma_p(D)$: the fixed point $L(D^2)(p)$ and all other points.*

The proof of this proposition, and the one following (Proposition 3.4) are based on the Lemma 3.2 immediately below. To appreciate the lemma, notice that the connected part of $PGL(2)$ consists of projective transformations of the form $exp(v)$ for some linear transformation v of $R^2 = D(p)/L(D)(p)$. Such a linear transformation can be viewed as a linear vector field on the plane, and hence a vector field v on the circle S^1. (The vector fields arising in this way are precisely the infinitesimal projective transformations.) The flow $exp(tv)$ of this vector field is a one-parameter group of projectivities connecting the identity to $exp(v)$. The set of such v forms the Lie algebra of $PGl(2)$, denoted $pgl(2)$.

Lemma 3.2. *The square of $\Gamma_p(D)$ is connected. In other words, if $g \in \Gamma_p(D)$, then $g^2 = g \circ g = exp\ (v)$ for some vector field $v \in pgl(2)$ on the circle $S^1_D(p)$ with the property that $exp\ (tv) \in \Gamma_p(D)$ for all $t \in R$.*

The proof of the Lemma is postponed to Appendix B.

We will now investigate the case in which $Fix_p(D)$ consists of two points: $L(D^2)(p)$ and some $V \neq L(D^2)(p)$.

Definition. *If $V \neq L(D^2)(p)$, let*

$$\sigma = \sigma(D, V, p) : \ S^1_D(p) \to S^1_D(p)$$

denote the projectivity induced by a reflection in the plane $D(p)/L(D)(p)$ whose fixed point set consists of the two points V and $L(D^2)(p)$ (mod $L(D(p))$).

We explain. Let $\alpha, \beta \in RP^1$ be two distinct points of the projective line. Choose coordinates for the plane R^2 so that α and β are the x and y coordinate axis, and let

$[x, y]$ be the standard homogeneous coordinates for RP^1 with respect to these axes. Then $\sigma([x, y]) = [x, -y]$, which corresponds to reflection about the x-axis. Note that $[x, -y] = [-x, y]$ so that we can also think of σ as reflection about the y-axis, β. One can characterize σ as the unique projectivity whose fixed point set is $\{\alpha, \beta\}$ and whose square is the identity.

If $Fix_p(D) = \{L(D^2)(p), V\}$ with $V \neq L(D^2)(p)$ then there are three alternative possibilities:

(a) $\Gamma_p(D)$ contains at least one more projectivity in addition to the identity and the reflection σ;

(b) $\Gamma_p(D)$ does not contain σ;

(c) $\Gamma_p(D) = \{id, \sigma\}$.

Proposition 3.4. *Suppose that $\#Fix_p(D)) = 2$, with $Fix_p(D) = \{L(D^2)(p), V\}$.*

(a). If $\Gamma_p(D)$ satisfies (a) above then it acts transitively on $S^1_D(p) \setminus Fix_p(D)$. The action has precisely three orbits, $\{L(D^2)(p)\}$, $\{V\}$, and $S^1_D(p) \setminus Fix_p(D)$.

(b). If $\Gamma_p(D)$ satisfies (b), then it acts transitively on each of the two connected components of $S^1_D(p) \setminus Fix_p(D)$, but does not mix points from the two components. The action has precisely 4 orbits, namely $\{L(D^2)(p)\}$, $\{V\}$ and the two connected components of $S^1_D(p) \setminus Fix_p(D))$.

(c). If $\Gamma_p(D)$ satisfies (c) then the number of distinct orbits is infinite. The orbit space is RP^1 modulo the action of the reflection σ, which is topologically a closed interval.

We summarize the results obtained so far into 5 cases:

(1). $Fix_p(D)$ consists of the single point $L(D^2)(p)$.

(2). $Fix_p(D)$ consists of two points, $L(D^2)(p)$ and some other point V. Then we have the following three subcases.

(2 a). $\sigma \in \Gamma_p(D)$ and $g \in \Gamma_p(D)$ for some $g \neq \sigma, id$.

(2b). $\sigma \notin \Gamma_p(D)$.

(2c). $\Gamma_p(D) = \{1, \sigma\}$ is the two-element group .

(3). $\Gamma_p(D) = \{id\}$ is the identity group. Every point of the circle $S^1_D(p)$ is fixed.

We reiterate that case (3) holds if and only if $Fix_p(D)$ contains at least 3 distinct points.

We recall that $\sqrt{D}$ denotes the set of all germs of Gours at distributions E of corank $s+1$ such that $E^2 = D$, where D is a given corank s Goursat distribution. The following statement is a corollary of Propositions 3.1 - 3.4.

Proposition 3.5. *Let D be the germ at a point p of a Goursat distribution, $rank(D) > 2$. Then one of the 5 cases (1), (2a), (2b), (2c), or (3) listed above holds. In each of these cases two germs $E, \tilde{E} \in \sqrt{D}$ are equivalent provided that $E(p) = \tilde{E}(p)$.*

In the case (3) E and $\tilde{E}$ are equivalent **only** *if $E(p) = \tilde{E}(p)$.*

Assume now that $E(p) \neq \tilde{E}(p)$. In cases (1) and (2,a) the germs E and $\tilde{E}$ are equivalent if and only if $E(p), \tilde{E}(p) \notin Fix_p(D)$. In case (2b) these germs are equivalent *if and only if $E(p)$ and $\tilde{E}(p)$ belong to the same connected component of the set $S^1_D(p) \setminus Fix_p(D)$. In case (2c) the germs are equivalent if and only if the reflection σ above takes $E(p)$ to $\tilde{E}(p)$.*

Write $\#\sqrt{D}$ for the number of distinct equivalence classes of germs for $E \in \sqrt{D}$. Consequent to the above analysis we have: $\#\sqrt{D} = 2$ in case (1), $\#\sqrt{D} = 3$ in case (2 a), $\#\sqrt{D} = 4$ in case (2 b), and $\#\sqrt{D} = \infty$ in cases (2 c) and (3).

This proposition does not solve the problem of classifying all Goursat distributions of any corank. Rather it reduces this problem to the problem of distinguishing among the 5 cases listed above. This reduction sheds light on the pre-existing classification results, as summarized in table 1. We expand on this theme in the next section.

We end this section by showing that Propositions 3.3 and 3.4 follow from Lemma 3.2. Consider the following subsets of $S^1(D)(p)$:

$$T = \{\alpha \in S^1(D)(p): \quad g^2.\alpha = \alpha \ for\ any\ g \in \Gamma_p(D)\}$$

$$T_1 = \{\alpha \in T: \quad g.\alpha \in T \ for\ any\ g \in \Gamma_p(D)\}.$$

Lemma 3.2 implies the following corollary.

Corollary to Lemma 3.2. *If $\beta \notin T_1$ then there exists a neighbourhood U of β in $S^1(D)(p)$ such that all points of U are $\Gamma_p(D)$-equivalent.*

Note that $Fix_p(D) \subset T_1 \subset T$ and that if T contains three different points then $T = S^1(D)(p)$. To prove Propositions 3.3 and 3.4 we consider the following cases.

1. Assume that $T \neq S^1(D)(p)$ and $Fix_p(D) = \{\alpha, \beta\}$. Then $T = T_1 = \{\alpha, \beta\}$. By the Corollary of Lemma 3.2 the group $\Gamma_p(D)$ either acts transitively on $S^1_D(p) \setminus Fix_p(D)$ or acts transitively on each of the two connected components of this set, but does not mix points from the two components. The first case holds if and only if the group $\Gamma_p(D)$ contains the reflection $\sigma = \sigma(\alpha, \beta)$ which fixes α and β. This corresponds to (a) and (b) of Proposition 3.4.

2. Assume that $T \neq S^1(D)(p)$ and $Fix_p(D) = \{\alpha\}$. If $T = \{\alpha, \alpha_1\}$, where $\alpha_1 \neq \alpha$ then $T_1 = \{\alpha\}$ since there exists $g \in \Gamma_p(D)$ such that $g.\alpha_1 \neq \alpha_1$ and $g.\alpha_1 \neq \alpha$ for any $g \in \Gamma_p(D)$. Thus $T_1 = \{\alpha\}$. By the Corollary of Lemma 3.2 the action of $\Gamma_p(D)$ is transitive away from α. This corresponds to Proposition 3.3.

3. Assume that $T = S^1(D)(p)$ and $Fix_p(D) = \{\alpha, \beta\}$. In this case the group $\Gamma_p(D)$ consists of the identity transformation and the reflection σ. The orbit space is the interval S^1/σ. This corresponds to (c) of Proposition 3.4.

4. Finally, let us show that the case $T = S^1(D)(p)$ and $Fix_p(D) = \{\alpha\}$ is impossible. Assume that this case holds. Then any projectivity $g \in \Gamma_p(D)$ has a fixed point α and satisfies the condition $g^2 = id$. It is easy to see that these conditions imply that any nonidenty $g \in \Gamma_p(D)$ is a reflection with two fixed points (one of them is α). $\Gamma_p(D)$ is a commutative group since $g^2 = id$ for any $g \in \Gamma_p(D)$. Now if two reflections with a common fixed point commute then they coincide. Therefore $\Gamma_p(D)$ consists of the identity

transformation and a single reflection. This contradicts the assumption that $Fix_p(D)$ consists of a single point.

Propositions 3.3 and 3.4 are proved.

4. Examples.

We give examples illustrating the notions of sections 2-3 and the classification table of section 1. Throughout this section all Goursat flags are germs at the origin in R^n.

Example 1. Let $D_s \subset D_{s-1} \subset \cdots \subset D_1$ be the Goursat flag described by 1-forms

$$\omega_1 = dy - z_1 dx,\ \omega_2 = dz_1 - z_2 dx,\ \ldots, \omega_s = dz_{s-1} - z_s dx. \qquad (C)$$

Using Lemma 2.2 we find:

$$L(D_s) = (dx, dy, dz_1, \ldots, dz_s)^\perp, \quad L(D_{s-1}) = (dx, dy, dz_1, \ldots, dz_{s-1})^\perp.$$

Since $D_s(0) = (dy, dz_1, dz_2, \ldots, dz_{s-1})^\perp$, the circle $S^1(D_s)(0)$ can be identified with the set of lines (1-dimensional subspaces) in the 2-space $span\ (\frac{\partial}{\partial x}, \frac{\partial}{\partial z_s}) \subset T_0R^n/L(D_s)(0)$. The line $span\ (\frac{\partial}{\partial z_s})$ corresponds to the space $L(D_{s-1})(0)$ and therefore it is a fixed point of $S^1(D_s)(0)$ with respect to the group $\Gamma_0(D_s)$. We show that this line is the only fixed point. The flag admits the local symmetry:

$$\Phi : \quad z_{s-i+1} \to z_{s-i+1} + x^i/i!,, \quad i = 1, 2 \ldots, s, \quad y \to y + x^{s+1}/(s+1)!,, \ x \to x.$$

This symmetry induces the projective transformation g_Φ of the circle $S^1(D_s)(0)$ which takes the line $span\ (a\frac{\partial}{\partial x} + b\frac{\partial}{\partial z_s})$ to the line $span\ (a\frac{\partial}{\partial x} + (b-a)\frac{\partial}{\partial z_s})$. These lines are different lines when $b \neq 0$.

