四元数ケーラー計量の無限小Einstein変形

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Matharxiv ???・・

§1 Review of QK mfds

$(n \geqq 2)$

> **Def** (M, g) $4n$-dim Riem mfd is QK mfd
> if $\mathrm{Hol}(M, g) \subset \underline{Sp(1)Sp(n)} \subset SO(4n)$
> $= (Sp(1) \times Sp(n))/\mathbb{Z}_2$

(1) $\Rightarrow$ Einstein mfd $\quad e := \frac{1}{4n}\mathrm{Scal}$ is const $(\mathrm{Ric} = eg)$

($e = 0 \rightsquigarrow$ locally hyper kähler)

QK-geometry is the case of $\underline{e > 0}$ or $e < 0$

(2) Known examples $(e > 0)$

> 1982 Salamon
> 1974 Ishihara

Wolf space (QK symm sp)

$Gr_2(\mathbb{C}^{m+2})$, $Gr_4^0(\mathbb{R}^{m+4})$, $\mathbb{H}P^n$, $G_2/SO(4)$

and 4-exceptional cases.

(3) LeBrun-Salamon (1994)

finiteness thm

$\#\{4n\text{-dim QK mfd}, \mathrm{scal}>0\} < \infty$ for each n

local rigidity as QK mfd

(M,g) QK mfd は QK mfdとして変形できない!!

L-S conjecture

(M,g) QK mfd, $\mathrm{Scal}>0 \Rightarrow (M,g)$ is Wolf space

$n \leqq 4$

$n=2$ O.K

$b_4=1$, $n \leqq 6$ O.K

$\left(\begin{array}{l}\text{Twistor sp} \\ \text{Fano Contact} \\ \quad \Rightarrow \text{homog}\end{array}\right)$

Question Is QK metric rigid as Einstein mfd?

$$\underbrace{\{\text{local defo as QK mfd}\}}_{=0} \subset \{\text{local defo as E mfd}\}_{??}$$

Question Is QK metric stable as Einstein metric?

Rigidity and Stability are classical but important problem for geometry of Einstein mfd (c.f. Besse)

If L·S conj is true, then QK is rigid

except $Gr_2(\mathbb{C}^{m+4})$

§2. Einstein deformation and Eigenvalues of Δ

M : cpt n-dim mfd

$\mathcal{M} := \{h \mid \text{metric on } M\} \subset \Gamma(S^2M)$ $\quad S^2M = S^2(TM)$

$$\mathcal{S} : \mathcal{M} \ni h \longmapsto \mathrm{Vol}(M.h)^{\frac{2-n}{n}} \int_M \mathrm{scal}_h \cdot \mathrm{vol}_h$$

critical pt of $\mathcal{S}$ $\Longleftrightarrow$ g : Einstein metric

$\mathrm{Ric}_g = cg$ $\quad {}^{\exists}c$ 定数

Then we have to study second variation $\mathcal{S}_g''$ at g

$$T_g\mathcal{M} \cong \Gamma(S^2M)_g$$

① Diffeo の方向 : $X \in \mathfrak{X}(M) \longrightarrow \phi_t^*(g)$ curve in $\mathcal{M}$

$$\xrightarrow{d/dt|_{t=0}} L_X g = \delta^* X$$

where $\delta^* : \Omega^1(M) \cong \mathfrak{X}(M) \longrightarrow \Gamma(S^2M)$ co-div

$\delta : \Gamma(S^2M) \longrightarrow \Gamma(TM)$ divergence $(\delta = \delta_g)$

