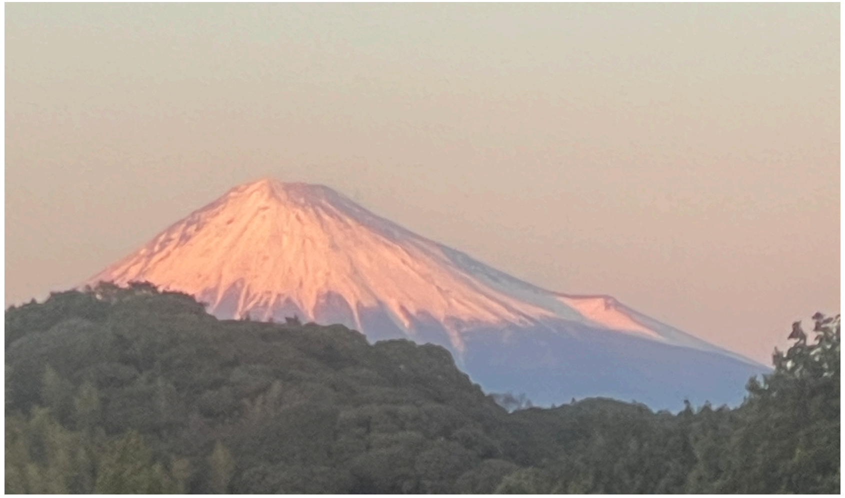
超楕円並進曲面の曲線グラフについて

四之宮 佳彦
(焼津)

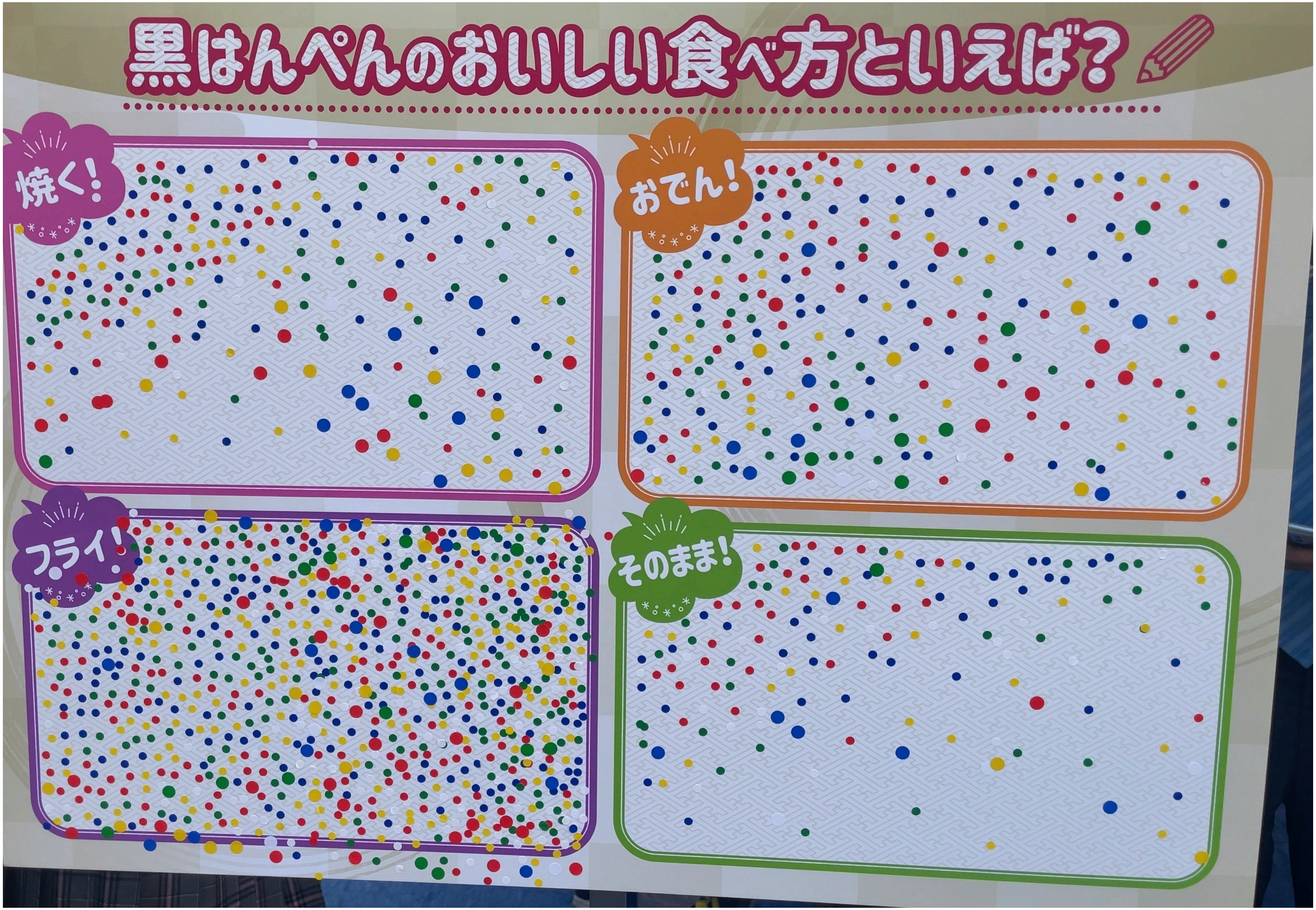
March 25, 2025

第 30 回沼津改め静岡研究会




焼津特産
黒はんぺん

↑天神屋さんHPより


黒はんぺんのおいしい食べ方といえば？
焼く！
おでん！
フライ！
そのまま！

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1. Introduction
2. Simple Cylinders and Main Theorem
3. Proof : Geometry of Flat Surface of Genus 0

$+ \alpha$

1. Introduction

Translation Surface

A translation surface (X, ω) is a pair of a Riemann surface X and holomorphic 1-form $\omega \neq 0$ on X.
If $p_0 \in X$ is not a zero of ω, there is a neighborhood U s.t.

$$z = \int_{p_0}^{p} \omega : U \to \mathbb{C}$$

is a chart of X. Let (U, z), (V, w) be such charts with $U \cap V \neq \emptyset$, the transition function is

$$w = z + (\text{const.}).$$

We may use terminologies of Euclidean geometry on (X, ω).
ex.) lengths and slopes of segments and area.

Flat Surface

A flat surface (X, q) is a pair of a Riemann surface X and mero. quad. diff. $q \neq 0$ on X.
If $p_0 \in X$ is not a zero or pole of q, there is a neighborhood U s.t.

$$z = \int_{p_0}^{p} \sqrt{q} : U \to \mathbb{C}$$

is a chart of X. Let (U, z), (V, w) be such charts with $U \cap V \neq \emptyset$, the transition function is

$$w = \pm z + (\text{const.}).$$

We may use terminologies of Euclidean geometry on (X, ω).
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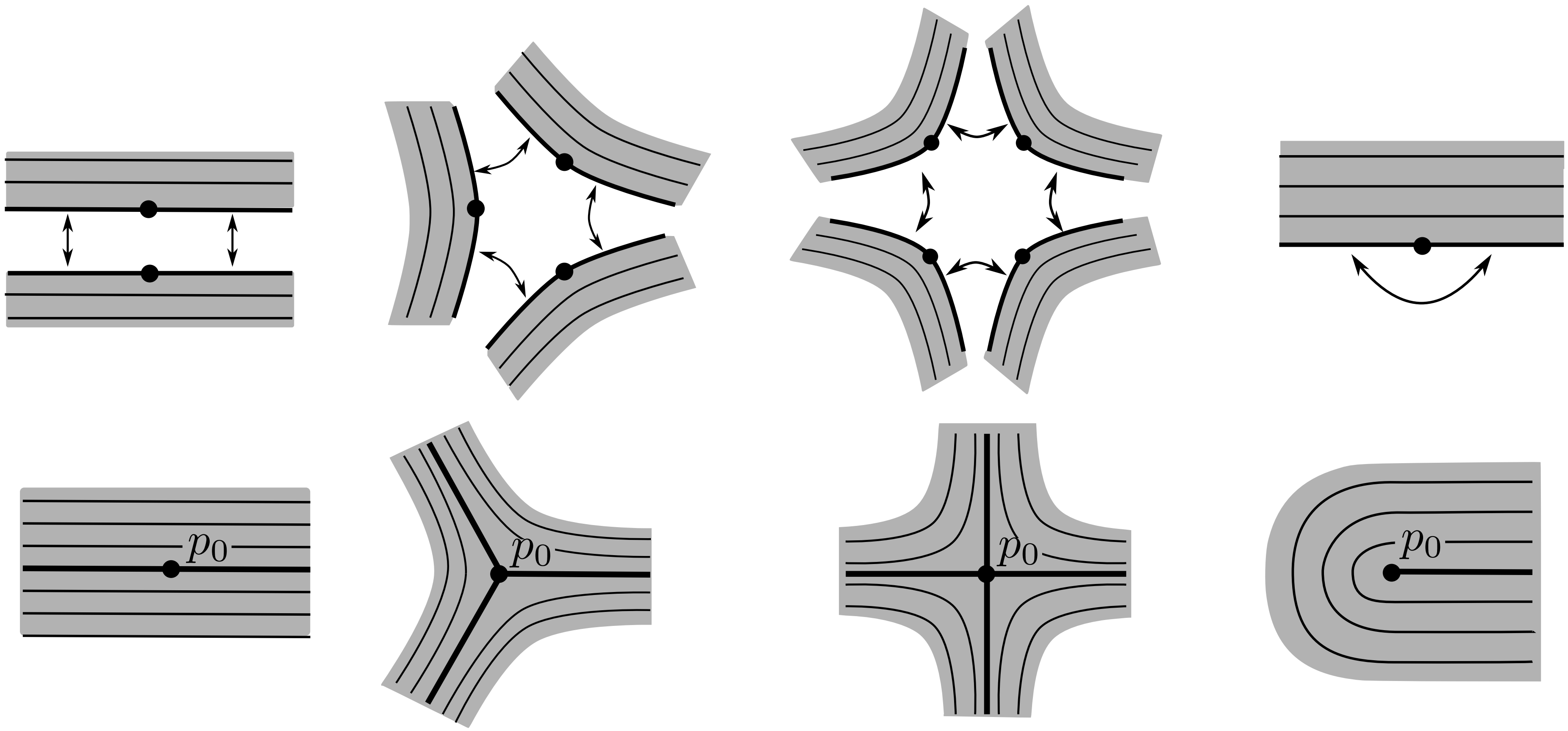
Charts around Zeros and Poles