This example, together with Proposition 3.3 has two immediate corollaries. Firstly, Proposition 2.1 (the geometric formulation of the Cartan theorem) follows by induction on s, with the Engel theorem $s = 2$ as the base of induction. Secondly, by restricting Example 1 to the case $s = 2$, and using Proposition 3.3 we can classify Goursat flags $D_3 \subset D_2 \subset D_1$ of length 3. Any such flag can be described either by the 1-forms

$$\omega_1 = dy - z_1 dx,\ \omega_2 = dz_1 - z_2 dx,\ \omega_3 = dz_2 - z_3 dx \qquad (4.1)$$

or by the 1-forms

$$\omega_1 = dy - z_1 dx,\ \omega_2 = dz_1 - z_2 dx,\ \omega_3 = dx - z_3 dz_2. \qquad (4.2)$$

The normal form (4.1) holds if $D_3(0) \neq L(D_1)(0)$ and the normal form (4.2) holds if $D_3(0) = L(D_1)(0)$.

Example 2. Consider the Goursat flag $D_3 \subset D_2 \subset D_1$ described by 1-forms (4.2). We have

$$L(D_3) = (dx, dy, dz_1, dz_2, dz_3)^\perp,\ L(D_2) = (dx, dy, dz_1, dz_2)^\perp,\ D_3(0) = (dy, dz_1, dx)^\perp.$$

Therefore the circle $S^1(D_3)(0)$ can be identified with the set of lines in the 2-space $span\ (\frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}) \subset T_0R^n/L(D_3)(0)$. The line $span\ (\frac{\partial}{\partial z_3})$ corresponding to $L(D_2)(0)$ is a fixed point with respect to the group $\Gamma_0(D_3)$. Let $Sing$ be the set of all singular points. We use the coordinate-free definition of a singular point from section 2. In this example $Sing$ consists of points p such that $D_3(p) = L(D_1)(p)$ and it is a smooth hypersurface given by the equation $z_3 = 0$. The space T_0Sing contains the space $L(D_3)(0)$, therefore the intersection $D_3(0) \cap T_0Sing$ is a point of the circle $S^1(D_3)(0)$. This point is the line $span\ (\frac{\partial}{\partial z_2})$. Since it is defined canonically, it is a fixed point with respect to the group $\Gamma_0(D_3)$. We have proved that the set $Fix_0(D_3)$ contains at least two points - $span\ (\frac{\partial}{\partial z_2})$ and $span\ (\frac{\partial}{\partial z_3})$. We show that there are no other fixed points. This follows from the existence of the local "scaling" symmetry

$$\Phi: \quad z_3 \to k^{-1}z_3,\ x \to kx,\ z_1 \to kz_1,\ y \to k^2y, \quad k \in R,\ k \neq 0.$$

This induces the projective transformation g_Φ of $S^1(D_3)(0)$ which takes the line $span\ (a\frac{\partial}{\partial z_2} + b\frac{\partial}{\partial z_3})$ to the line $span\ (a\frac{\partial}{\partial z_2} + kb\frac{\partial}{\partial z_3})$. These two lines are different provided $a, b \neq 0$, and $k \neq 1$. Finally, we note that the group $\Gamma_0(D_3)$ contains the reflection σ with fixed points $span\frac{\partial}{\partial z_2}$ and $span\frac{\partial}{\partial z_3}$. Indeed, $\sigma = g_\Phi$ where Φ is the scaling symmetry for $k = -1$.

Examples 1,2 and Propositions 3.3- .5 imply a complete classification of Goursat flags $D_4 \subset D_3 \subset D_2 \subset D_1$ of length 4: there are exactly 5 orbits with respect to the group of local diffeomorphisms corresponding to the following cases:

(A) $D_3(0) \neq L(D_1)(0),\ D_4(0) \neq L(D_2)(0)$
(B) $D_3(0) \neq L(D_1)(0),\ D_4(0) = L(D_2)(0)$
(C) $D_3(0) = L(D_1)(0),\ D_4(0) \neq L(D_2)(0),\ D_4(0) \not\subset T_0Sing$
(D) $D_3(0) = L(D_1)(0),\ D_4(0) \subset T_0Sing$
(E) $D_3(0) = L(D_1)(0),\ D_4(0) = L(D_2)(0)$

These cases do not intersect since the above coordinate computation showed that $L(D_2)(0) \not\subset T_0Sing$. The orbit (A) is open and corresponds to Cartan's normal form. Orbits (B) and (C) have codimension 1. Orbits (D) and (E) have codimension 2. The adjaciences are:

$$\begin{array}{ccccccc} & & & & A & & \\ & & & \nearrow & & \nwarrow & \\ & & C & & & & B \\ & \nearrow & & \nwarrow & & \nearrow & \\ D & & & & E & & \end{array}$$

Given a global Goursat flag of length 4 on a manifold, denote by $Sing_A, \ldots, Sing_E$ the set of points at which the corresponding singularity holds. It follows from Examples 1 and 2 that for any (not necessarily generic) global Goursat flag of length 4 on a manifold M the set $Sing_A$ is open and dense, that $Sing_B$ and $Sing_C$ are smooth hypersurfaces in M which intersect transversally forming $Sing_E$, and that $Sing_D$ is a smooth surface of codimension 1 within $Sing_C$ and disjoint from $Sing_E$.

The orbits A-E can be easily described by normal forms, using Lemma 2.2. Any Goursat flag of length 4 can be described locally by 1-forms $\omega_1, \ldots, \omega_4$, where $\omega_1, \omega_2, \omega_3$ have the form (4.1) for A- and B-singularities and the form (4.2) for the 3 other singularities, and where the 1-form ω_4 has the form

$$dz_3 - z_4 dx, \quad dx - z_4 dz_2, \quad dz_3 - (1 + z_4) dz_2, \quad dz_3 - z_4 dz_2, \text{ or } \quad dz_2 - z_4 dz_3$$

for the A,B,C,D,E-singularities respectively.

Example 3. To classify Goursat flags $D_5 \subset D_4 \subset D_3 \subset D_2 \subset D_1$ we find the set of fixed points of the circle $S^1(D_4)(0)$ under the action of $\Gamma_0(D_4)$. We start by assuming that the flag $D_4 \subset D_3 \subset D_2 \subset D_1$ has one of the 5 normal forms described above. Arguing in the same way as in Examples 1 and 2 we come to the following conclusions.

1. If the flag $D_4 \subset D_3 \subset D_2 \subset D_1$ has singularity A or singularity C then the set of fixed points of $S^1(D_4)(0)$ consists of the single point $L(D_3)(0)$ and therefore the space of germs of flags $D_5 \subset D_4 \subset D_3 \subset D_2 \subset D_1$ consists of two orbits corresponding to the cases

$(A_1 \text{ and } C_1)$ $D_5(0) \neq L(D_3)(0)$
$(A_2 \text{ and } C_2)$ $D_5(0) = L(D_3)(0)$

2. If the flag $D_4 \subset D_3 \subset D_2 \subset D_1$ has the singularity B (respectively D, E) then the set $Fix_0(D_4)$ consists of the point $L(D_3)(0)$ and the point $\alpha = D_4(0) \cap T_0 Sing_B$ (respectively $\alpha = D_4(0) \cap T_0 Sing_D$, $\alpha = D_4(0) \cap T_0 Sing_E$). The hypersurface $Sing_B$ and the codimension two submanifolds $Sing_D, Sing_E$ are tangent to the foliation $L(D_4)$, therefore the point α is a well-defined point of the circle $S^1(D_4)(0)$. The points α and $L(D_3)(0)$ are always different, and the group $\Gamma_0(D_4)$ admits the reflection with these two fixed points. Therefore the space of germs of flags $D_5 \subset D_4 \subset D_3 \subset D_2 \subset D_1$ such that the flag $D_4 \subset D_3 \subset D_2 \subset D_1$ has a fixed singularity within the singularities B,D,or E consists of 3 orbits corresponding to the cases

(B_1, D_1, E_1) $D_5(0) \neq L(D_3)(0)$, $D_5(0) \not\subset T_0 Sing_U$, $U = B, D, E$
(B_2, D_2, E_2) $D_5(0) \subset T_0 Sing_U$, $U = B, D, E$
(B_3, D_3, E_3) $D_5(0) = L(D_3)(0)$.

Thus the space of germs of Goursat flags of length 5 consists of 13 orbits. The A- and C-singularity each "decompose" into two new singularities. The B-, D- and E-singularities each decompose into three. The orbit A_1 is open. Orbits A_2, B_1, C_1 have codimension 1. Orbits B_2, B_3, E_1, C_2, D_1 have codimension 2. The deepest singularities are E_2, E_3, D_2, D_3. They have codimension 3. The graph of adjaciences can be easily derived. It is rather complicated and we do not present it here.

Example $s+1$**.** Of course, we could continue to get the classification of flags of length 6 or longer, or to get the classification of flags of any length satisfying certain genericity assumptions. The principle remains the same. If we know the normal form for a certain orbit Or_s of flags $D_s \subset D_{s-1} \subset \cdots \subset D_1$ of length s then we should find the set $Fix_0(D_s) \subset S^1(D_s)(0)$. If this set consists of two points, we should determine whether or not the group $\Gamma_0(D_s)$ admits the reflection σ with these two fixed points. This information together with the results of section 3 would then yield the classification of all flags $D_{s+1} \subset D_s \subset \cdots \subset D_1$ of length $s+1$ for which the subflag $D_s \subset D_{s-1} \subset \cdots \subset D_1$ belongs to the orbit Or_s. In many cases the necessary information regarding fixed points can be obtained without using normal forms for the orbit Or_s. This was the case in the description of Goursat flags of length ≤ 5 given above in examples 1, 2 and 3.