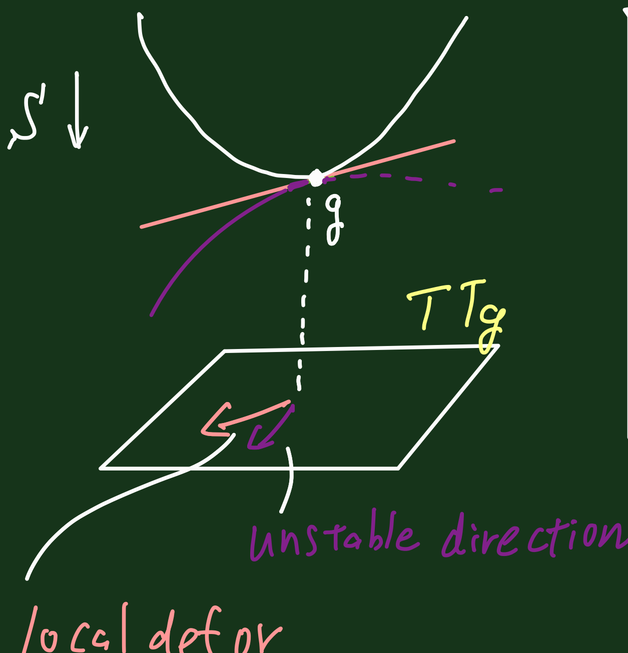
② 共形変形の方向 : $g_t = e^{tf} g \overset{d/dt|_{t=0}}{\rightsquigarrow} f \cdot g \quad f \in C^\infty(M)$

③ essential part (transversal - traceless)

$$TT_g := \{ h \in \Gamma(S^2M) \mid tr_g(h) = 0,\ \delta_g(h) = 0 \}$$

$$\underset{\Gamma(S^2M)}{\overset{\cup\!\mid}{T_g\mathcal{M}}} = \underbrace{\delta_g^*(\mathfrak{X}(M))}_{\text{diffeo}} \oplus \underbrace{C^\infty(M)\,g}_{\text{conf defo}} \oplus \underbrace{TT_g}$$

diffeo $S_g'' = 0$ $\quad$ conf defo $S_g'' > 0$ $\quad$ $TT_g \to S_g'' : ??$



Fact $\quad g$: Einstein $\quad Ric = e\,g$
$\quad (e : \text{const})$

For $h \in TT_g$,

$$S_g''(h,h) = \frac{1}{2}\int_M \left(-(\Delta - 2e)h,\, h\right) \mathrm{vol}_g$$

Δ : standard Laplacian on $\Gamma(S^2TM)$

$$\Delta = \delta\delta^* - \delta^*\delta + 2(\underbrace{2R^\circ - Ric}_{=e})$$

$\therefore$ $-(\Delta - 2e)$ is negative except finite dim subsp

i.e. g is stable "

Koiso

Def (1) g is stable if $S_g'' \leqq 0$ on TT_g

(i.e. $\Delta \geqq 2e$ on TT_g)

(2) g is infinitesimally deformable (locally) if $\exists h_{x_0} \in TT_g$ $\Delta h = 2eh$

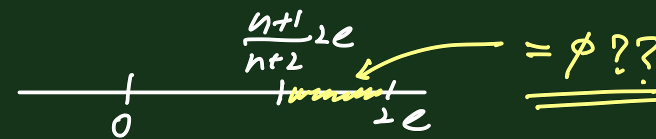
注 locally deformable $\not\Rightarrow$ deformable.

stability $\rightsquigarrow$ Δ on $\Gamma(S_0^2 M)$ の eigenvalue を調べる.

§3 Eigenvalues of Δ on positive QK mfds.

QK mfd with $Ric = \underset{\substack{\vee \\ 0}}{e} \cdot g$, $\Delta \geqq \frac{n+1}{n+2} \cdot 2e$ on $S^2 M$

$Hol(M, g) \subset Sp(1)Sp(n)$



HE - formalism

$E = (\mathbb{C}^{2n}, \sigma_E)$: natural rep sp of $Sp(n)$.

$H = (\mathbb{C}^2, \sigma_H)$: natural rep sp of $Sp(1)$

$Sp(1)Sp(n) \curvearrowright \mathbb{R}^{4n} \otimes \mathbb{C} = H \cdot E \; (= H \otimes E)$ with $\underset{\substack{\| \\ \sigma_H \otimes \sigma_E}}{g^c}$

$(Sp(1)Sp(n) \subset SO(4n))$

We can extend this onto QK mfd

$P(M)$ o.n. frame bundle $\curvearrowleft Sp(1)Sp(n)$

$H := P(M) \times_{\rho_1} H$, $E := P(M) \times_{\rho_2} E$ (locally defined)

Then $TM^c = H \cdot E \; (= H \otimes E)$ (globally defined)