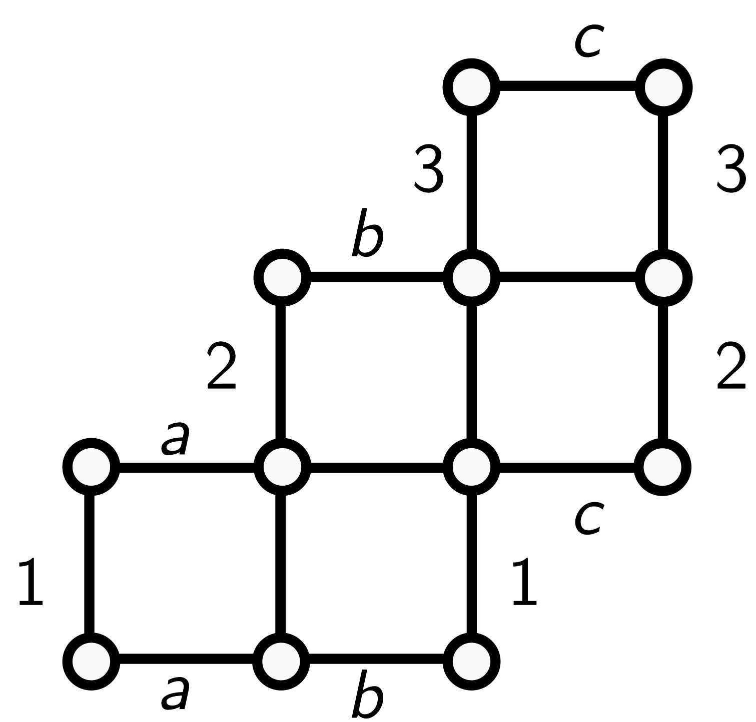
If $p_0 \in X$ is a zero of q of order n, $n+2$ half planes are glued around p_0.



not zero | simple zero | zero of order 2 | simple pole

Construction of Translation Surface (St_5)

Gluing edges by the same labels of the following polygon, the resulting surface X is of genus 3. We give a complex structure on X so that the differential dz on this polygon induces a holo. 1-form ω. Then, $\mathrm{St}_5 = (X, \omega)$ is a translation surface with only one zero.



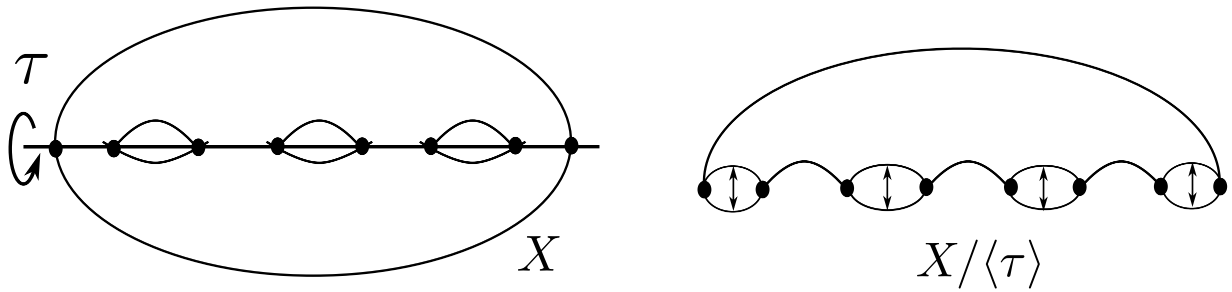
◯ : angle $= 10\pi$, zero of order 4

Hyperelliptic Involution and $\mathcal{H}^{\text{hyp}}(2g-2)$

Definition

Let X be a compact Riemann surface of genus $g \geq 2$.
The hyperelliptic involution τ of X is an order 2 automorphism which has $2g+2$ fixed points.

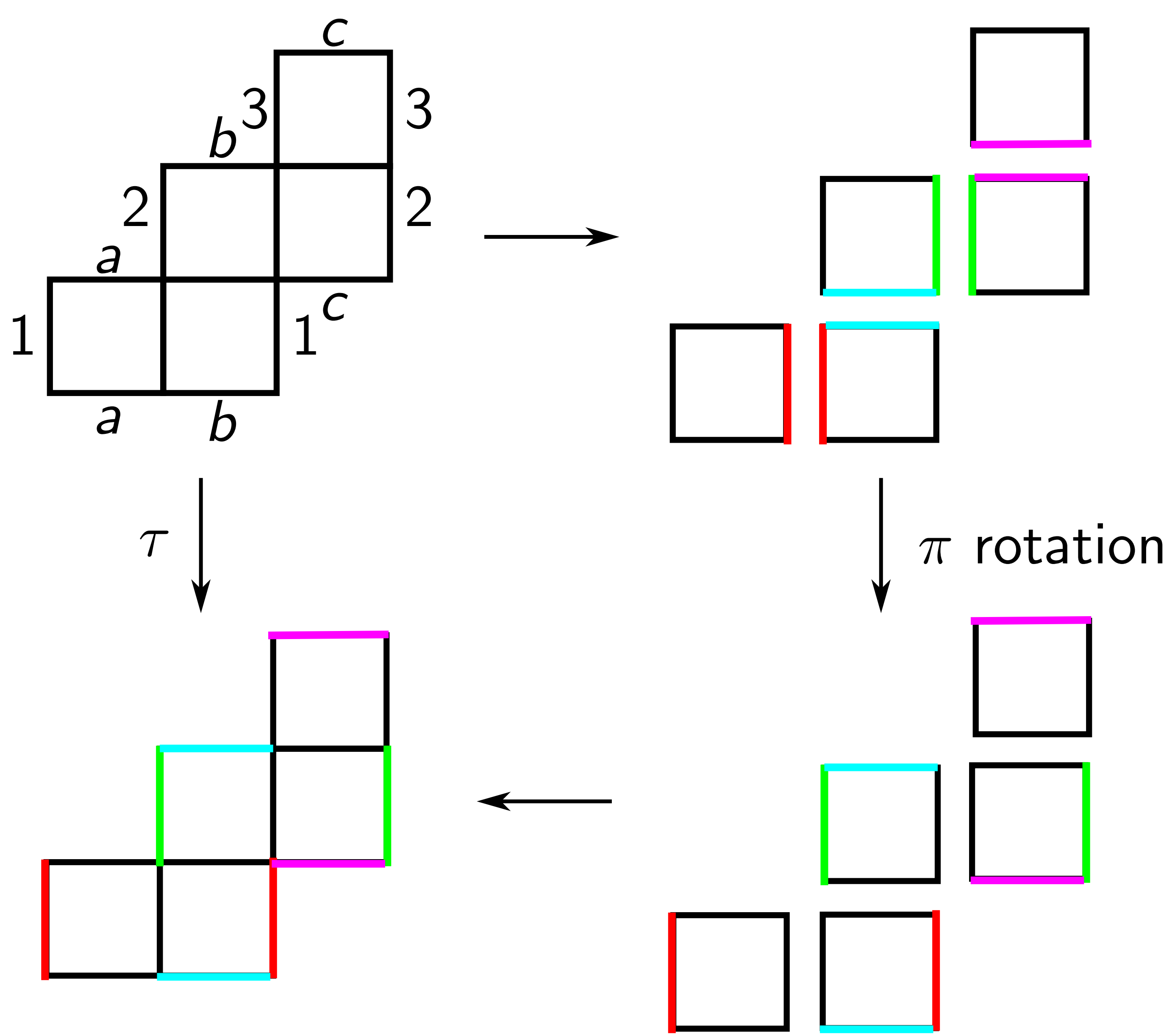
By Riemann-Hurwitz formula, $X/\langle\tau\rangle \cong \hat{\mathbb{C}}$.



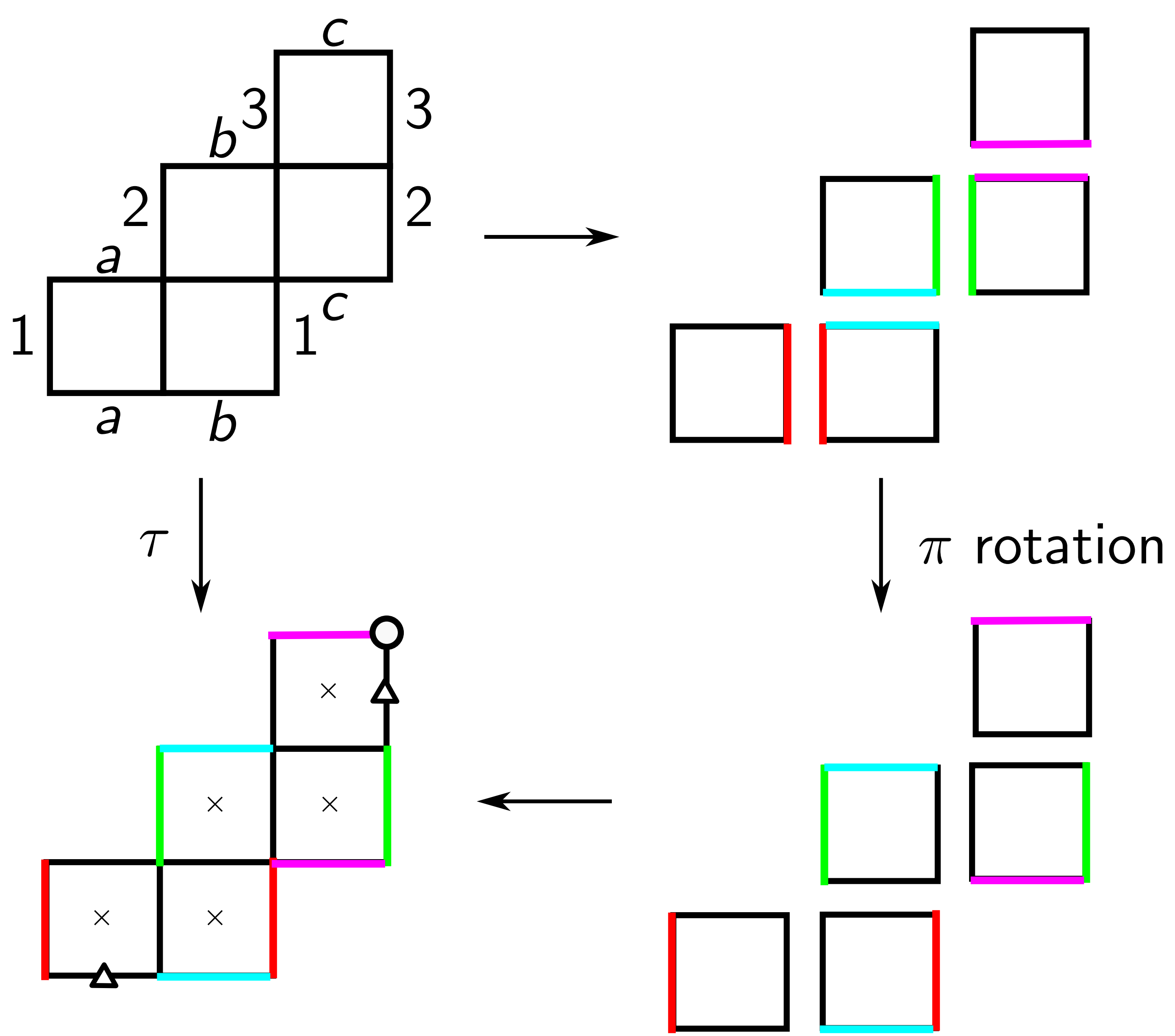
- $\mathcal{H}^{\text{hyp}}(2g-2) := \{(X,\omega) : X$ is hyperelliptic, ω has a unique zero, $\tau^*\omega = -\omega\}$
- $\mathcal{H}^{\text{hyp}}(g-1,g-1) := \{(X,\omega) : X$is hyperelliptic, ω has two zeros permuted by$\tau, \tau^*\omega = -\omega\}$