As an example of results for general length s, suppose that $D_s(0)$ is one of the fixed points of the previous circle $S^1(D_{s-1})(0)$. Then the next circle $S^1(D_s)(0)$ contains at least two fixed points, namely $L(D_{s-1})(0)$ together with the intersection of $D_s(0)$ with $T_0 Sing_*$, where $Sing_*$ is the subvariety of points where the germ of the flag $D_s \subset D_{s-1} \subset \ldots$ is equivalent to its germ at the origin. $Sing_*$ is a smooth submanifold which is tangent to $L(D_s)$ and transversal to $L(D_{s-1})$ as well as to $D_s(0)$, consequently this second fixed point is is well-defined and distinct from $L(D_{s-1})(0)$. Therefore, upon "prolonging" the D_s flag in order to investigate flags of length $s+1$, the resulting longer set of flags decompose into either 3, 4 or an infinite number of singularities. There are 3 if these two fixed points are the only two fixed points and if the group $\Gamma_0(D_s)$ admits the reflection σ. There are 4 if they are the only two fixed points but the reflection is not in $\Gamma_0(D_s)$. There are an infinite number of different germs if there is at least one more fixed point.

Unfortunately, for Goursat flags of arbitrary length we do not know of a general way of distinguishing the cases with of 1,2, or an infinite number of fixed points, nor of determining the presence or absence of the reflection in the case of 2 fixed points. If we knew such a method, then the whole "Goursat tree" would be completely classified.

The examples show that for flags of length $s \leq 4$ the number of fixed points of $S^1(D_s)(0)$ is either 1 or 2. In the latter case the group $\Gamma_0(D_s)$ admits the reflection σ with these two fixed points. This corresponds to the cases (1), (2a) in section 3. Interpreting Mormul's results [Mormul, 1987, 1988] in our language (see Appendix C) we see that the same holds for flags of length 5. The case (2b) of exactly two fixed points but no reflection is realized for a unique singularity of flags of length 6. This decomposes into 4 singularities of flags of length 7. The case (3) in which the group $\Gamma_0(D)$ consists of only the identity transformation is realized for at least one singularity of flags of length 7. It follows that upon prolongation of such flags to length 8, *the point $D_8(0)$ of the circle $S^1(D_7)(0)$ is a continuous modulus.* This accounts for the entry $or(8) = \infty$ in the table of section 1

We do not know if the case (2,c) in section 3 is realized. According to Mormul it is.

5. Prolongation and deprolongation. Monster Goursat manifold.

5.1. Prolongation

Prolongation builds new distributions from old. Let D be a rank 2 distribution on a manifold M. Its prolongation is a distribution on the new manifold

$$PD := \bigcup_{m \in M} P(D(m))$$

where $P(D(m))$ is the projectivization – the set of lines through the origin– of the two-plane $D(m)$. If D is a Goursat distribution of any rank we set

$$PD := \bigcup_{m \in M} S^1_D(m),$$

where the $S^1_D(m) = P(D(m)/L(D)(m))$ are the circles of section 2. If D is a rank 2 Goursat distribution then $L(D) = 0$ so that $S^1_D(m) = P(D(m))$ so that these two definitions match up. PD is a circle bundle over M.

We endow PD with a distribution E as follows. It is enough to describe what it means for a curve in PD to be tangent to E. A curve in PD consists of a moving pair $(m(t), V(t))$ where $m(t)$ is a point moving on M, and where $V(t)$ is a moving family of hyperplanes in $D(m(t))$, sandwiched as in the sandwich lemma in section 2: $L(D(m(t)) \subset V(t) \subset D(m(t))$. We declare the curve to be tangent to the distribution if and only if $\frac{dm}{dt} \in V(t)$. Equivalently, let

$$\pi : PD \to M$$

be the projection and $d\pi$ be its differential. Then

$$E(m, V) := d\pi_q^{-1}(V), \quad q = (m, V).$$

Definition. The manifold PD with distribution E is the *prolongation* of the distribution D on M.

Example. Let M be a surface and let $D = TM$, the whole tangent bundle to M. Then $PD = PTM$ consists of the space of tangent lines. Let x, y be local coordinates on M near a point m. Then a line $\ell \subset T_m M$ is described by its slope: $dy = zdx$. The new coordinate z is a fiber affine coordinate on $PTM \to M$. The distribution on PTM is defined by $dy - zdx = 0$. This is the standard contact form in three-dimensions. Indeed, PTM is canonically isomorphic to PT^*M, which has a well-known contact structure, and which is this prolongation.

Returning to the general rank 2 prolongation PD, let $\omega^1, \ldots, \omega^s$ be one-forms whose vanishing defines D. Complete these forms to a local co-framing of all of T^*M by adding two other one-forms, say dx and dy. Restricted to D_m, the forms dx and dy form a linear coordinate system. Then any line $\ell \subset D_m$ can be expressed in the form $adx + bdy = 0$, with $(a,b) \neq 0$. Thus $[a,b]$ form homogeneous coordinates on the projective line PD_m. One obtains a fiber affine coordinate by writing $[a,b] = [z,1]$. This z is defined away from the "vertical line" $dx = 0$ and is the negative of the slope: $z = -dy/dx$. Therefore z forms an affine fiber coordinate for the bundle $PD \to M$. The Pfaffian system describing the prolonged distribution on PD is $\pi^*\omega^i$, $i = 1, \ldots, s$ together with

$$\omega^{s+1} = dx + zdy.$$

The coordinate z breaks down in a neighborhood of the vertical lines. There we must switch to the other affine coordinate $\tilde{z}$ which is related to z by $\tilde{z} = -dx/dy = 1/z$ in their common domain. In such a "vertical" neighborhood we must use the form $\tilde{z}dx + dy$ instead of $dx + zdy$.

Proposition 5.1. *The prolongation E of a Goursat distribution D of rank k and corank s on a manifold M is a Goursat distribution of rank k and corank $s+1$ on the manifold PD. It satisfies $E^2 = \pi^*D$. If $rank(D) = 2$ then $L(E^2) = ker(d\pi)$, the vertical space for the fibration $PD \to M$.*

Proof. We only give the proof in the case $rank(D) = 2$. E is rank 2, so E^2 has rank at most 3. Now $E \subset \pi^*D$, where π^*D is the rank 3 distribution on PD defined by the vanishing of the $\pi^*\omega^i$ as above. Indeed, in terms of our coordinates

$$E = \{v \in \pi^*D : \omega^{s+1}(v) = 0\}$$

with $\omega^{s+1} = dx + zdy$ as above. $E^2 = \pi^*D$ because $d\omega^{s+1} = dz \wedge dy \neq 0 \mod \omega^{s+1}$. (See the proof of lemma 2.2.) Now $E^j = \pi^*D^{j-1}$, $j = 3, \ldots$, and they have the right rank, so the rest of the Goursat conditions follow. E is Goursat.

By definition, the vertical space $ker(d\pi)$ belongs to π^*D, and is involutive. Thus $ker\ (d\pi) \subset L(E^2)$. The equality $ker(d\pi) = L(E^2)$ now follows from the sandwich lemma and a dimension count. Alternatively, to get equality, use the fact that $E = \pi^*D$ is defined by the vanishing of the $\pi^*\omega^i$, and these forms are independent of the vertical direction. Consequently $L(E^2) = ker(d\pi)$. Q.E.D.

5.2. Deprolongation

The reverse of prolongation is deprolongation. Suppose that E is a distribution on a manifold Q, and that $L(E^2)$ is a constant rank foliation. Let us suppose that the leaf space

$$M = Q/L(E^2)$$

is a manifold, and that the projection

$$\pi : Q \to M$$

is a submersion. In this case we will say that the foliation $L(E^2)$ is *nice*. Since the vector fields in $L(E^2)$ leave E^2 invariant, the distribution E^2 pushes down to M. Set

$$D = \pi_* E^2$$

meaning that $D_{\pi(q)} = d\pi_q(E^2(q))$, $q \in Q$. To reiterate, the fact that the flows of $L(E^2)$ are symmetries of E^2 implies that the value of D at $m = \pi(q)$ is independent of the representative $m \in \pi^{-1}(q)$ which we choose. Note that we have a natural identification:

$$D_{\pi(q)} = E^2(q)/L(E^2(q)),$$

since $ker(d\pi_q) = L(E^2(q))$.

Suppose now that E is Goursat. Then $L(E^2)$ has codimension two within E^2, so that D is a two-plane field on M.

Proposition 5.2. *Assume that E is a Goursat distribution on a manifold Q with corank $s+1$ and arbitrary rank, and whose leaf space with respect to $L(E^2)$ is nice in the sense above. Then its deprolongation $D = \pi_* E^2$ is a corank s Goursat distribution of rank 2 on the quotient manifold $M = Q/L(E^2)$.*

Proof. The distributions E^k, $k \geq 2$, defined by the inductive relation $D^{k+1} = [D^k, D^k]$, also are invariant under the flows of $L(E^2)$, since $L(E^2) \subset L(E^k)$ for $k \geq 2$. It follows that these E^k push down to M. One easily checks that $D^j = \pi_* E^{j+1}$ and that $rank(D^j) = 2 + j$. Q.E.D.

Local deprolongation. If the foliation by $L(E^2)$ is not nice, we can still deprolong locally. To proceed, restrict E to a small enough open subset of $U \subset Q$. For example we could take U to be a flow-box for $L(E^2)$, in which case $U \cong U_1 \times U_2$ with the leaves of $L(E^2)$ corresponding to $U_1 \times \{m\}$. (U_1 is an interval when $dim(L(E^2)) = 1$.) The restriction of $L(E^2)$ to U is nice, so that we can proceed with deprolongation. We will call the deprolongation $\pi_* E^2$ of $E|_U$ a *local deprolongation*. The germ of a local deprolongation near a particular leaf of $L(E^2)$ is independent of the choice of neighborhood U since the flows along $L(E^2)$ preserve E^2. Thus we can speak of the **deprolonged germ** of any Goursat distribution.

5.3. Prolongation and deprolongation are inverses

Deprolongation changes rank from r to 2, whereas prolongation preserves the rank of the distribution, so these two constructions cannot literally be inverses. Rather they are inverses "modulo trivial factors". We say that two distribution germs D on M and $\tilde{D}$ on $\tilde{M}$ are *the same modulo trivial factors* if there are integers k, m such that the distribution germs $D \times R^k$ on $M \times R^k$ and $\tilde{D} \times R^m$ on $\tilde{M} \times R^m$ are diffeomorphic . Recall that Zhitomirskii's theorem (section 1, following the table) asserts that any Goursat germ is the same, modulo a trivial factor, to one of rank 2.

Proposition 5.3. *The deprolongation of the prolongation of a rank 2 distribution is diffeomorphic to the original. The converse is true locally: modulo trivial factors, the germ of the prolongation of the deprolonged germ of a Goursat distribution of any rank is diffeomorphic to the original.*

Proof. Let E be the prolongation of the Goursat distribution D on M. The leaves of $L(E^2)$ are the fibers PD_m of the fibration $\pi : PD \to M$, so that M itself is canonically identified with the leaf space $PD/L(E^2)$. Now $\pi^* D = E^2$ by the previous proposition, and $\pi_* \pi^* D = D$. This proves that the deprolongation of the prolongation is the original.