$\Lambda^2(M)^c \cong S^2H \oplus S^2H \cdot \Lambda_0^2 E \oplus S^2 E$

irr decomp

$S^2M^c \cong S^2HS^2E \oplus \Lambda_0^2 E \oplus \underset{= M \times \mathbb{C}}{\mathbb{C}}$

Here

S^kH : symm k tensor of H (S^kE : ... of E)

(h.w $= (k)$) (h.w. $= (k, 0, \cdots, 0)$)

$\Lambda^p_0 E$: top irr comp of $\Lambda^p E$ $(0 \leqq p \leqq n)$

(h.w. $= (\underbrace{1, \cdots 1}_{k}, 0, \cdots, 0)$)

d, d^* on $\Lambda^p(M)$
δ, δ^* on $S^p M$

decompose w.r.t. $Sp(1)Sp(n)$.

Qk-gradients

V : irr vect bundle assoc to $P(M)$

$$D_i : \Gamma(V) \xrightarrow{\nabla} \Gamma(\underline{V \otimes TM^c}) \xrightarrow[\text{proj}]{\pi_i} \Gamma(V_i)$$

$$\underline{V \otimes TM^c} = \bigoplus_{i \in I} V_i$$

Qk-gradient

$$B_i := D_i^* D_i : \Gamma(V) \xrightarrow{D_i} \Gamma(V_i) \xrightarrow{D_i^*} \Gamma(V)$$

Fact $\quad \nabla^* \nabla = \sum_{i \in I} B_i$ on $\Gamma(V)$

Identities of casimirs of $Sp(n)$

Fact Weitzenböck formulas (Sem-Weingart, 2002; Hom, 2006)

$$\sum_{i \in I} a_i B_i = \text{curvature action on } \Gamma(V)$$

i.e. certain linear combinations of 2nd order B_i are order zero.

Def The Standard Laplacian

$$\Delta = \Delta_V := \nabla^* \nabla + R_V \quad \text{on } \Gamma(V)$$

Ex
- $\Delta = dd^* + d^* d$
- Lichenowiz on $S^p M$
- $\Delta = Cas_2$ on G/K
- $\Delta = D^2 + Scal/8$ spinor

Fact commutator relations (Sem-Wein, 2019)

今回, 利用する irr bdle V 上で,

$$\begin{tikzcd} \Gamma(V) \arrow[r, "\Delta"] \arrow[d, "D_i"'] \arrow[dr, phantom, "\circlearrowright"] & \Gamma(V) \arrow[d, "D_i"] \\ \Gamma(U_i) \arrow[r, "\Delta"'] & \Gamma(U_i) \end{tikzcd}$$

Cor $\Gamma(V)_\lambda := \{ \varphi \in \Gamma(V) \mid \Delta\varphi = \lambda\varphi \}$ eigen sp

Then $D_i : \Gamma(V)_\lambda \longrightarrow \Gamma(U_i)_\lambda$

$\because)$ $\Delta\varphi = \lambda\varphi \Rightarrow \Delta D_i\varphi = D_i\Delta\varphi = \lambda(D_i\varphi)$

How to use them

① $\Delta = \nabla^* \nabla + q(R) = \sum_{i \in I} (1 + w_i) B_i$ (by Gauduchon)

$$\boxed{\text{W-formulas } \sum_{i \in I} a_i B_i = c \cdot Scal}$$

$$= \cdots = \sum_{i \in I} b_i \underset{0}{\underset{\shortparallel}{B_i}} + d_V \cdot Scal \quad (b_i \geqq 0)$$

$\therefore \ \Delta \geqq d_V \cdot Scal$ (eigenvalue estimate)

$\tau < 1 \Leftarrow d_V > 0 \Rightarrow$ harmonic part $= 0$ (vanishing)