$$\mathcal{H}_g^{\text{hyp}} := \mathcal{H}^{\text{hyp}}(2g-2) \cup \mathcal{H}^{\text{hyp}}(g-1,g-1)$$

Example : $\mathrm{St}_5 \in \mathcal{H}^{\mathrm{hyp}}(4)$



Example : $\mathrm{St}_5 \in \mathcal{H}^{\mathrm{hyp}}(4)$

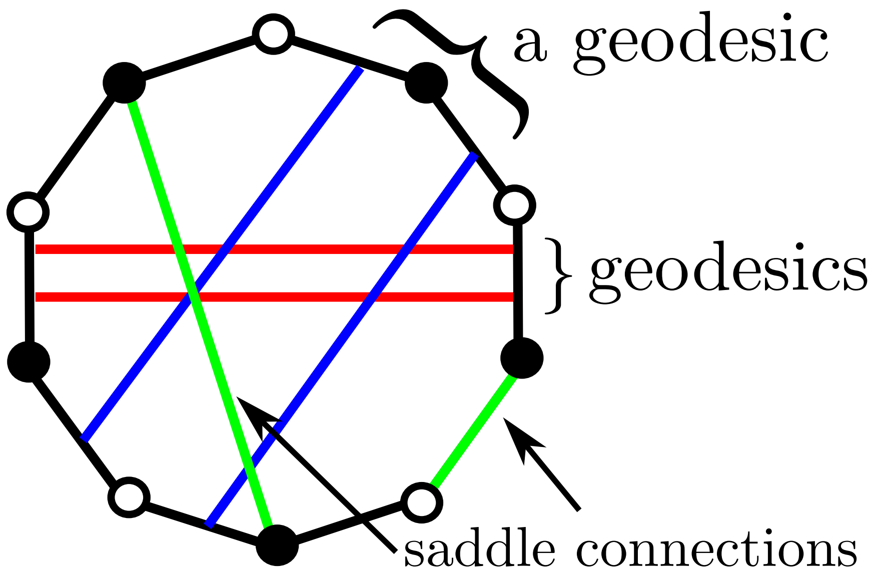


Geodesics on (X,ω)

Definition

Let (X,ω) be a translation surface.

- A simple closed geodesic on (X,ω) is called regular if it does not pass through singular points.
- A saddle connection on (X,ω) is a segment which connects singular points and does not contain singular points in its interior.



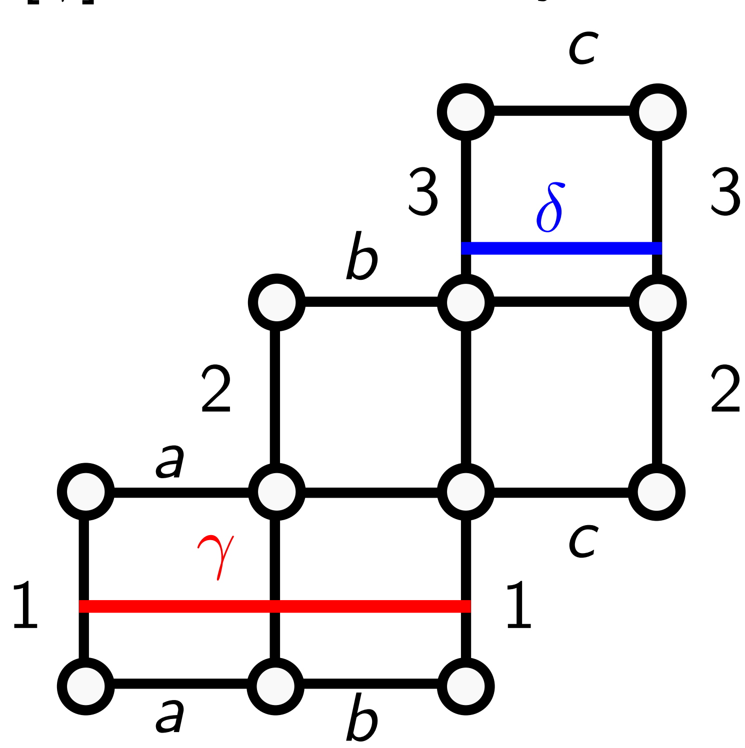
Regular geodesics are not unique in a homotopy class.

Regular geodesics → Cyliners

Let γ be a regular geodesic on a translation surface (X, ω). Then,

$$[\gamma] := \bigcup \{\delta : \delta \text{ is a regular geodesic}, \delta \sim \gamma\}$$

is an open Euclidean cylinder each of whose boundary components is a concentration of saddle connections which are parallel to γ.
We call $[\gamma]$ the maximal cylinder for γ.



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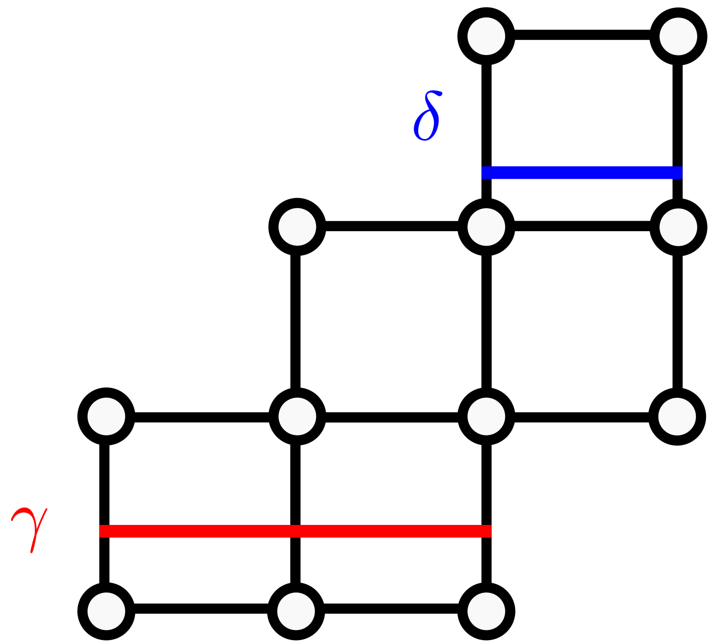

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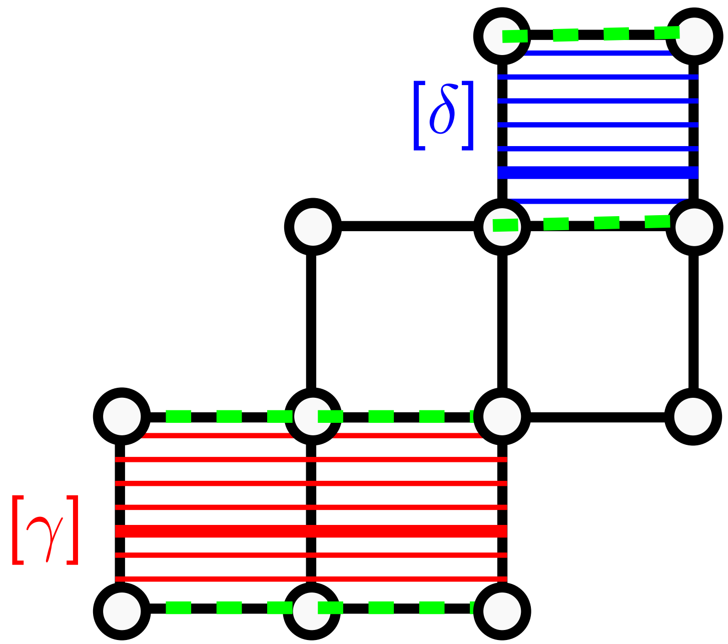

Regular geodesics $\to$ Cyliners

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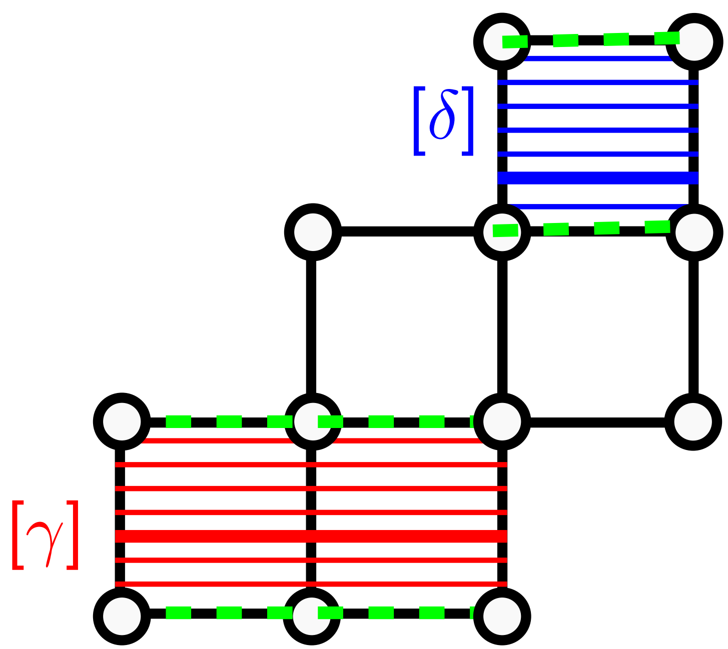


A maximal cylinder is called simple if each boundary component is a saddle connection.
Example : $[\delta]$

Curve graphs for translation surfaces

The curve graph $\mathcal{C}(X,\omega)$ for a translation surface (X,ω):

- the vertices are the cylinders $[\gamma]$ of regular closed geodesics γ of (X,ω),
- two vertices $[\gamma],[\delta]$ are connected by an edge if $[\gamma]$ and $[\delta]$ are disjoint i.e. $i(\gamma,\delta)=0$ (geometric intersection number).