Conversely, suppose that $\pi : U \to M$ is a local deprolongation, where E is the rank 2 Goursat distribution on U, and $D = \pi_*(E^2)$ is its deprolonged distribution. Write $m = \pi(u)$, with $u \in U$. Then

$$d\pi_u(E_u) \subset D_m$$

is a one-dimensional subspace – an element of PD_m. Thus $u \to d\pi_u(E_u)$ defines a map

$$\Phi : U \to PD$$

from the original Goursat manifold to the prolongation PD of its (local) deprolongation. We claim that Φ is a local diffeomorphism. Indeed, Φ is a fiber bundle map over M, so all we need to check is that the restriction of its differential to $L(E^2)_u$, the tangent space to the fiber of $\pi : U \to M$ at u is onto. Moving along the leaf $\ell = \pi^{-1}(m)$ of $L(E^2)$ corresponds to flowing with respect to a nonzero vector field $W \in L(E^2)$. So we want to show that $d\Phi_u(W_u) \neq 0$. Complete W to a local frame $\{W, X\}$ for E near u. Then $[W, X](u) \neq 0$, mod E_u since $E_u^2 \neq E_u$. This is equivalent to the condition that $d\Phi_u(W_u) \neq 0$. Finally, one easily checks that Φ maps E to the prolongation of D. Q.E.D.

5.4. Monster Goursat manifold. Proof of Theorem 1.

Suppose that we had a Goursat distribution of corank s on a manifold M with the property that **every** corank s Goursat germ was represented by some point of the manifold. Then the prolongation of M would enjoy the same property, but now among corank $s+1$ Goursat distribution germs! For if we are given any corank $s+1$ Goursat distribution, its deprolongation is represented by some point of M, by hypothesis. And by proposition 5.3, upon prolonging this deprolongation we arrive at a germ diffeomorphic to the original. **There is such an M in the corank 2 case.** Indeed, in this case, there is only one corank 2, rank 2 Goursat germ up to diffeomorphism. This is the Engel germ. Thus any Engel distribution on a 4-manifold will serve for M, with $s = 2$. It follows that **every Goursat germ of corank** $s+2$ is realized within the s-fold prolongation of an Engel distribution!

Now an Engel distribution can be obtained by prolonging a contact structure on a three-manifold. And a contact three-manifold can be obtained by prolonging the tangent bundle to a surface (see the example of section 5.1). We have proved

Every corank s Goursat germ can be found, up to a diffeomorphism, within the s-fold prolongation of the tangent bundle to a surface.

We have called this s-fold prolongation the "monster manifold". It is a very tame monster in many respects. Theorem 1 is proved.

Remark. The direction of this section is in some sense opposite to that of sections 3 and 4. In this section we imagine building Goursat distributions up from below by prolonging, beginning with a surface. In sections 3 and 4 we think of building Goursat distributions "down from above" by taking a corank s Goursat flag, beginning with $s = 2$, and examining all possible "extensions" or "square roots" of its corank s generator D_s, thus filling out out the Goursat flag to one of length $s+1$. Now, the prolongation E of a Goursat distribution D is a square root of $\pi^* D$ (see Proposition 5.1), so the two approaches are really the same.

6. Proof of Theorems 2 and 3.

In this section we prove Proposition 3.1 and Theorems 2 and 3. We will use the following notation. Given a distribution D and 1-form ω on a manifold M, with $\omega|_D \neq 0$, (D, ω) will denote the subbundle $E \subset D$ for which $E(p) = \{X_p \in D(p) : \ \omega(X_p) = 0.\}$. (If $\omega|_D$ is allowed to vanish at some points, then (D, ω) is not a subbundle, but rather a a subsheaf.)

The proof of Theorem 3 is based on our generalized Gray's theorem (Theorem A.2 in Appendix A) and the following proposition.

Proposition 6.1. *Let $F : \ D_s \subset D_{s-1} \subset \cdots \subset D_1$ and $F_N : \ D_{N,s} \subset D_{s-1} \subset \cdots \subset D_1$ be two Goursat flags on the same manifold whose distributions agree except at the largest corank, corank s. Suppose that $D_s = (D_{s-1}, \omega)$ and that $D_{N,s} = (D_{s-1}, \omega_N)$, for 1-forms ω and ω_N. Assume that $\omega_N \to \omega$ in the C^l-Whitney topology, $l \geq 1$. Then there exist global diffeomorphisms Φ_N such that $\Phi_N \to id$ in the C^l-Whitney topology and $(\Phi_N)_* F_N = F$ for sufficiently big N.*

We also need the following local version of this Proposition.

Proposition 6.2.

Part 1. (for germs at a nonfixed point). *Assume the flags F and F_N are the same as in Proposition 6.1, but the condition $\omega_N \to \omega$ is replaced by the condition $j^l_p \omega_N \to j^l_p \omega$ for some point p. Let U be any neighbourhood of the point p. Then for sufficiently large N there exist open sets (possibly disjoint) $U_1^N, U_2^N \subset U$ with $p \in U_1^N$ and a diffeomorphism $\Phi_N : U_1^N \to U_2^N$ which sends the flag F_N restricted to U_1^N to the flag F restricted to U_2^N, and satisfies $j^l_p \Phi_N \to j^l_p id$ as $N \to \infty$.*

Part 2. (for germs at a fixed point). *Fix N and assume that the Goursat flags F and F_N are the same as in Proposition 6.1. Assume also that $j^l_p \omega_N = j^l_p \omega$ for some point p and $l \geq 0$. Then there exists a local diffeomorphism Φ preserving the point p, sending the germ at p of F_N to the germ at p of F and such that if $l \geq 1$ then $j^l_0 \Phi = id$.*

Remarks.

1. Note that in part 1 we may have $\Phi_N(p) \neq p$ for all N. To make sense of the condition $j_p^{l-1}\Phi_N \to j_p^{l-1} id$ one should take U to be a coordinate neighborhood and identify the ℓ-th jet with the ℓ-th order Taylor expansion of Φ_N.

2. Proposition 6.1 and the first part of Proposition 6.2 hold for $l \geq 1$ whereas the second part of Proposition 6.2 also covers the case $l = 0$. This difference is essential. The case $l = 0$ is necessary for the proof of Proposition 3.1 and the proof of s-determinacy in Theorem 2.

Proof of Proposition 3.1. This is the case $l = 0$ of Proposition 6.2, part 2.

Proof of Theorem 3. Let $F : D_s \subset D_{s-1} \subset \cdots \subset D_2 \subset D_1$ and $\tilde{F} : \tilde{D}_s \subset \tilde{D}_{s-1} \subset \cdots \subset \tilde{D}_2 \subset \tilde{D}_1$ be Goursat flags on manifold M described by C^{s+1}-close tuples $\omega_1, \ldots, \omega_s$ and $\tilde{\omega}_1, \ldots, \tilde{\omega}_s$ of 1-forms. Assume that the foliations $L(D_1)$ and $L(\tilde{D}_1)$ are the same. By Theorem A.2 (Appendix A) there exists a C^s-close to the identity diffeomorphism Φ_1 of M which brings $\tilde{D}_1$ to D_1. This diffeomorphism brings the flag $\tilde{F}$ to the flag $(\Phi_1)_*\tilde{F} : (\Phi_1)_*\tilde{D}_s \subset (\Phi_1)_*\tilde{D}_{s-1} \subset \cdots \subset (\Phi_1)_*\tilde{D}_2 \subset D_1$ described by the tuple of 1-forms $\omega_1, \Phi_1^*\tilde{\omega}_2, \ldots, \Phi_1^*\tilde{\omega}_s$ which is C^{s-1}-close to the tuple $\omega_1, \omega_2, \ldots, \omega_s$. Now we apply Proposition 6.1 with $s = 2$ there and the ℓ there equal to the current $s - 1$. It guarantees the existence of a C^{s-1}-small diffeomorphism Φ_2 which brings the length 2 flag $(\Phi_1)_*\tilde{D}_2 \subset D_1$ to the flag $D_2 \subset D_1$. This diffeomorphism brings the flag $(\Phi_1)_*\tilde{F}$ to the flag $(\Phi_2\Phi_1)_*\tilde{F} : (\Phi_2\Phi_1)_*\tilde{D}_s \subset (\Phi_2\Phi_1)_*\tilde{D}_{s-1} \subset \cdots \subset (\Phi_2\Phi_1)_*\tilde{D}_3 \subset D_2 \subset D_1$ described by the tuple of 1-forms $\omega_1, \omega_2, (\Phi_2\Phi_1)^*\tilde{\omega}_3, \ldots, (\Phi_2\Phi_1)^*\tilde{\omega}_s$ which is C^{s-2}-close to the tuple $\omega_1, \omega_2, \omega_3, \ldots, \omega_s$. Continue applying Proposition 6.1 $(s - 3)$ times more we to obtain a sequence of diffeomorphisms $\Phi_3, \ldots, \Phi_{s-1}$ for which the composition $\Phi_{s-1}\Phi_{s-2}\cdots\Phi_1$ brings the flag $\tilde{F}$ to the flag $\hat{F}$ described by 1-forms $\omega_1, \omega_2, \ldots, \omega_{s-1}, \hat{\omega}_s$, where $\hat{\omega}_s = (\Phi_{s-1}\Phi_{s-2}\cdots\Phi_1)^*\tilde{\omega}_s$. The 1-forms $\hat{\omega}_s$ and ω_s are C^1-close. Using Proposition 6.1 for one last time we obtain a diffeomorphism Φ_s which brings the flag $\hat{F}$ to the flag F. The diffeomorphism $\Phi_s\Phi_{s-1}\Phi_{s-2}\cdots\Phi_1$ brings the flag $\tilde{F}$ to the flag F. Q.E.D.

Proof of Theorem 2 – structural stability. This follows from Theorem A.3 part 1 and the Proposition 6.2 part 1 in the same way that Theorem 3 followed from Theorem A.2 and Proposition 6.1.

Proof of Theorem 2– s-determinacy.

The proof is essentially the same as the proof of theorem 3 above, except we use Theorem A.3, part 2 instead of theorem A.2, and the second part of Proposition 6.2 instead of Proposition 6.1. Namely, we start with two germs F and $\tilde{F}$ at a fixed point p of Goursat flags of length s described by s-tuples of 1-forms $\omega_1, \ldots, \omega_{s-1}, \omega_s$ and $\tilde{\omega}_1, \ldots, \tilde{\omega}_{s-1}, \tilde{\omega}_s$ as in the proof of theorem 3 above, and having the same s-jets at p. Using Theorem A.3, part 2 and then Proposition 6.2, part 2, $s - 2$ times we conclude that $\tilde{F}$ is equivalent to the germ of another Goursat flag $\hat{F}$ at p, where $\hat{F}$ is described by the tuple of 1-forms $\omega_1, \ldots, \omega_{s-1}, \hat{\omega}_s$ and where $\hat{\omega}_s(p) = \omega(p)$. Now apply Proposition 6.2, part 2, with $l = 0$ to conclude that the germ of $\hat{F}$ is equivalent to the germ of F.