② $\varphi \in \Gamma(V)_\lambda \Rightarrow D_i\varphi \in \Gamma(V_i)_\lambda$

If $\lambda <$ smallest eigenvalue of V_i,

$$\Gamma(V)_\lambda \xrightarrow{D_i} \Gamma(V_i)_\lambda \text{ is zero map}$$

③ $\varphi \in \Gamma(V)_\lambda$, by W-formulas,

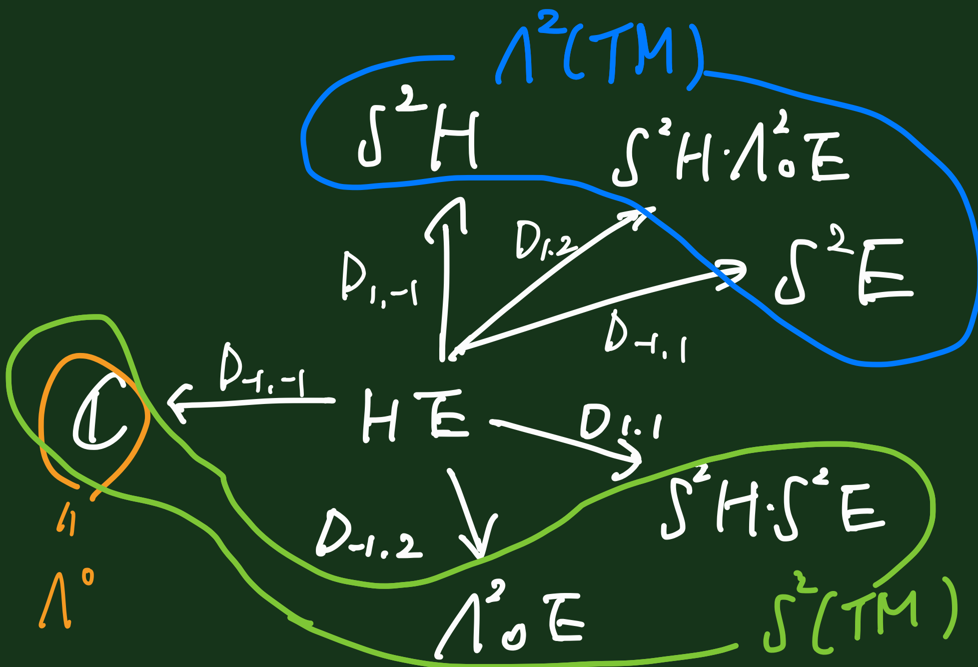
$$\rightsquigarrow \quad B_i\varphi = c_1(\overbrace{\lambda + c_2}^{\neq 0})\varphi$$

Then $D_i : \Gamma(V)_\lambda \longrightarrow \Gamma(V_i)_\lambda$ is injective

Example on $TM^c = HE$

$$HE\otimes HE = \left(S^2H \oplus S^2H\cdot\Lambda^2_0E \oplus S^2E\right) \oplus \left(S^2H\cdot S^2E \oplus \Lambda^2_0 E \oplus \mathbb{C}\right)$$

$$\nabla : \Gamma(HE) \to \Gamma(HE\otimes HE)$$

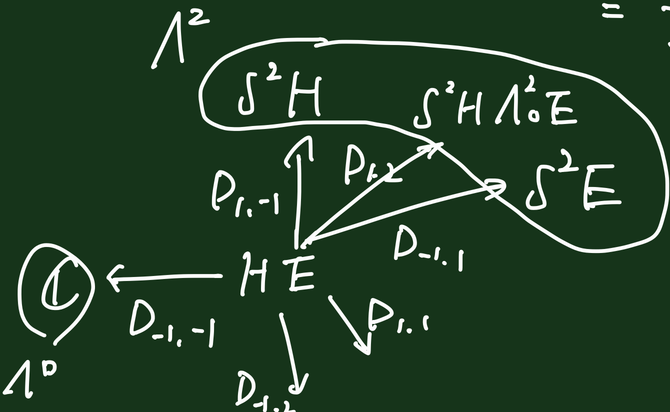


$$\nabla^*\nabla = B_{1,2} + B_{1,1} + B_{1,-1} + B_{-1,2} + B_{-1,1} + B_{-1,-1}$$

$$\begin{cases} -B_{1,1} + B_{1,2} + (2n+1)B_{1,-1} - B_{-1,1} + B_{1,2} + (2n+1)B_{-1,-1} = \dfrac{2n+1}{4n(n+2)}\,Scal \\ -B_{1,1} - B_{1,2} - B_{1,-1} + 3(B_{-1,1} + B_{1,2} + B_{-1,-1}) = \dfrac{3}{4(n+2)}\,Scal \qquad * \\ B_{1,2} + (2n+1)B_{1,-1} - 3B_{-1,1} = 0 \end{cases}$$

$$\Delta = d^*d + dd^* \overset{*}{=} 2(B_{-1,1} + B_{1,-1} + B_{1,2}) + 4n\,B_{-1,-1}$$

$$\overset{*}{=} \frac{4}{3}(B_{1,1} + B_{1,2} + B_{1,-1}) + \frac{n+1}{2n(n+2)}\,Scal$$