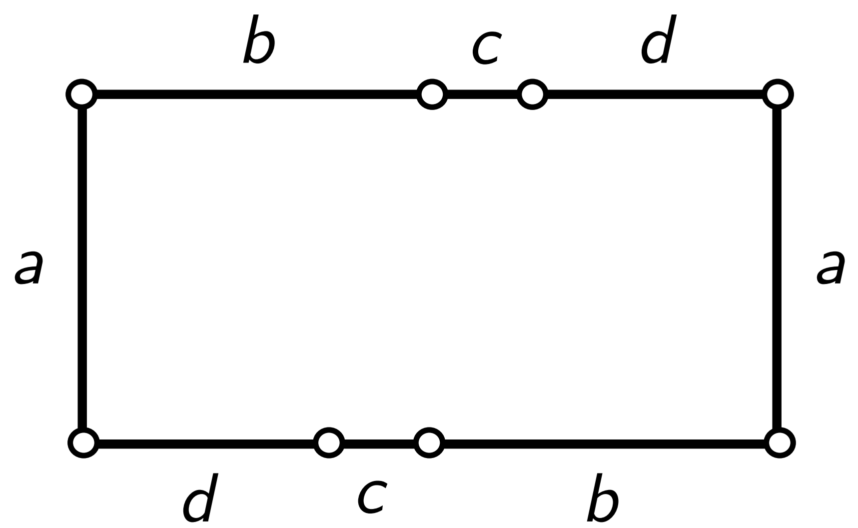
Theorem(Nguyen:$g=2$, S:$g\geq 3$)

- If $(x,\omega)\in\mathcal{H}^{\mathrm{hyp}}(2g-2)$, then $\mathcal{C}(X,\omega)$ contains K_g but not K_{g+1}.
- If $(x,\omega)\in\mathcal{H}^{\mathrm{hyp}}(g-1,g-1)$, then $\mathcal{C}(X,\omega)$ contains K_{g+1} but not K_{g+2}.

Remark

The curve graph $\mathcal{C}(X,\omega)$ is not always connected.

Example The case where $X = \overline{[\gamma]}$.

$(X,\omega) =$ 

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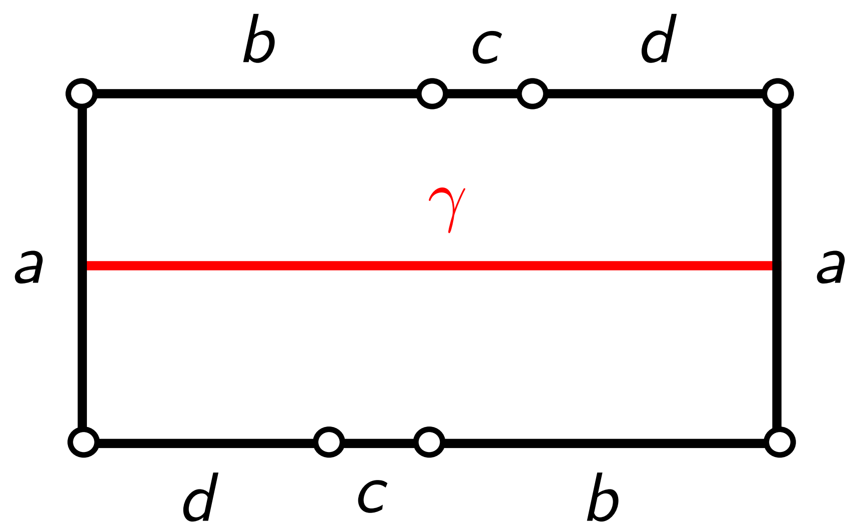
$(X,\omega) =$ 

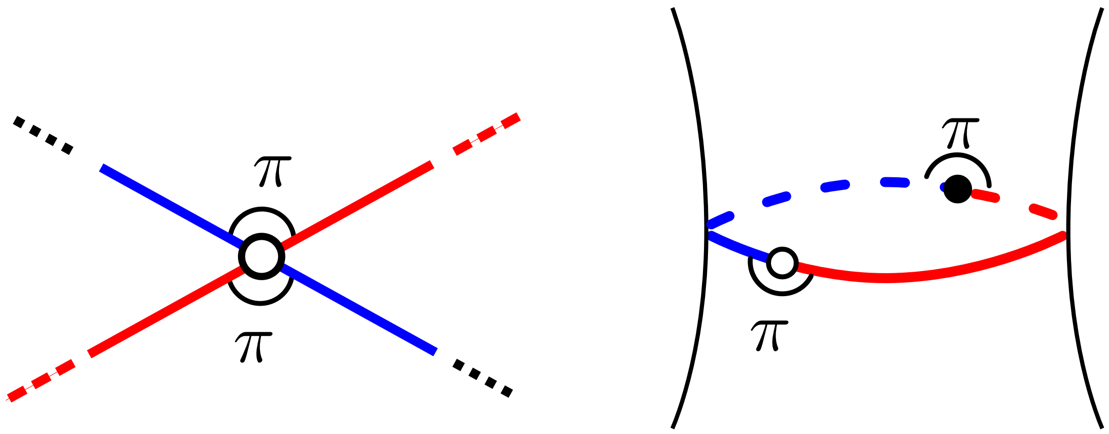
The cylinder $[\gamma]$ intersects with any closed geodesics on (X,ω).
Thus, $[\gamma]$ is an isolated vertex of $\mathcal{C}(X,\omega)$.

Degenerate cylinder

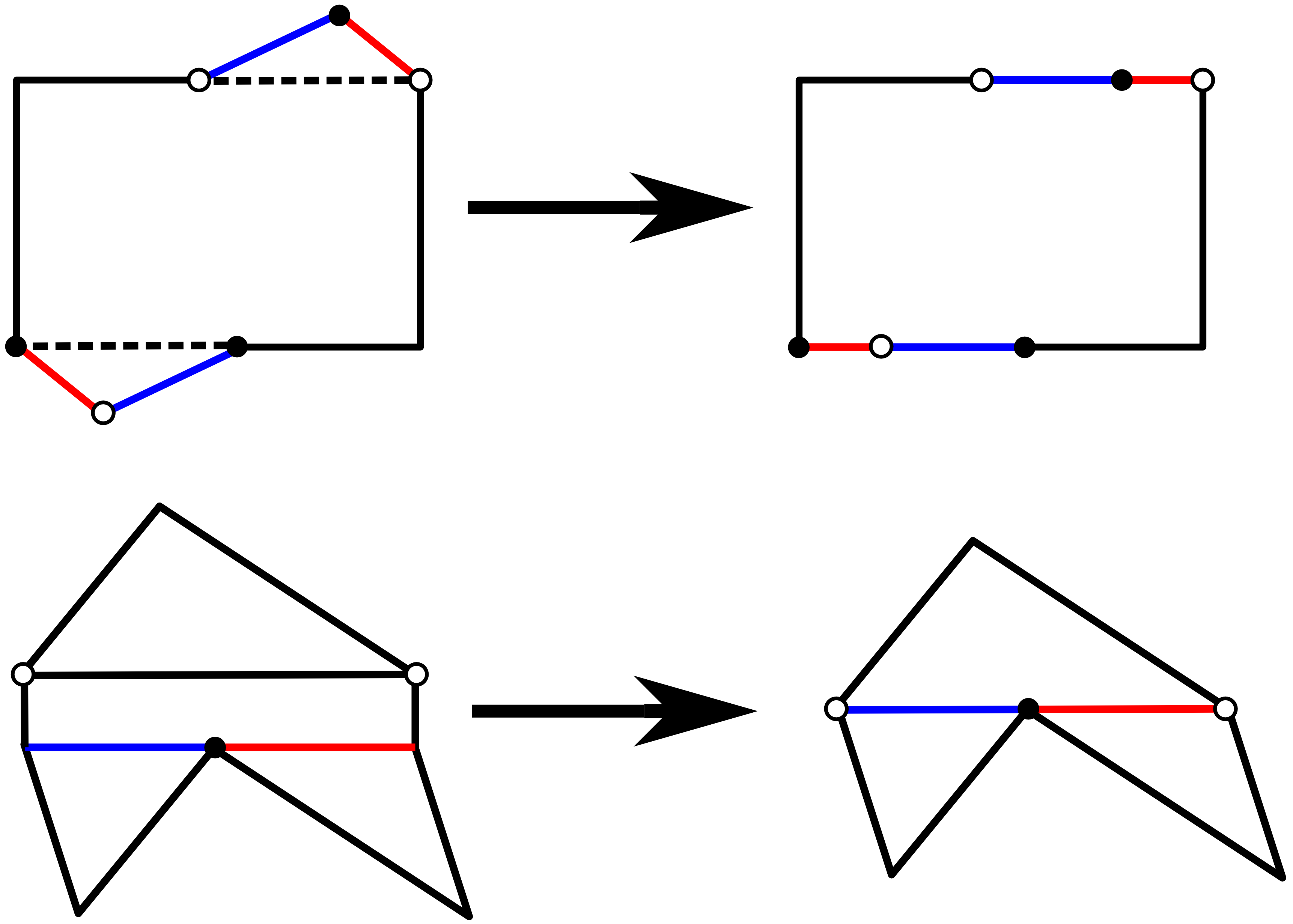
Definition (Degenerate cylinder)

A degenerate cylinder on $(X, \omega) \in \mathcal{H}_g^{\mathrm{hyp}}$ is the union of two saddle connections $[\gamma] := s_1 \cup s_2$ s.t.