Proof of Proposition 6.1. The proof will consist of three steps.

First step. We will show that for sufficiently large N the flag

$$F_{N,t}: \ D_{s,N,t} \subset D_{s-1} \subset \cdots \subset D_1, \quad D_{s,N,t} = (D_s, \omega_{N,t}), \quad \omega_{N,t} = \omega + t(\omega_N - \omega)$$

is a Goursat flag for any $t \in [0,1]$. To show this we have to check the following statements:

(a) $\omega_{N,t}|_{D_{s,N,t}(p)}$ is a nonzero 1-form for any $p \in M$, $t \in [0,1]$ and sufficiently large N;

(b) $d\omega_{N,t}|_{D_{s,N,t}(p)}$ is a nonzero 2-form for any $p \in M$, $t \in [0,1]$ and sufficiently large N;

(c) if μ is a 1-form annihilating the distribution D_{s-1} then $d\mu|_{D_{s,N,t}(p)} = 0$ for any N, any $p \in M$ and $t \in [0,1]$.

The statements (a) and (b) follow from the fact that they are valid for $t = 0$, the condition that ω_N tends to ω in the C^1-Whitney topology (here we use that $l \geq 1$ in the formulation of Proposition 6.1), and the observation that the hyperplane $D_{s,N,t}(p)$ as well as the restrictions of the forms $\omega_{N,t}$ and $d\omega_{N,t}$ to this hyperplane depend on the 1-jet at p of the form $\omega_{N,t}$ only.

To prove (c) we consider the space $L(D_{s-1})(p)$. By the sandwich lemma 2.1 it is a codimension 2 subspace of $D_{s-1}(p)$ and the 1-forms ω and ω_N annihilate this space. Therefore $\omega_{N,t}$ annihilates $L(D_{s-1})(p)$ for all t, i.e. $L(D_{s-1})(p)$ is a hyperplane in $D_{s,N,t}(p)$, independent of N. Because $L(D_{s-1})(p)$ is the kernel of the 2-form $d\mu$ restricted to $D_{s-1}(p)$, where μ annihilates D_{s-1} but not D_{s-2} (see Lemma 2.2), any hyperplane in $D_{s-1}(p)$ containing $L(D_{s-1})(p)$ is isotropic for $d\mu$. In particular, $D_{s,N,t}(p)$ is isotropic for $d\mu$.

Second step. We have proved that $F_{N,t}$ is a Goursat flag for sufficiently large N and all $t \in [0,1]$. In what follows assume that N is sufficiently large. Now we start to construct a path $\Phi_{N,t}$ of global diffeomorphisms such that $(\Phi_{N,t})_* F_{N,t} = F_{N,0} = F$ and in particular $(\Phi_{N,1})_* F_N = F$. We use the homotopy method. The second step of the proof is to reduce the construction of $\Phi_{N,t}$ to the construction of a path $X_{N,t}$ of global vector fields satisfying the linear equations

$$(X_{N,t} \rfloor d\omega_{N,t} + \omega_N - \omega)|_{D_{s,N,t}} = 0, \quad X_{N,t} \in L(D_{s-1}). \tag{6.1}$$

Assume that $X_{N,t}$ satisfies (6.1). Consider the following ordinary differential equation and the initial condition with a parameter $p \in M$:

$$\frac{d\Phi_{N,t}(p)}{dt} = X_{N,t}(\Phi_{N,t}(p)), \quad \Phi_{N,0}(p) = p, \ p \in M. \tag{6.2}$$

Since M is a compact manifold and t varies on the compact segment $[0,1]$, the solution of (6.2) is a path $\Phi_{N,t}$ of global diffeomorphisms on M. Let us show that $(\Phi_{N,t})_* F_{N,t} = F_{N,0}$. The condition $X_{N,t} \in L(D_{s-1})$ implies that $\Phi_{N,t}$ preserves the distribution D_{s-1}. Therefore to show that $(\Phi_{N,t})_* F_t = F_0$ it is suffices to show that there exists a path $H_{N,t}$ of nonvanishing functions such that

$$(H_{N,t} \Phi^*_{N,t} \omega_{N,t} - \omega_0)|_{D_{s-1}} \equiv 0. \tag{6.3}$$

We will seek for $H_{N,t}$ in the form $H_{N,t} = e^{h_{N,t}}$, where $h_{N,0}$ is a function identically equal to 1. Let $A_{N,t} = H_{N,t}\Phi^*_{N,t}\omega_{N,t} - \omega_0$. Then $A_{N,0}$ is the zero 1-form and therefore (6.3) can be replaced by the equation $(\frac{dA_{N,t}}{dt})|_{D_{s-1}} \equiv 0$. We have

$$\frac{dA_{N,t}}{dt} = H_{N,t}\frac{dh_{N,t}}{dt}\Phi^*_{N,t}\omega_{N,t} + H_{N,t}\Phi^*_{N,t}(\mathcal{L}_{X_{N,t}}\omega_{N,t} + \frac{d\omega_{N,t}}{dt}),$$

where $\mathcal{L}_{X_{N,t}}$ is the Lie derivative for the vector field $X_{N,t}$. Let $q_{N,t}$ be a path of functions on M such that $\frac{dh_{N,t}}{dt} = q_{N,t}(\Phi_{N,t})$. Then the equation $(\frac{dA_{N,t}}{dt})|_{D_{s-1}} \equiv 0$ is equivalent to the equation

$$(q_{N,t}\omega_{N,t} + \mathcal{L}_{X_{N,t}}\omega_{N,t} + \omega_N - \omega)|_{D_{s-1}} = 0 \tag{6.4}$$

with respect to the path of functions $q_{N,t}$. By the sandwich lemma $L(D_{s-1})$ is a subset of $D_{s,t}$ for all t. Therefore $\omega_{N,t}$ annihilates $X_{N,t} \in L(D_{s-1})$. It follows that $\mathcal{L}_{X_{N,t}}\omega_{N,t} = X_{N,t}\rfloor d\omega_{N,t}$. Then (6.4) can be written in the form

$$(q_{N,t}\omega_{N,t} + X_{N,t}\rfloor d\omega_{N,t} + \omega_N - \omega)|_{D_{s-1}} = 0.$$

This equation has a solution $q_{N,t}$ due to the relation (6.1), and the definition of $D_{s,N,t}$.

Third step. Note that the diffeomorphisms $\Phi_{N,t}$ defined by the ordinary differential equation (6.2) tend to the identity diffeomorphism as $N \to \infty$ in the same topology in which $X_{N,t} \to 0$. Therefore to finish the proof of Proposition 6.1 it suffices to prove that (6.1) has a solution $X_{N,t}$ tending to the zero vector field as $N \to \infty$ in the C^l-Whitney topology. The third step of the proof is to construct such $X_{N,t}$.

Fix a Riemannian metrics on M. Let $V_{N,t}(p) \subset D_{s,N,t}(p)$ be the orthogonal complement to $L(D_{s,N,t})(p)$ within $D_{s,N,t}(p)$ with respect to this metric. By Lemmata 2.1 and 2.2 $dim\ V_{N,t}(p) = 2$ and $rank(d\omega_{N,t})|_{V_{N,t}(p)} = 2$. Therefore there is a unique vector $X_{p,N,t} \in V_{N,t}(p)$ such that

$$(X_{p,N,t}\rfloor d\omega_{N,t} + \omega_N - \omega)|_{V_t(p)} = 0, \quad p \in M, t \in [0,1]. \tag{6.5}$$

Set $X_{N,t}(p) = X_{p,N,t}$. Since $\omega_N - \omega$ tends to 0 in the C^l-Whitney topology, $X_{N,t} \to 0$ as $N \to \infty$ in the same topology. We will show that the path $X_{N,t}$ satisfies (6.1). This will complete the proof of Proposition 6.1.

Since $L(D_{s,t}) \oplus V_t = D_{s,N,t}$ the first condition in (6.1), which is to say the validity of equation there, follows immediately from (6.5) once we have shown that all the forms in that equation, namely $d\omega_{N,t}, \omega_N$ and ω annihilate $L(D_{s,t})$. The fact that $d\omega_{N,t}$ annihilates any vector in $L(D_{s,t})$ is contained in Lemma 2.2. To prove that ω and ω_N annihilate, use the sandwich lemma 2.1 twice to conclude that $L(D_{s-1})(p)$ is contained in both $D_s(p)$ and in $D_{s,N}(p)$. Therefore ω and ω_N annihilate the space $L(D_{s-1})(p)$. But the sandwich lemma also gives $L(D_{s,t})(p) \subset L(D_{s-1})(p)$, and therefore these forms annihilate $L(D_{s,t})(p)$.

It remains to prove the inclusion $X_{N,t} \in L(D_{s-1})$ of equation (6.1). The validity of the first equation in (6.1) and the fact that ω and ω_N annihilate the space $L(D_{s-1})(p)$ imply

$$(X_{p,N,t}\rfloor d\omega_{N,t})|_{L(D_{s-1})(p)} = 0. \tag{6.6}$$

By the sandwich lemma $L(D_{s-1})(p)$ is a hyperplane in $D_{s,N,t}$. Every such hyperplane is isotropic, so (6.6) implies that either $X_{p,N,t} \in L(D_{s-1})(p)$ or that $X_{p,N,t}$ is a nonzero vector in the kernel of the 2-form $(d\omega_{N,t})|_{D_{s,N,t}}$. The latter possibility is excluded by the condition $X_{p,N,t} \in V_{N,t}(p)$, the orthogonal complement to $L(D_{s,N,t})(p) = ker(d\omega_{N,t})|_{D_{s,N,t}}$. Proposition 6.1 is now proved. Q.E.D.

Proof of Proposition 6.2. The proofs of the statements of Proposition 6.2 with $\ell > 0$ are almost the same as as the proof we have just given. The difference occurs mainly in the construction of the diffeomorphism $\Phi_{N,t}$ by the ordinary differential equation (6.2). Concerning the case of part 1, the problem is that if $X_{N,t}$ is a time-dependent vector fields on a neighborhood U of a point p then its flow will typically map out of that neighborhood – hence the business with domains U_i^N in part 1. Although there may be no single flow $\Phi_{N,t},\ t \in [0,1]$ of diffeomorphisms on a single neighborhood of p, nevertheless, for N large the vector $X_{N,t}(p)$ is sufficiently close to zero so that its solution defines diffeomorphisms $\Phi_{N,t} : U_1^N \to U_{2,t}^N$, $t \in [0,1]$, where U_1^N is a neighbourhood of p contained in U and $U_{2,t}^N$ is an open subset of U (which may or may not contain p).