$\therefore$ $\Delta \geqq \dfrac{n+1}{n+2}\,2e$ on $\Omega^1(M)$

(M,g) QK with $e = \frac{1}{4n}Scal > 0$

Thm (Alekseevsky et al. (1994), LeBrun (1995), ⋯)

M : QK mfd with $Ric = e \cdot g > 0$

Then nonzero eigenvalue λ of Δ on $C^\infty(M)$ satisfies

$$\lambda \geqq \frac{n+1}{n+2} 2e$$

(They also showed "$=$" $\Longleftrightarrow$ $M = \mathbb{H}P^n$)

proof リヒネロビッツ-小幡 の QK-version

or twistor space を見る

Example Eigenspace $\Gamma(HE)_{2e} = \Omega^1(M)^c_{2e}$, $e = \frac{Scal}{4n}$

($Ric = eg$)

By Hodge-de Rham on cpt M,

$$\Gamma(HE)_{2e} = \underbrace{\Gamma(HE)_{2e} \cap \ker d^*}_{X \in} \oplus \Gamma(HE)_{2e} \cap \operatorname{Im} d$$

$$\Delta = \delta\delta^* - \delta^*\delta + 2e$$

(on Einstein)

$$2e X = \Delta X = \delta\delta^* X - \underbrace{\delta^*\delta X}_{= \, dd^* X = 0} + 2e X,$$

$$\therefore \quad \delta^* X = 0 \qquad \therefore \ X \text{ is killing v.f.}$$

$$\boxed{\underline{\text{prop}} \qquad \Gamma(HE)_{2e} = \operatorname{isom}(M, g) \oplus d\, C^\infty(M)_{2e}}$$

$D_{1,2} : \Gamma(HE)_{2e} \longrightarrow \Gamma(S^2H\Lambda^2_0E)_{2e} = \{0\}$, $\quad D_{1,2}X = 0$

$\therefore$ X killing v.f $\iff D_{-1,2}X = 0$, $D_{1,1}X = 0$, $D_{-1,-1}X = 0$ $\quad$ /☆

$(\Rightarrow D_{1,2}X = 0)$

$\because)$ $\delta\delta^* = 2(B_{1,1} + B_{-1,2} + B_{-1,-1})$, $\quad \delta^*\delta = 4n\,B_{-1,-1}$

By ☆ & $*$, $\quad B_{1,-1}X = \underset{\neq 0}{c_1}X$, $\quad B_{-1,1}X = \underset{\neq 0}{c_2}X$

$$\therefore \begin{cases} D_{1,-1} : \Gamma(HE)_{2e} \longrightarrow \Gamma(S^2H)_{2e} \subset \Omega^2(M), \text{ inj} \\ D_{-1,1} : \Gamma(HE)_{2e} \longrightarrow \Gamma(\Lambda^2_0E)_{2e} \subset \Omega^2(M), \text{ inj} \end{cases}$$

(In fact, isom)

prop $S^2H \subset \Lambda^2(M)$, the bundle of self dual 2-forms.

Then (1) $\Gamma(S^2H)_{2e} \cong isom(M)$ (Alekseevsky et al.)