- s_i is invariant under the hyperelliptic involution τ $(i = 1, 2)$,
- $[\gamma]$ satisfies the following angle condition;



Degenerate cylinder



Extended curve guraph

Definition (Extended curve graph)

The extended curve graph $\hat{\mathcal{C}}(X, \omega)$:

- the vertices are the cylinders $[\gamma]$ or degenerate cylinders of (X, ω),
- two vertices $[\gamma], [\delta]$ are connected an edge if they are disjoint.

Let $d(\cdot, \cdot)$ be the graph distance for $\hat{\mathcal{C}}(X, \omega)$ and $i(\cdot, \cdot)$ the geometric intersection number function for closed curves.

Main results

Main Theorem (Nguyen:$g = 2$, S:$g \geq 3$)

Let $[\gamma], [\delta]$ be vertices of $\hat{\mathcal{C}}(X, \omega)$. Then,

$$d([\gamma], [\delta]) \leq 3i([\gamma], [\delta]) + 6.$$

Here, $i([\gamma], [\delta]) = i(\gamma, \delta)$.

Corollary

For $(X, \omega) \in \mathcal{H}_g^{\text{hyp}}$, the curve graph $\hat{\mathcal{C}}(X, \omega)$ is connected.

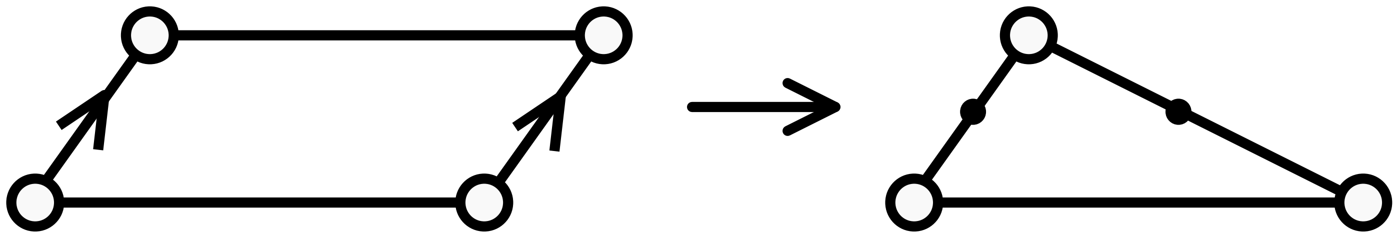
3. Proof: Geometry of Flat Surface of Genus 0

Flat Surface (Y, q) of genus 0

Let $(X, \omega) \in \mathcal{H}_g^{\mathrm{hyp}}$ with the hyperelliptic involution τ.
$(Y, q) = (X, \omega)/\langle\tau\rangle$: a flat surface of genus 0. Here, q is the mero. quadratic differential induced by ω^2on Y. The flat surface (Y, q) has a unique singular point $\circ$ and some poles $\bullet$.

Proposition

For any regular geodesic γ on (X, ω), the maximal cylinder $[\gamma]$ is invariant under τ. Especially, the "mid" core curve of $[\gamma]$ passes through two fixed points of τ other than $\circ$.

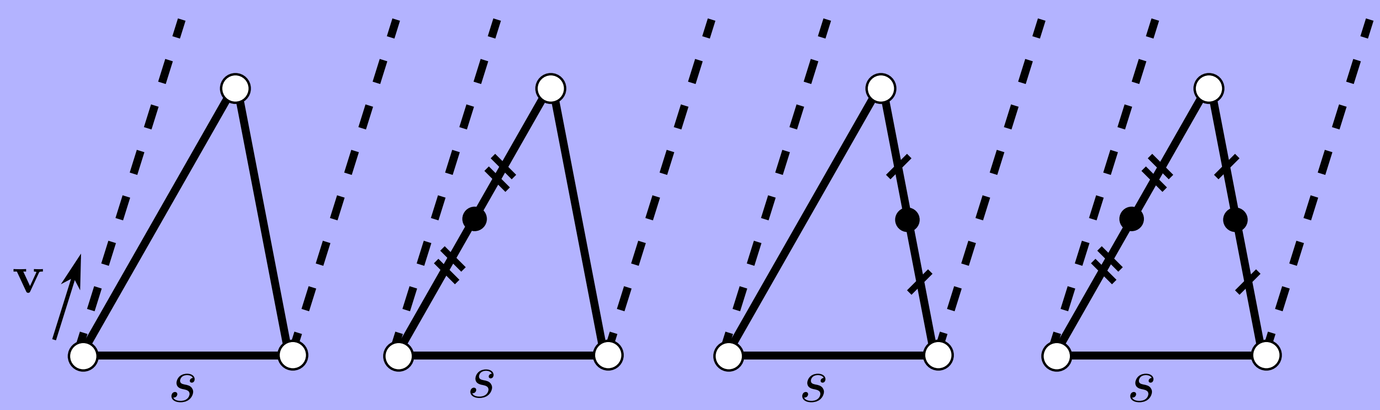


(X, ω) (Y, q)

Proposition

Let s be a saddle connection on (Y, q) and $\mathbf{v} \in \mathbb{R}^2$ with $\mathbf{v} \not\parallel s$.

There exists a triangle Δ satisfying the following:

- one of the edges is s,
- there are no singularities in its interior,
- only the vertices are $\circ$ and
- 

Note: One of the sides of Δ may be parallel to $\mathbf{v}$.

Remark

We construct "good" triangulations of (Y, q) by this proposition.

The "ends" of this triangulation are triangles corresponding to simple cylinders on (X, ω).

Reduce the problem to simple cylinders

Lemma 1

Assume that $\overline{[\gamma]} \neq X$ and $i([\gamma], [\delta]) > 0$. Then, there exists a simple cylinder $[\gamma']$ such that $d([\gamma], [\gamma']) \leq 1$ and $i([\gamma'], [\delta]) \leq i([\gamma], [\delta])$.

Lemma 2

Assume that $\overline{[\gamma]} = X$. Then, there exists a simple cylinder $[\gamma']$ such that $d([\gamma], [\gamma']) = 2$ and $i([\gamma'], [\delta]) \leq i([\gamma], [\delta])$.

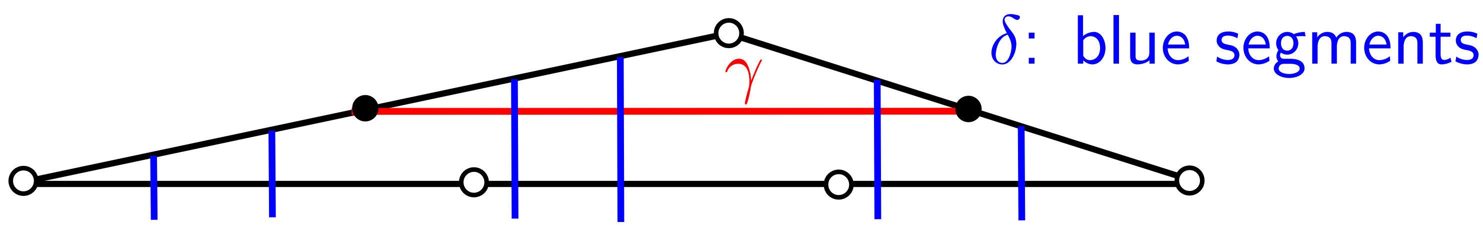
- $d([\gamma], [\delta])$
  $\leq d([\gamma], [\gamma'])d([\gamma'], [\delta']) + d([\delta'], [\delta]) \leq d([\gamma'], [\delta']) + 4$
- $i([\gamma], [\delta]) \geq i([\gamma'], [\delta]) \geq i([\gamma'], [\delta'])$

Proof of Lemma 1

Lemma 1

Assume that $\overline{[\gamma]} \neq X$ and $i([\gamma],[\delta]) > 0$. Then, there exists a simple cylinder $[\gamma']$ such that $d([\gamma],[\gamma']) \leq 1$ and $i([\gamma'],[\delta]) \leq i([\gamma],[\delta])$.

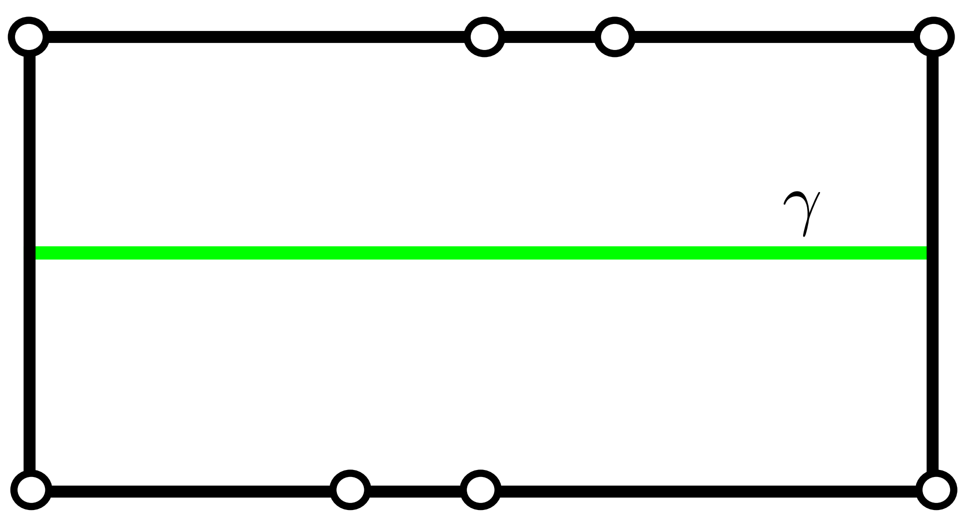
<u>Proof</u> Let $(Y,q) = (X,\omega)/ < \tau >$. Then q has a unique zero $\circ$ and some simple poles $\bullet$. If $[\gamma]$ is a simple cylinder, then $[\gamma'] = [\gamma]$. Assume that $[\gamma]$ is not a simple cylinder.