In the case of part 2 we have to show that $\Phi_t(p) = p$ and $U_{2,t}$ contains p. This follows because $X_{N,t}(p) = 0$ for all t.

The proof of Proposition 6.2 part 2 with $l = 0$ is also the same, except that we meet a difficulty in the first step of the proof. We have to show that the restriction $\theta_t(p)$ of the form $d\omega + t(d\tilde{\omega} - d\omega)$ to the space $D_s(p) = \tilde{D}_s(p)$ does not vanish for all $t \in [0,1]$. This is true for $t = 0$ and $t = 1$, but if $l = 0$ then $d\omega(p)$ might not be close to $d\tilde{\omega}(p)$ even in the C^0-topology and consequently $\theta_t(p)$ might vanish for some $t \in (0,1)$. Since $\theta_t(p)$ depends linearly on t, this is impossible if θ_0 and θ_1 define the same orientation of the 2-space $D_s(p)/L(D_s)(p)$ (the orientations are well-defined since $L(D_s)(p)$ is the kernel of $\theta_0(p)$ and $\theta_1(p)$). If the orientations are different then we have to show the existence of a symmetry of the germ at p of the distribution D_{s-1} which also preserves $\omega(p)$ and the foliation $L(D_s)(p)$ and changes the defined above orientation. We can find local coordinates centered at p such that $L(D_{s-1}) = (dx_1, \ldots, dx_{s+1})^\perp$ and $L(D_s) = (dx_1, \ldots, dx_{s+1}, dx_{s+2})^\perp$, and such that the forms defining the D_i can be taken to be independent of x_j, $j \geq s+2$. It follows from the sandwich lemma that the diffeomorphism $x_{s+2} \to -x_{s+2}$ is a symmetry of the required type.

Appendix A. Generalization of the Gray Theorem

Gray's theorem states that for any path of global contact structures $D_t,\ t \in [0,1]$ on an odd-dimensional manifold M there exists a family of global diffeomorphisms $\Phi_t : M \to M$ such that $(\Phi_t)_* D_t = D_0,\ t \in [0,1]$. See [Gray, 1959]. It follows that two global contact structures D and $\tilde{D}$ are equivalent provided that $\tilde{D}$ is sufficiently close to D in the Whitney C^1-topology.

In this section we generalize Gray's theorem to corank one distributions D of any *constant class*. Let ω be any nonvanishing 1-form describing D near p. By the *class* of D at p we will mean the odd number $2r+1$ such that $\omega \wedge (d\omega)^r(p) \neq 0$ and $\omega \wedge (d\omega)^{r+1}(p) = 0$. The even integer $2r$ is the rank of the restriction of the two-form $d\omega_p$ to D_p.

A corank one distribution has *constant class* if this class $2r+1$ does not depend on the point $p \in M$. The definition of the class is due to [Frobenius, 1887] and [Cartan, 1899].

For example, the class of a contact structure is the dimension of the underlying manifold. The maximal possible class of a corank one distribution on a manifold of even dimension $2k$ is $2k-1$. Such a distribution is called a quasi-contact , or even-contact, structure. A foliation of codimension one has class 1, the minimal possible class. In section 2 we proved that the corank one distribution D_1 of a Goursat flag has constant class 3.

Recall that the *characteristic foliation* $L(D)$ of the distribution D is the foliation generated by vector fields $X \in D$ such that $[X, D] \subset D$, i.e. $[X, Y] \in D$ for any $Y \in D$. The characteristic foliation $L(D) \subset D$ for a corank 1 distribution D of constant class $2r+1$ has codimension $2r$ **within** D. It is the kernel of the 2-form $d\omega|_{D(p)}$, where ω is as above. (See the proof of Lemma 2.2.) This kernel coincides with the kernel of the $(2r+1)$-form $\omega \wedge (d\omega)^r(p)$ on the space T_pM. (By the kernel of an exterior q-form on a vector space we mean the subspace of vectors v such that the form annihilates every q-tuple of vectors containing v.)

For example, the characteristic foliation of a quasi-contact structure is a line field. The characteristic foliation of a contact structure is trivial: it is the zero section of the tangent bundle. The characteristic foliation of an involutive corank one distribution is the distribution itself. The characteristic foliation of the corank one distribution of a Goursat flag has codimension 3 within the manifold.

The following theorems generalizes Gray's theorem. By a *cooriented corank one distribution* we mean a distribution which can be globally described by a 1-form.

Theorem A.1. *Let D_t be a path of cooriented corank one distributions on a compact manifold M of constant class $2r+1$ such that $L(D_t) = L(D_0)$, $t \in [0,1]$. Then there exists a path Φ_t of global diffeomorphisms of M such that $(\Phi_t)_* D_t = D_0$, $t \in [0,1]$.*

For quasi-contact structures Theorem A.1 is known to specialists, although is unpublished to our knowledge. ,

Using Theorem A.1 we obtain Theorem A.2 below. We need it for our proofs of Theorems 2 and 3 in the body of the present paper, where it is applied to the case of corank one distributions of constant class 3.

Theorem A.2. *Let D and D_N, $N = 1, 2, \ldots$ be cooriented corank one distributions on a compact manifold M of constant class $2r+1$ such that $D_N \to D$ as $N \to \infty$ in the C^{l+1}-Whitney topology, $l \geq 1$, and $L(D_N) = L(D)$ for all N. Then there exists a sequence Φ_N of global diffeomorphisms of M such that $\Phi_N \to id$ as $N \to \infty$ in the C^l-Whitney topology and $(\Phi_N)_* D_N = D_0$ for sufficiently big N.*

Proof of Theorem A.1. Fix a Riemannian structure on M. For $p \in M$, denote by $V_t(p) \subset D_t(p)$ the $2r$-dimensional subspace of $D_t(p)$ which is the orthogonal complement to $L(D_t)(p)$ with respect to this metric. Let ω_t be the path of 1-forms describing D_t. The form 2-form $d\omega_t|_{V_t(p)}$ is nondegenerate because $L(D_t) = ker\ d\omega_t|_{D_t(p)}$. Therefore the equation $(X_t(p) \rfloor d\omega_t)|_{V_t(p)} = \mu_t(p)$ has a unique solution $X_t(p) \in V_t(p)$ for any 1-form $\mu_t(p)$ on $V_t(p)$. We need this solution when $\mu_t = -\frac{d\omega_t}{dt}|_{V_t(p)}$. The solution $X_t(p)$ depends smoothly (analytically) on the point p and on t, and so defines a smooth (analytic) path

X_t of vector fields on M. The relation $X_t \rfloor d\omega_t = -\frac{d\omega_t}{dt}$ in fact holds upon restriction to the entire space $D_t(p)$. This is because $L(D_t)(p) = kerd\omega_t(p)$ and because the 1-form $\frac{d\omega_t}{dt}$ vanishes on $L(D_t)(p)$. The latter fact is a consequence of the condition that $L(D_t) = L(D_0)$ does not depend on t. *This is the only place in the proof where this condition is used.*

Now define the path Φ_t of global diffeomorphisms to be the solution to the ordinary differential equation $\frac{d\Phi_t}{dt} = X_t(\Phi_p)$ with the initial condition $\Phi_0 = id$. We will show that $(\Phi_t)_* D_t = D_0$. We have $\frac{d}{dt}((\Phi_t)^*\omega_t) = \Phi_t^*(\mathcal{L}_{X_t}\omega_t + \frac{d\omega_t}{dt})$, where $\mathcal{L}$ is the Lie derivative along X_t. Since X_t is annihilated by ω_t the Lie derivative is equal to $X_t \rfloor d\omega_t$. We showed that $(X_t \rfloor d\omega_t + \frac{d\omega_t}{dt})|_{D_t(p)} = 0$ for any point p. This implies that $X_t \rfloor d\omega_t + \frac{d\omega_t}{dt} = h_t\omega_t$ for some path of functions h_t. Therefore the path of 1-forms $A_t = (\Phi_t)^*\omega_t$ satisfies the **linear** ordinary differential equation $\frac{dA_t}{dt} = \tilde{h}_t A_t$ with $\tilde{h}_t = h_t(\Phi_t)$ with initial condition $A_0 = \omega_0$. We can integrate this equation. Indeed the ansatz $A_t = H_t\omega_0$ yields the scalar differential equation $\frac{dH_t}{dt} = \tilde{h}_t H_t$ with solution $H_t = \exp\{\int_0^t \tilde{h}_s ds\}$. We have shown that $A_t := \Phi_t^*\omega_t = H_t\omega_0$ which means that $(\Phi_t)_* D_t = D_0$. Q.E.D.

Proof of Theorem A.2. Let ω be a global 1-form describing D, and let $\hat{\omega}_N$ be global 1-forms describing D_N and such that $\hat{\omega}_N \to \omega$ in the Whitney C^{l+1}-topology. Since the $(2r+1)$-forms $\hat{\omega}_N \wedge (d\hat{\omega}_N)^r$ and $\omega \wedge (d\omega)^r$ have the same kernel $L(D_N) = L(D)$ of codimension $2r+1$ then $\hat{\omega}_N \wedge (d\hat{\omega}_N)^r = H_N \omega \wedge (d\omega)^r$, where H_N is a nonvanishing function. Replace $\hat{\omega}_N$ by $\omega_N = \frac{\hat{\omega}_N}{H_N^{r+1}}$. . The forms ω_N also describe distributions D_N, and we have

$$\omega_N \wedge (d\omega_N)^r = \omega \wedge (d\omega)^r \tag{A.1}$$

The value of H_N at any point depends on the values of $\omega, \hat{\omega}_N$ and their differentials at the same point only, therefore $H_N \to 1$ in the Whitney C^l-topology. Consequently $\omega_N \to \omega$ in the same topology.

Define the path

$$\omega_{N,t} = \omega + t(\omega_N - \omega), \quad t \in [0,1]$$

of one-forms. Let $D_{N,t}$ be the field of kernels of $\omega_{N,t}$. We show that for sufficiently big N the distribution $D_{N,t}$ is a corank one distribution of the same constant rank $2r+1$ and with the same characteristic foliation $L(D_{N,t}) = L(D)$ for all $t \in [0,1]$. This follows immediately from the following two statements:

(a) $\omega_{N,t} \wedge (d\omega_{N,t})^r(p) \neq 0$ (for sfficiently big N, any $t \in [0,1]$, and any $p \in M$);

(b) $d\omega_{N,t}(Z, Y_{N,t}) = 0$ for any vector field $Z \in L(D)$ and any vector field $Y_{N,t} \in D_{N,t}$.