(2) $\lambda < 2e,\ 2e < \lambda < 4e \Rightarrow \Gamma(S^2H)_\lambda = \{0\}$ (H-S)

$\because$)

$$H\cdot E \xleftarrow{D_{-1.1}} S^2H \xrightarrow{D_{1.1}} S^3H\cdot E \qquad \begin{cases} \Delta = \frac{3}{2}B_{1.1} + 2e \\ -B_{1,1} + 2B_{-1.1} = \frac{e}{n(n+2)} \end{cases}$$

min eijen $\frac{n+1}{n+2}\cdot 2e$ $2e$ $4e$

inj for $\lambda > 2e$

$$B_{1.1}\varphi = \frac{2}{3}(\lambda - 2e)\varphi$$

//

§4. Main thm

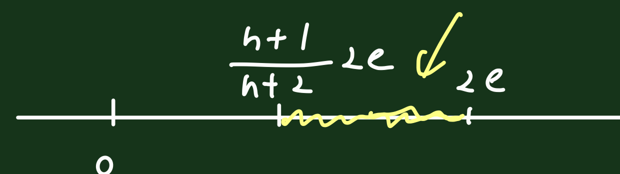
$$TT_g = \{ \varphi \in \Gamma(S^2 M) \mid \operatorname{tr} \varphi = 0,\ \delta\varphi = 0 \}$$

$$S^2 M = \Lambda^2_0 E \oplus S^2 H S^2 E \oplus \underline{\Lambda^0} \leftarrow \text{trace part}$$

$$(TT)_\lambda := \{ \varphi \in TT \mid \Delta \varphi = \lambda \varphi \}$$

$$\Gamma(\Lambda^2_0 E)_\lambda, \quad \Gamma(S^2 H S^2 E)_\lambda, \quad C^\infty(M)_\lambda, \ \cdots$$

Our concern is eigenspaces with $\lambda \leq 2e$



Thm (H-Semmelmann 2024)

(1) for $\lambda = \frac{n+1}{n+2} 2e$, $\begin{cases} \Gamma(S^2 H S^2 E)_\lambda = (TT)_\lambda \\ \Gamma(\Lambda^2_0 E)_\lambda \cong C^\infty(M)_\lambda \cong \Gamma(HE)_\lambda \end{cases}$

$$(M \neq \mathbb{H}P^n \Rightarrow C^\infty(M)_\lambda = \{0\})$$

(2) for $\frac{n+1}{n+2} \cdot 2e < \lambda < 2e$

$$\underline{C^\infty(M)_\lambda} \cong \Gamma(HE)_\lambda \underline{\cong (TT)_\lambda} \cong \Gamma(S^2 H S^2 E)_\lambda \cong \Gamma(\Lambda^2_0 E)_\lambda$$

(3) for $\lambda = 2e$

$$\underline{C^\infty(M)_\lambda \cong} \; \underline{(TT)_\lambda} \cong \Gamma(S^2 H S^2 E)_\lambda \cong \Gamma(\Lambda^2_0 E)_\lambda$$

$$\Gamma(HE)_\lambda \cong \mathrm{isom}(M, g) \oplus C^\infty(M)_\lambda$$

Cor Let M be QK with $Ric = e > 0$, $M \neq \mathbb{H}P^n$

(1) If M is stable as Einstein mfd,

then the nonzero eigenvalue on $C^\infty(M) \geqq 2e$

(2) If $i^{1,n+1} := \operatorname{index}(D_{S^{n+1}HE}) = 0$,

M is stable $\Longleftrightarrow$ nonzero eigen on $C^\infty(M) \geqq 2e$

(3) M is rigid $\Longleftrightarrow$ $2e$ is not eigenvalue on $C^\infty(M)$

Thus, Einstein Stability (eigenvalue on TT_g) problem

corresponds to minimal eigenvalue prob on $C^\infty(M)$ //

Ex Wolf spaces M $(\neq \mathbb{H}P^n, Gr_2(\mathbb{C}^{n+2}))$

$$\begin{cases} \text{Einstein strictly stable} \\ \Delta \text{ on } C^\infty(M) \text{ satifies } \Delta > 2e \end{cases}$$

Ex $M = \mathbb{H}P^n$

- $\lambda = \frac{n+1}{n+2} 2e$ $\quad E_\lambda(TT) = 0 \quad E_\lambda(M) \neq 0$
- Einstein strictly stable
- The second nonzero eigenvalue on $C^\infty(M) > 2e$

Ex $M = Gr_2(\mathbb{C}^{n+2})$

- Einstein stable. $\quad$ • Δ on $C^\infty(M) \geq 2e$
- $E_{2e}(M) \cong \{\text{infinitesimal deform of } g\} \neq 0$

Thank you!