Lemma 2

Assume that $\overline{[\gamma]} = X$. Then, there exists a simple cylinder $[\gamma']$ such that $d([\gamma],[\gamma']) = 2$ and $i([\gamma'],[\delta]) \leq i([\gamma],[\delta])$.

Proof

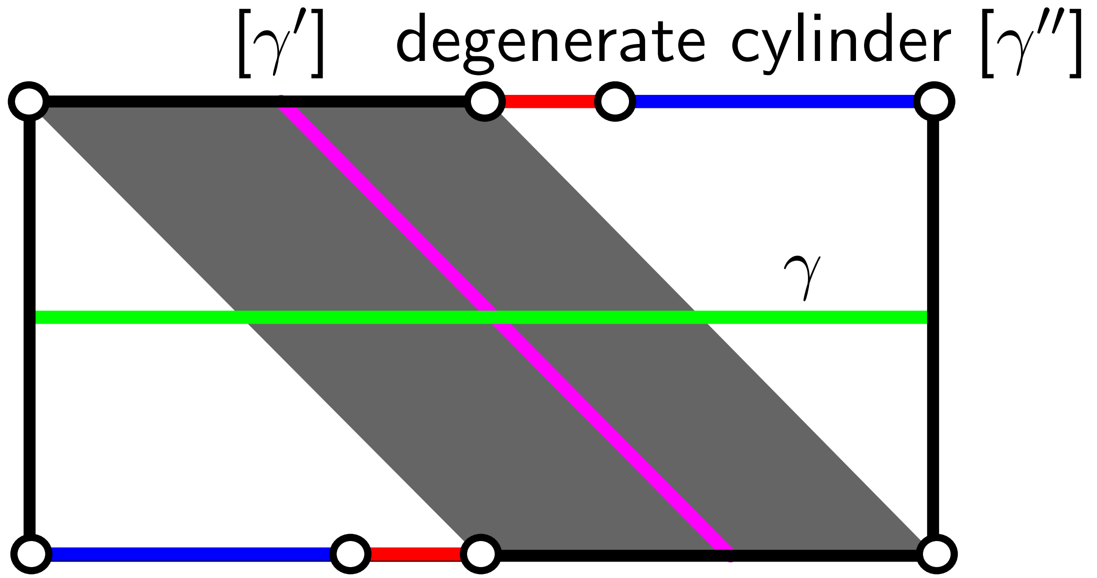


Proof of Lemma 2

Lemma 2

Assume that $\overline{[\gamma]} = X$. Then, there exists a simple cylinder $[\gamma']$ such that $d([\gamma],[\gamma']) = 2$ and $i([\gamma'],[\delta]) \leq i([\gamma],[\delta])$.

Proof



- $1 < d([\gamma],[\gamma']) \leq d([\gamma],[\gamma'']) + d([\gamma''],[\gamma']) = 2$
  $\rightarrow d([\gamma],[\gamma']) = 2$
- $i([\gamma'],[\delta]) \leq i([\gamma],[\delta])$ by construction.

Hereafter, the direction of a cylinder $C = [\gamma]$ is a direction of γ.

Lemma 3

Let C' and D' be simple cylinders. If there exists a leaf of direction D' which does not intersect C', then $d(C', D') \leq 2$.

Lemma 4

Let C' and D' be simple cylinders. If all leaves of direction D' intersect C', then there exists a simple cylinder C'' such that $d(C', C'') \leq 3$ and $i(C'', D') < i(C', D')$.

If C' and D' satisfy the assumption of Lemma 3, then $d(C', D') \leq 2$. If not, apply Lemma 4; there exists a simple cylinder C'' such that

- $d(C', D') \leq d(C', C'') + d(C'', D') \leq 3 + d(C'', D')$
- $i(C'', D') < i(C', D')$

$\to \cdots \to d(C, D) \leq d(C', D') + 4 \leq \cdots \leq 3i(C, D) + 6.$

Further Problem: finiteness of vertices of quotient graph

$\mathrm{Aff}^+(X,\omega)$
$:= \{h : X \to X : \text{ori. pres. homeo} : \text{h is an affine map of } (X,\omega)\}$
i.e. h is locally of the form $h(z) = Az + c$ on (X,ω).

$\Gamma(X,\omega) :=$
$\left\{A \in \mathrm{SL}(2,\mathbb{R}) : h \text{ is locally } h(z) = Az + c \ (h \in \mathrm{Aff}^+(X,\omega))\right\}$
A map $h \in \mathrm{Aff}^+(X,\omega)$ induces an automorphism $\rho(h)$ of $\hat{\mathcal{C}}(X,\omega)$.

Theorem (Nguyen)

Let $(X,\omega) \in \mathcal{H}_g^{\mathrm{hyp}}$. Then, $\mathrm{Aut}\left(\hat{\mathcal{C}}(X,\omega)\right) = \rho\left(\mathrm{Aff}^+(X,\omega)\right)$.

Theorem (Nguyen)

Let $(X,\omega) \in \mathcal{H}_2^{\mathrm{hyp}}$. Then, $\Gamma(X,\omega)$ is a lattice in $\mathrm{SL}(2,\mathbb{R})$ iff the number of vertices of $\hat{\mathcal{C}}(X,\omega)/\mathrm{Aut}\left(\hat{\mathcal{C}}(X,\omega)\right)$ is finite.

$\to$ How about $(X,\omega) \in \mathcal{H}_g^{\mathrm{hyp}}$?

Further Problem: Gromov hyperbolicity and inclusion of K_n

Theorem (Nguyen)

If $(X, \omega) \in \mathcal{H}_2^{\mathrm{hyp}}$, then the extended graph $\hat{\mathcal{C}}(X, \omega)$ is Gromov hyperbolic.

$\rightarrow$ How about $(X, \omega) \in \mathcal{H}_g^{\mathrm{hyp}}$?

Theorem(Nguyen:$g = 2$, S:$g \geq 3$)

- If $(x, \omega) \in \mathcal{H}^{\mathrm{hyp}}(2g-2)$, then $\mathcal{C}(X, \omega)$ contains K_g but not K_{g+1}.
- If $(x, \omega) \in \mathcal{H}^{\mathrm{hyp}}(g-1, g-1)$, then $\mathcal{C}(X, \omega)$ contains K_{g+1} but not K_{g+2}.

$\rightarrow$ How about $\hat{\mathcal{C}}(X, \omega)$?

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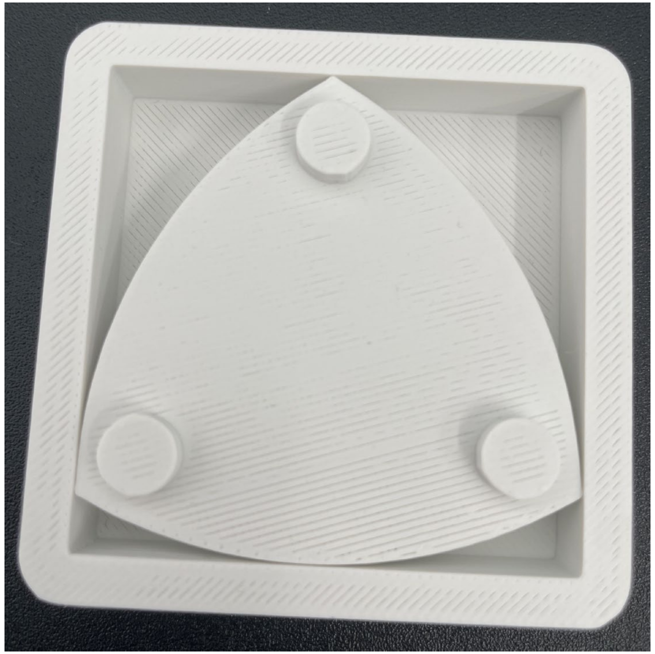
3Dプリンタ

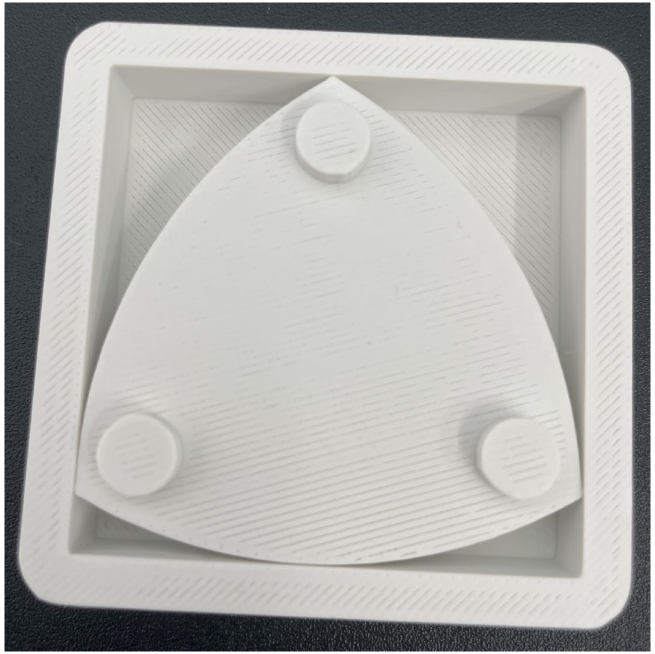
- QIDI TECH QI Pro
- 46.7奥行きx 47.7幅x 48.9高さ(cm)
- 8万円くらい
- 材料：フィラメント
（数種類の素材がある）