Statement (a) follows from the C^l-closeness of $\omega_{N,t}$ to ω, the compactness of the segment $[0,1]$ and the condition $l \geq 1$.

To prove the second statement we use the equality (A.1). Fix a vector field $Z \in L(D)$. We know that $Z(p)$ belongs to the kernel of $d\omega(p)|_{D(p)}$ for any point p of the manifold. This condition implies that $Z \rfloor d\omega = h\omega$ for some function h. Similarly $Z \rfloor d\omega_N = h_N\omega_N$ for some function h_N. To prove (b) it suffices to show that $h_N = h$. Indeed, if $h_N = h$ then for any vector field $Y_{N,t} \in D_{N,t}$ we have:

$$d\omega_{N,t}(Z, Y_{N,t}) = (1-t)d\omega(Z, Y_{N,t}) + td\omega_N(Z, Y_{N,t}) =$$

$$= (1-t)h\omega(Y_{N,t}) + th\omega_N(Y_{N,t}) = h\omega_{N,t}(Y_{N,t}) = 0.$$

To prove that $h_N = h$ we take the Lie derivative $\mathcal{L}_Z$ of the relation (A.1) along the vector field Z. Since Z belongs to the kernel of each of the $(2r+1)$-forms in (A.1), we obtain $\mathcal{L}_Z(\omega \wedge (d\omega)^r) = Z\rfloor(d\omega)^{r+1} = (r+1)(d\omega)^r \wedge (Z\rfloor d\omega) = (r+1)h\omega \wedge (d\omega)^r$ and, in the same way, $\mathcal{L}_Z(\omega_N \wedge (d\omega_N)^r) = (r+1)h_N\omega_N \wedge (d\omega_N)^r$. But (A.1) holds, and hence so does the Lie derivative of (A.1) with respect to Z. We conclude that $h_N = h$.

We have proved that the path of distributions $D_{N,t}$ satisfies the conditions of Theorem A.1. By this theorem there exists a diffeomorphism Φ_N sending $D_N = D_{N,1}$ to $D = D_{N,0}$. Tracing the proof of Theorem A.1 we see that as $N \to \infty$ the diffeomorphism Φ_N tends to the identity diffeomorphism in the same topology in which the 1-form $\frac{d\omega_{N,t}}{dt}$ tends to zero 1-form. Since $\frac{d\omega_{N,t}}{dt} = \omega_N - \omega$ and $\omega_N \to \omega$ in the C^l-Whitney topology, we have that $\Phi_N \to id$ in the same topology. Q.E.D.

We also need the following local version of Theorem A.2. Its proof is the almost the same.

Theorem A.3.

Part 1 (for germs at a nonfixed point). *Let D and D_N be corank one distributions on a manifold M of constant class $2r+1$ described by 1-forms ω and ω_N such that $j^l_p\tilde{\omega}_N \to j^l_p\omega$ for some point $p \in M$, and for $l \geq 1$. Let U be any neighbourhood of the point p. Then for sufficiently large N there exist open sets (possibly disjoint) $U_1^N, U_2^N \subset U$ with $p \in U_1^N$ and a diffeomorphism $\Phi_N : U_1^N \to U_2^N$ which sends the distribution D_N restricted to U_1^N to the distribution D restricted to U_2^N, and satisfies $j_p^{\ell-1}\Phi_N \to j_p^{\ell-1}id$ as $N \to \infty$.*

Part 2 (for germs at a fixed point). *Let D and $\tilde{D}$ be germs at a point p of corank one distributions of constant class $2r+1$ with the same l-jets at p, $l \geq 1$. Then there exists a local diffeomorphism Φ such that $j_p^{l-1}\Phi = j_p^{l-1}id$ and $\Phi_*\tilde{D} = D$.*

Note that in Part 1 in general $\Phi_N(p) \neq p$. To make sense of the condition $j_p^{l-1}\Phi_N \to j_p^{l-1}id$ one should take U to be a coordinate neighborhood and identify the ℓ-th jet with the ℓ-th order Taylor expansion of Φ_N.

Appendix B. Proof of Lemma 3.2.

This lemma is based on the following statement.

Proposition B.1. *Let D be any Goursat distribution of corank $s \geq 2$. All eigenvalues of the linearization at p of any local symmetry $\Phi \in Symm_p(D)$ are real.*

We prove this Proposition at the end of this Appendix. To show how it implies Lemma 3.2 we need several reduction steps.

Step 1. The projectivity g_Φ of the circle $S^1(D)(p)$ depends on $j^1_p\Phi$ only. Therefore to prove Lemma 3.2 it suffices to prove the following statement:

R1. *Let $\Phi \in Symm_p(D)$. Then we can express Φ^2 in the form $\Phi^2 = \Psi_1 exp(V)$ where $\Psi_t exp(tV) \in Symm_p(D)$, V is a vector field germ at p, vanishing at p and Ψ_t is a family of local diffeomorphisms such that $j^1_p \Psi_t = id$, $t \in R$.*

Note that we are not asserting that Ψ_t or $exp(tV)$ lie in $Symm_p(D)$.

Step 2. Proof of (R1). Fix any $k > s = corank(D)$. It follows from Proposition 6.2, part 2 that if $\tilde{D}$ is a germ at p of a Goursat distribution such that $j^k_p \tilde{D} = j^k_p \tilde{D}$ then there exists a local diffeomorphism Φ such that $\Phi_* \tilde{D} = D$ and $j^{k-s}_p \Phi = id$. In particular $j^1_p \Phi = id$. Therefore to prove (R1) it suffices to prove the following statement:

R2. *Let $\Phi \in Symm_p(D)$. Then there exists a local vector field V such that*:

$$j^{k+1}_p \Phi^2 = j^{k+1}_p exp(V), \tag{B.1}$$

$$j^k_p exp(tV) * D = j^k_p D, \quad t \in R. \tag{B.2}$$

Step 3. We show that (B.1) implies (B.2). It is clear that (B.1) implies (B.2) for all integer t. By Proposition B.1, the eigenvalues of $j^1_p \Phi$ are real, therefore the eigenvalues of $j^1_p \Phi^2$ are positive and consequently those of $j^1 V_p$ are real. Therefore the relation (B.2) can be expressed in the form $F_1(t) \equiv \cdots \equiv F_m(t) \equiv 0$, where each of the functions $F_1, \ldots, F_m$ is a linear combination of real exponential functions with polynomial coefficients. Since $F_i(t) = 0$ for any integer t then $F_i(t) \equiv 0$ and (B.2) holds.

Step 4. We have reduced Lemma 3.2 to the proof of the existence of a vector field V satisfying (B.1). Let J^{k+1}_p be the space of the $(k+1)$-jets at p of functions vanishing at p. Consider the linear operator $A : \ J^{k+1}_p \to J^{k+1}_p$ such that $A(f) = j^{k+1}_p f(\Phi^2)$, $f \in J^{k+1}_p$. To prove that (B.1) holds for some vector field V it suffices to show that the operator A admits a logarithm, i.e. that there exists a linear operator $B : \ J^{k+1}_p \to J^{k+1}_p$ such that $A = exp(B)$. To show this it suffices to prove that the eigenvalues of A are real positive numbers. It is known that the eigenvalues of A have the form $\lambda_1^{\alpha_1} \cdot \cdots \cdot \lambda_n^{\alpha_n}$, where λ_i are eigenvalues of the linearization of Φ^2 at p, and where the α_i are non-negative integers which sum to $k+1$. By Proposition B.1 these λ_i's are real positive numbers. Therefore the same is true for the eigenvalues of the operator A. The proof of Lemma 3.2 is completed.

Proof of Proposition B.1. To prove Proposition B.1 we will show that in suitable coordinate system the matrix of $j^1_p \Phi$ is triangular. Let $D = D_s \subset D_{s-1} \subset \cdots \subset D_2 \subset D_1$ be the Goursat flag generated by D. Take a local coordinate system $x_1, \ldots, x_n$ centered at the point p such that the Engel subflag $D_2 \subset D_1$ is described by 1-forms $\omega_1 = dx_1 - x_2 dx_3$ and $\omega_2 = dx_2 - x_4 dx_3$ and the characteristic foliations $L(D_i)$ have the form $(dx_1, \ldots, dx_{i+2})^\perp$, $\ i = 1, \ldots, s$. Denote $\Phi_i = \Phi(x_i)$. The form of the characteristic foliations and the fact that they are preserved by Φ implies that $\frac{\partial \Phi_i}{\partial x_j}(0) = 0$ for $j > i$ *and* $j > 3$. To show that the matrix of the linear approximation of Φ is triangular in the chosen coordinate system we have to prove that

$$\frac{\partial \Phi_1}{\partial x_2}(0) = \frac{\partial \Phi_1}{\partial x_3}(0) = \frac{\partial \Phi_2}{\partial x_3}(0) = 0. \tag{B.3}$$

To prove (B.3) we use the relations $\Phi^*\omega_1 = H\omega_1$ and $\Phi^*\omega_2 = H_1\omega_1 + H_2\omega_2$ that hold for some functions H, H_1, H_2. Write these relations in the coordinate system $x_1, \ldots, x_n$. We obtain

$$d\Phi_1 - \Phi_2 d\Phi_3 = H(dx_1 - x_2 dx_3), \quad d\Phi_2 - \Phi_4 d\Phi_3 = H_1(dx_1 - x_2 dx_3) + H_2(dx_2 - x_4 dx_3).$$

Since $\Phi_2(0) = \Phi_4(0) = 0$ we obtain (B.3). Q.E.D.

Appendix C. Kumpera-Ruiz normal forms, Mormul's codes, and growth vector

The Kumpera-Ruiz normal forms are preliminary normal forms for corank s Goursat flags. They are parametrized by a subsets $I \subset \{3, 4, \ldots, s\}$ and provide representatives for the Kumpera-Ruiz singularity classes

$$D_i(0) = L(D_{i-2})(0), \quad i \in I; \quad D_i(0) \neq L(D_{i-2})(0), \quad i \notin I$$

described in section 2. Using Proposition 3.1, Lemma 2.2 and arguing by induction, it is easy to prove that any such flag germ can be described by s 1-forms $\omega_1, \ldots, \omega_s$ of the type

$$\omega_i = df_i - g_i dh_i, \quad i > 2,$$

together with

$$\omega_1 = dy - z_1 dx, \ \omega_2 = dz_1 - z_2 dx.$$

The functions f_i, g_i, h_i, $i > 2$ are as follows:

$$f_i = g_{i-1}, \ h_i = h_{i-1}, \ g_i = z_i + c_i \quad if \ i \notin I,$$

$$f_i = h_{i-1}, \ h_i = g_{i-1}, \ g_i = z_i \quad if \ i \in I.$$

The constants $c_i, i \notin I$ are real parameters arising in the Kumpera-Ruiz normal forms. The number of these parameters is equal to s minus the cardinality of the set I. These parameters are not invariants in general. For example when I is the empty set all of the parameters can be reduced to zero according to the Cartan theorem.