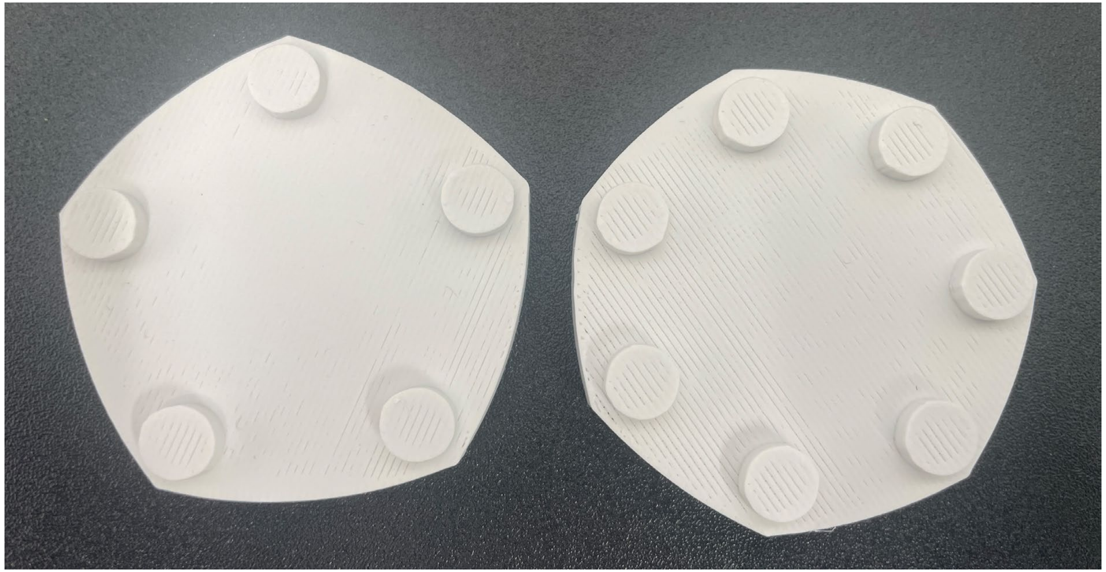
色々作ってみた！

ルーローの三角形

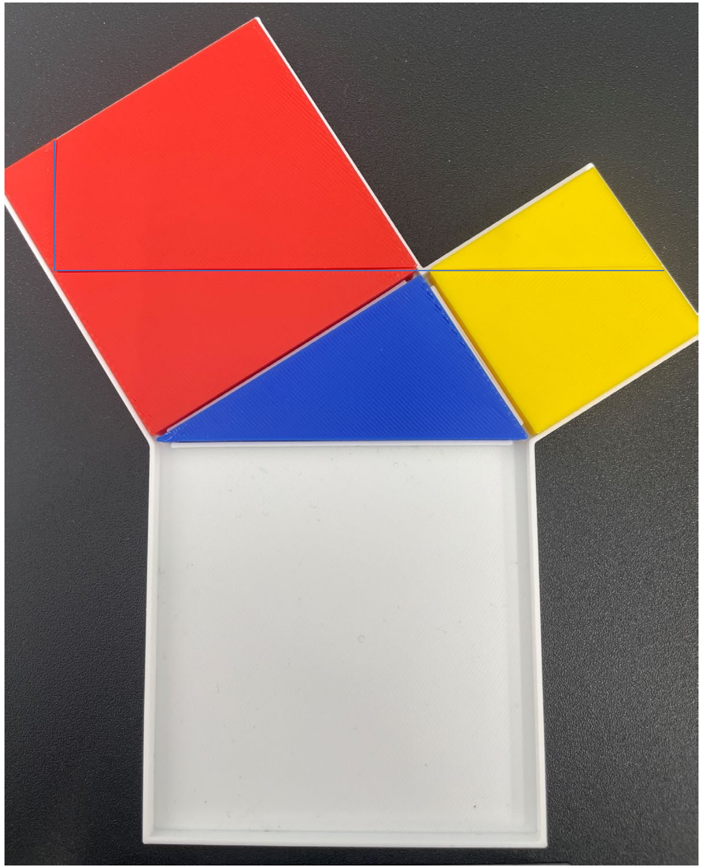


ルーローの多角形





三平方の定理



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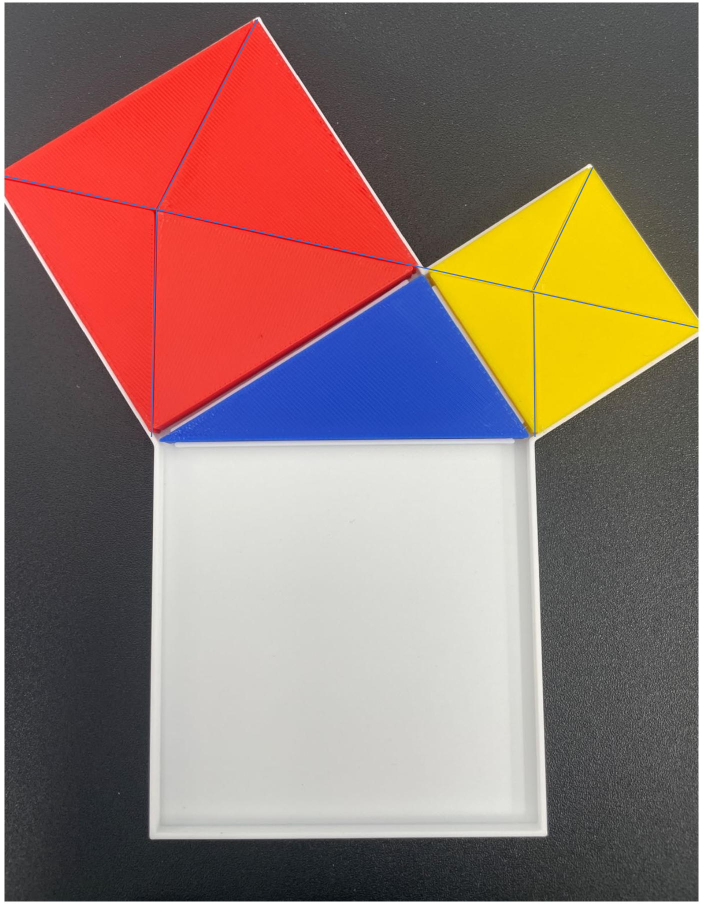
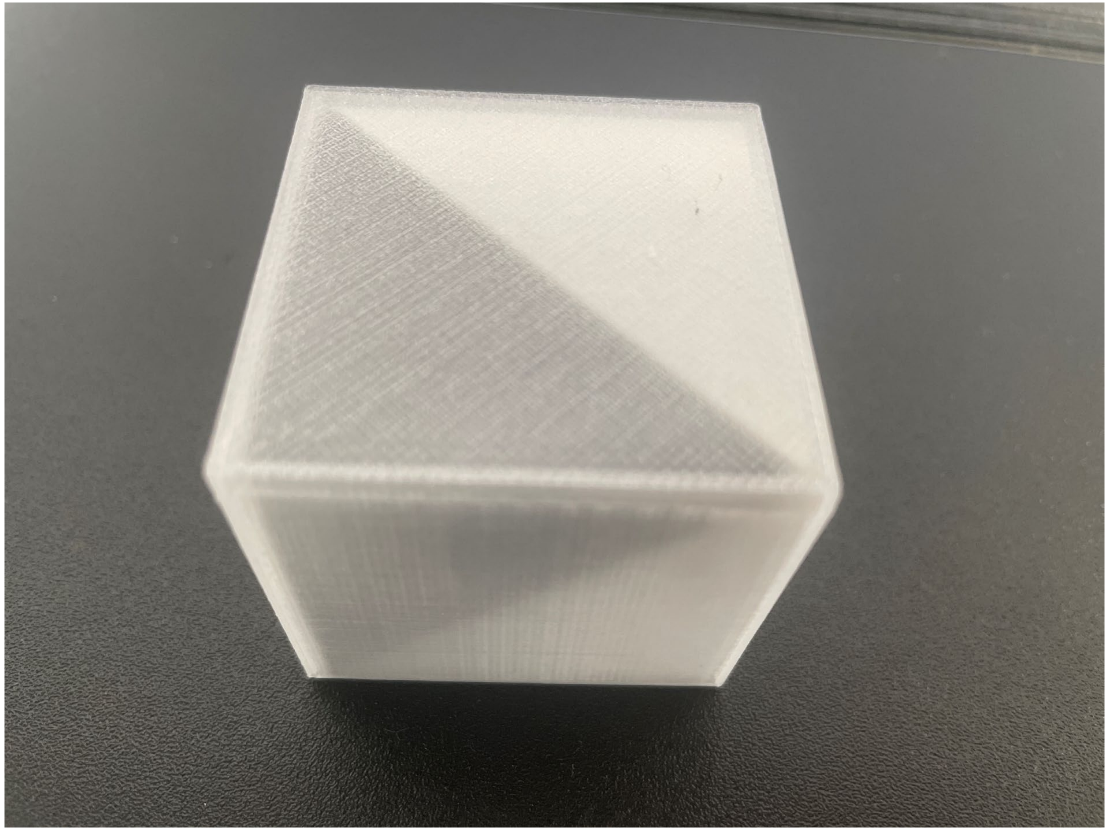


二項分布

三角錐の体積



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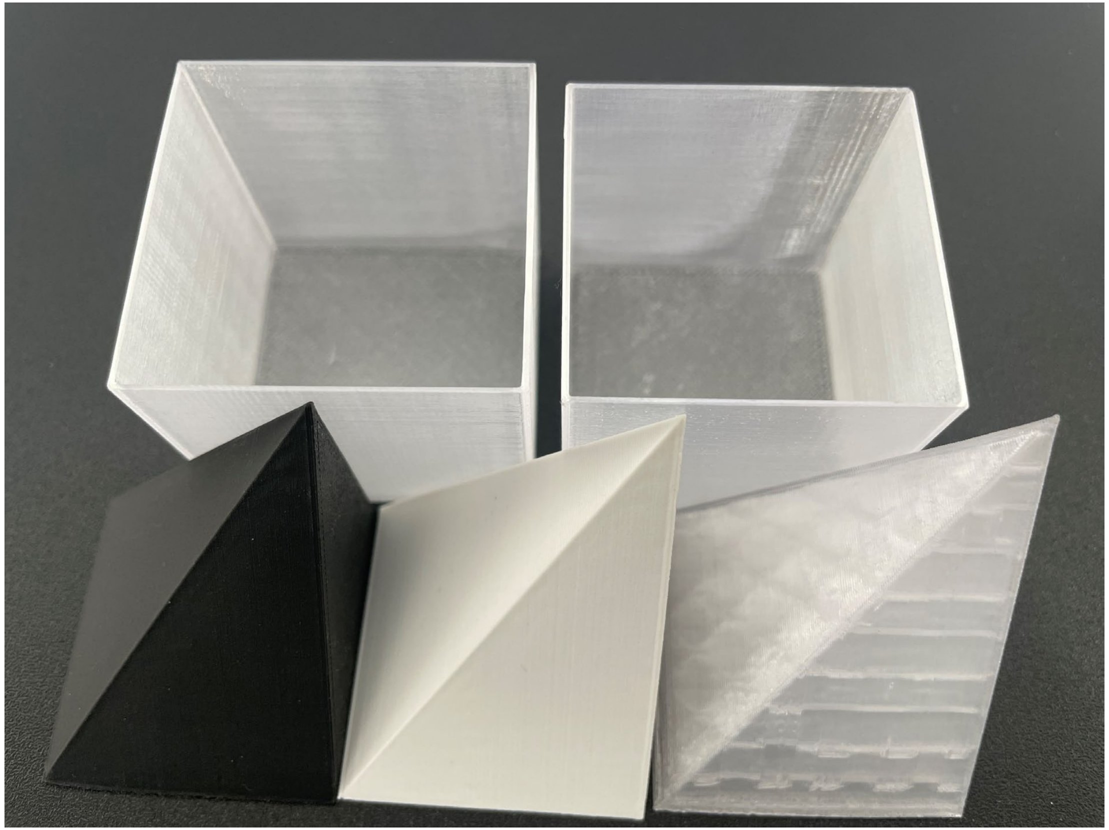
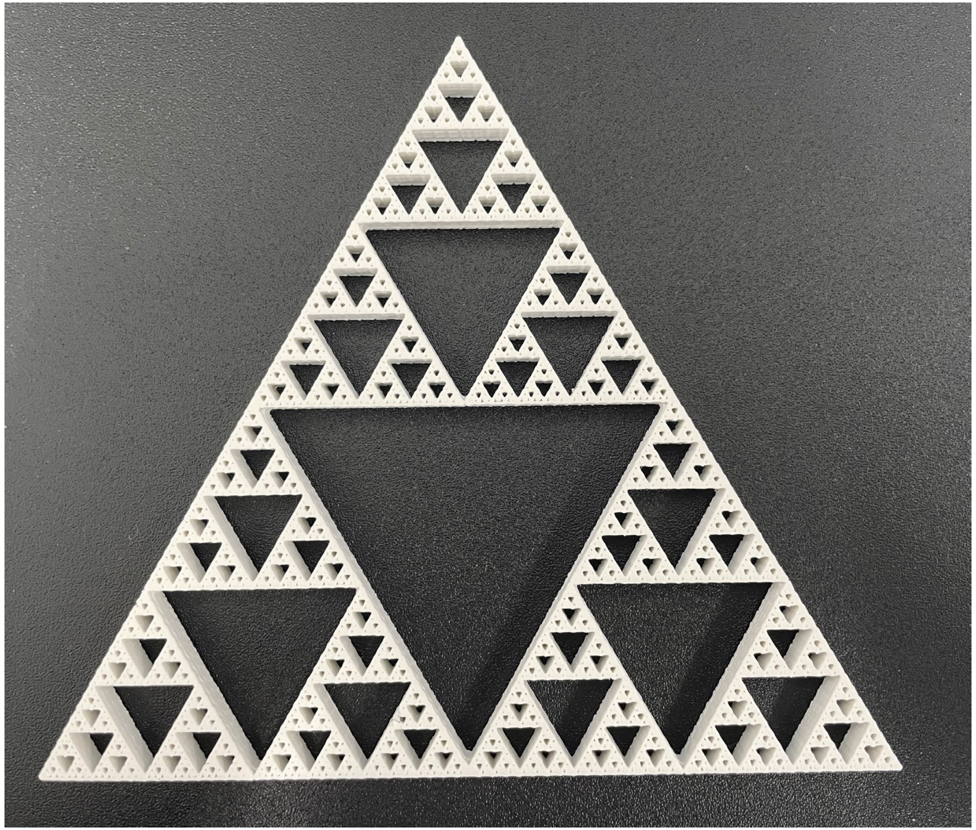
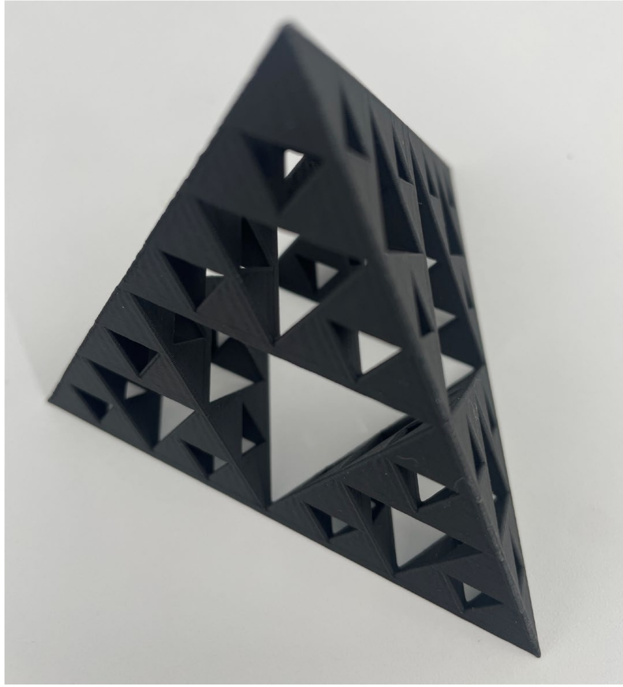
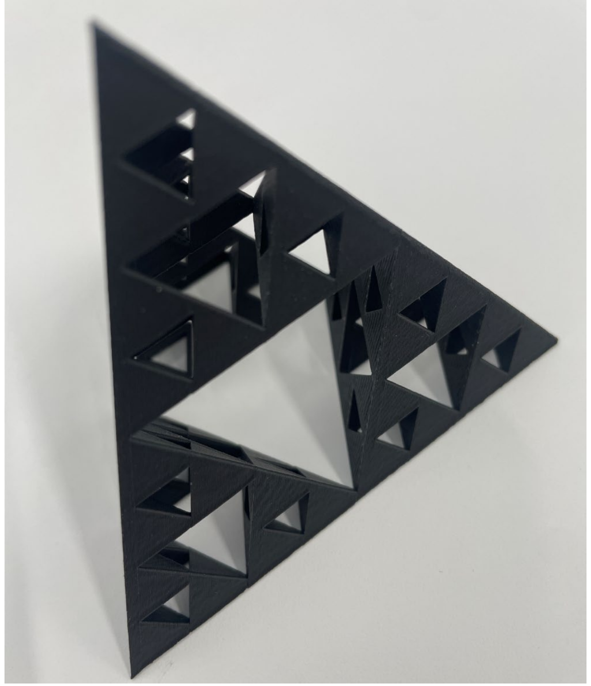


シェルピンスキーのギャスケット



シェルピンスキー四面体





立体：投影図

Q. 上から見ると円，正面から見ると正三角形，
横から見ると四角形であるような立体は？

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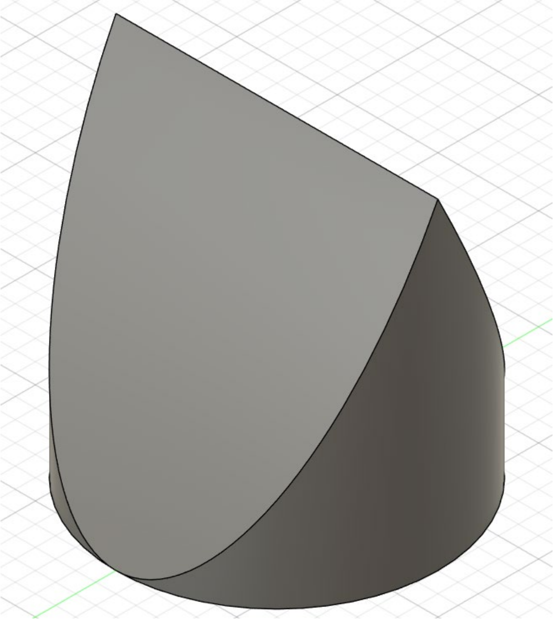
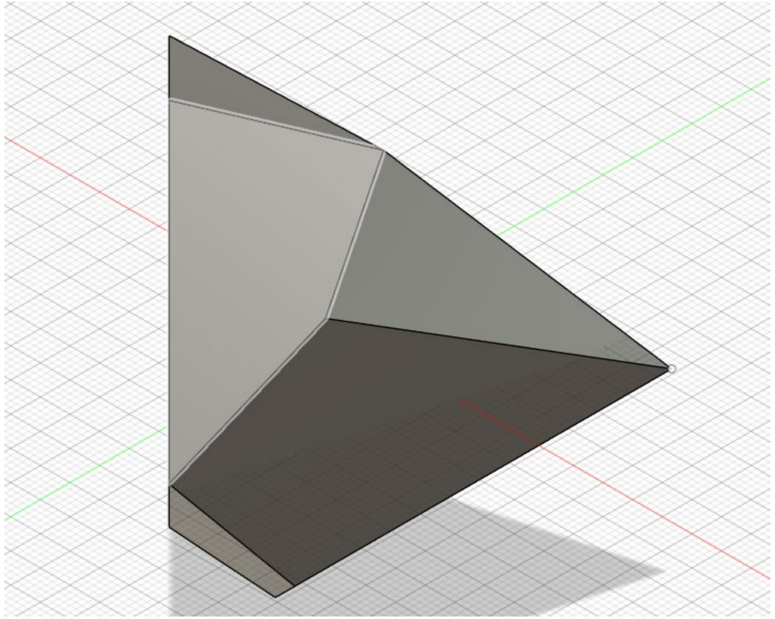


立体：投影図

Q. 上から見ると正三角形，正面から見ると正方形，
横から見ると正五角形であるような立体は？

立体：投影図

Q．上から見ると正三角形，正面から見ると正方形，
横から見ると正五角形であるような立体は？



楕円の面積公式：関孝和の方法