P.Mormul treats the problem of local classification of Goursat distributions on R^n of rank 2 as the problem of normalizing the parameters c_i by changes of coordinates. To systematize his results Mormul introduced the following codes. The Kumpera-Ruiz normal form corresponding to a subset $I \subset \{3, 4, \ldots, s\}$ is coded by the tuple of $s-2$ digits, where the i-th digit is a 2 if $i+2 \notin I$ and is a 3 if $i+2 \in I$. The digit 2 acts like an indeterminant: if the constant c_{i+2} in the Kumpera-Ruiz normal form can be normalized to 0 then Mormul changes it to 1, if c_{i+2} cannot be normalized to 0 but can be normalized to either 1 or to -1 then Mormul replaces the 2 by either a **bold 2** or a **2-**. However, if $i+2 \notin I$, but one does not know, or does not want to specify whether or not the c_{i+2} can be normalized, then Mormul leaves it as a 2.

These codes allow Mormul to formulate his results in a very compact way. For example the assertion " 3.3.1.**2**.2 $\equiv$ 3.3.1.**2**.1" of [Mormul, first paper of 1988, p.15]) means that in the Kumpera-Ruiz normal form for Goursat flags of length 7 corresponding to the set $I = \{3,4\} \subset \{3,4,\ldots,7\}$, one can reduce the constant c_7 to 0 provided that the parameters c_5, c_6 have been normalized to 0 and 1 respectively. Translating this result to our language we obtain the following. If D is a Goursat distribution of corank 6 on R^n (any $n \geq 8$) generating the flag $D = D_6 \subset \cdots \subset D_1$ with singularity $D_3(0) = L(D_1)(0)$, $D_4(0) = L(D_2)(0)$ and such that $D_5(0)$ is tangent to the submanifold of points at which this singularity holds whereas $D_6(0)$ is generic, then the space $L(D_5)$ is the only fixed point of the circle $S^1(D)(0)$ and therefore the set $\sqrt{D}$ consist of two orbits.

The Cartan theorem admits an alternative formulation in terms of the *growth vector*. The growth vector at a point p of a distribution D (not necessarily Goursat) is the sequence $g_1, g_2, \ldots$, where g_k is the dimension of the space spanned by all vectors of the form $[X_1,[X_2,[X_3,\ldots X_j]]]...](p)$ with $X_1,\ldots,X_j \in D$, and $j \leq k$. For nonholonomic distributions on an n-manifold $g_l = n$ for some finite l and so the growth vector is an l-tuple $g = (r,\ldots,n)$ starting with the rank r of D and ending with n. The number l as well as the growth vector g may depend on the point p. At generic points of a Goursat distribution, as described by Cartan's normal form (C) of section 1, this growth vector is $g = (r, r+1, r+2, r+3, \ldots, n)$. This is the growth vector with the fewest number of components ($s = n-r$), or fastest growth, given the constraint that it is that of a Goursat distribution. Murray [Murray, 1994] proved the converse: a point of a Goursat distribution with this growth vector is a nonsingular point.

This, together with other computations, suggested the conjecture that the growth vector is a complete invariant of Goursat distributions, i.e. that two germs of Goursat distributions at a point p are equivalent if and only if they have the same growth vectors at p. Mormul showed that this conjecture is false for $s > 6$, although it is valid for $s \leq 6$. The growth vectors of Goursat distributions can be quite complicated. For example using normal forms Mormul found a Goursat 2-distributions on R^9 whose growth vector at the origin is $2,3,4,4,5,5,5,5,6,6,6,6,6,7,\ldots 7,8,\ldots 8,9$ where 7 is repeated 8 times and 8 is repeated 13 times.

The number $gr(s)$ of all possible growth vectors for Goursat distributions of a fixed corank s is finite. (Computing the growth vector from the normal form is a straightforward tedious job.) Mormul obtained the following table comparing $gr(s)$ with the number $or(s)$ of orbits in the space of germs of Goursat distributions of the same corank s.

s	2	3	4	5	6	7	8	9
$or(s)$	1	2	5	13	34	93	∞	∞
$gr(s)$	1	2	5	13	34	89	*not known*	*not known*

The tuple $gr(2), gr(3), ..., gr(7)$ is the list of the first 6 odd Fibonacci numbers F_{2s-3}. Conjecturally, this pattern continues: $gr(s)$ is the $(2s-3)$-d Fibonacci number for all s. In particular $gr(8) = 233, gr(9) = 610$. Results in this direction have been obtained by [Jean, 1996], [Sordalen, 1993] and [Luca, Risler, 1994] for the Goursat distribution corresponding to the kinematic model of a truck pulling $s-1$ trailers.

In the next Appendix we use our Theorem 1 to give a simple proof that *the local classification of Goursat distributions corresponding to the model of a truck with s trailers* *and the local classification of arbitrary Goursat flags of length $s+1$ are the same problem.* This allows us to extend some of these truck-trailer results on $gr(s)$ to arbitrary Goursat distributions.

Appendix D. The kinematic model of a truck with trailers.

In this Appendix we use Theorem 1 to give a simple proof that

the local classification of Goursat distributions corresponding to the model of a truck with s trailers and the local classification of arbitrary Goursat flags of length $s+1$ are the same problem.

The kinematic model of a truck towing s trailers can be described by a 2-distribution on $R^2 \times (S^1)^{s+1}$ generated by vector fields

$$X_1^s = \frac{\partial}{\partial \theta_s},$$

$$X_2^s = cos\theta_0 f_0^s \frac{\partial}{\partial x} + sin\theta_0 f_0^s \frac{\partial}{\partial y} + sin(\theta_1 - \theta_0) f_1^s \frac{\partial}{\partial \theta_0} + \cdots + sin(\theta_s - \theta_{s-1}) f_s^s \frac{\partial}{\partial \theta_{s-1}},$$

where

$$f_i^s = \Pi_{j=i+1}^s cos(\theta_j - \theta_{j-1}), \quad i \leq s-1, \quad f_s^s = 1,$$

(x, y) are the coordinates of the last trailer (trailer number s), θ_s is the angle between the truck and the x-axis, and θ_i is the angle between the trailer number $s-i$ and the x-axis. See [Fliess et al, 1992], [Sordalen, 1993] and [Jean, 1996]. This representation holds under the condition that the distance between the truck and the first trailer is equal to the distance between the i-th and the $(i+1)$-st trailers. The distribution (X_1^s, X_2^s) generated by X_1^s and X_2^s satisfies the Goursat condition. (See [Jean, 1996].)

Proposition D1. *The Goursat distribution spanned by X_1^s, X_2^s and defining the kinematics of a truck pulling s trailers is diffeomorphic to the $(s+1)$-fold Cartan prolongation of the tangent bundle to the Euclidean plane.*

Combining this proposition with Theorem 1 and the reduction from Goursat k-distributions to Goursat 2-distributions given in section 1, we obtain the following corollary.

Corollary D1. *All corank $s+1$ Goursat germs occur within the truck-trailer model with s trailers. Namely, any germ D of any Goursat 2-distribution on R^{s+3} is equivalent to the germ of the distribution spanned by (X_1^s, X_2^s) at some point $p = p(D)$ of $R^2 \times (S^1)^{s+1}$. More generally, any germ of any rank k Goursa -distribution on R^{k+s+1} is equivalent to the germ of the distribution $span\{X_1^s, X_2^s\} \oplus R^{k-2}$ on $R^2 \times (S^1)^{s+1} \times R^{k-2}$.*

Remark. We now can state Theorem 1 in the following picturesque way. Every singularity for a corank s Goursat distribution corresponds to some way of jacknifing a truck towing $s-1$ trailers.

Proof of Proposition D1. We show that the distribution spanned by (X_1^{s+1}, X_2^{s+1}) on $R^2 \times (S^1)^{s+2}$ is the Cartan prolongation of the distribution spanned by (X_1^s, X_2^s) on $R^2 \times (S^1)^{s+1}$. Let $p \in R^2 \times (S^1)^{s+1}$. The set of directions in the space spanned by $X_1^s(p)$ and $X_2^s(p)$ is parametrized by an angle $\phi \in [0, \pi)$ by representing each direction by the span of the vector $cos\phi X_1^s(p) + sin\phi X_2^s(p)$. The Cartan prolongation of the distribution spanned by (X_1^s, X_2^s) is the distribution on $R^2 \times (S^1)^{s+2}$ spanned by $Y_1^{s+1} = \frac{\partial}{\partial\phi}$ and $Y_2^{s+1} = cos\phi X_1^s + sin\phi X_2^s$. Replace ϕ by the angle $\theta_{s+1} = \phi + \theta_s$. In the new coordinates $x, y, \theta_1, \ldots, \theta_s, \theta_{s+1}$ we have $Y_1^{s+1} = \frac{\partial}{\partial\theta_{s+1}} = X_1^{s+1}$ and $Y_2^{s+1} = X_2^{s+1} \ mod \ X_1^{s+1}$. Therefore (Y_1^{s+1}, Y_2^{s+1}) and (X_1^{s+1}, X_2^{s+1}) span the same 2-distribution. Q.E.D.

Now we can extend known results on the growth vector of the truck-trailer distributions $T_s = span\{X_1^s, X_2^s\}$ to arbitrary Goursat flags. Jean [Jean,1996] proved that the number of distinct growth vectors $g(p)$ for T_s, as p varies over the truck-trailer configuration space $R^2 \times (S^1)^{s+1}$), does not exceed F_{2s-1}. Here F_i denotes the i-th Fibonacci number. Sordalen [Sordalen, 1993] and Luca-Risler [Luca, Risler, 1994] estimated the degree of nonholonomy of the T_s from above. Recall that this is the length $\ell = \ell(p)$ (the number of components) of the growth vector $g(p)$ at p. They proved $\ell(p) \leq F_{s+3}$ at any point $p \in R^2 \times (S^1)^{s+1}$ and that there exist certain points where equality is achieved. (These certain points correspond to the case where each trailer, except the last , is perpendicular to the one in front of it.) These results, combined with Corollary D1 have the following corollaries.

Corollary D2. *Let D be a Goursat distribution of corank s on an n-dimensional manifold M. Then the degree of nonholonomy of D at any point of M does not exceed the Fibonacci number F_{s+2}.*

Corollary D3. *The number $gr(s)$ of all possible growth vectors of Goursat distributions of corank s does not exceed the Fibonacci number F_{2s-3}.*

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