Discrete Integrable System and Invariant Variety of Periodic Points

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＊ 本講演は齋藤暁氏（首都大)、脇本佑紀君（首都大）との共同研究に基づく

Outline

Conclusion

IVPP Theorem :

IVPP Theorem $\sim$ Integrable/Non Integrable Conjecture

$\Downarrow$ Sufficient condition of integrability

Derivation of IVPPs by SC $\sim$ "Projective resolution" of IDP

$\Downarrow$ Invariant/Parameter duality

Integrable ADE $\sim \bigoplus_n$ Recurrence Equation(RE) of period n

Intersections of IVPPs :

Intersections of IVPPs $\sim$ Problems of IVPP Theorem?

$\Downarrow$ Some conjectures and analysis of some examples

Origins of intersections of IVPPs $\sim$ IDP

$\Downarrow$ Integrable/Non Integrable Transition

Fate of Julia set $\Rightarrow$ IDP?

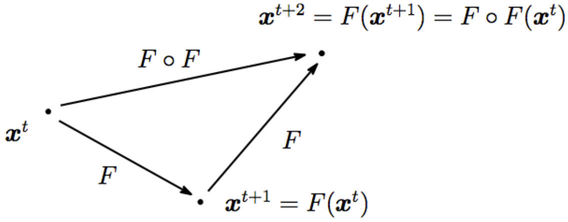
1. Introduction

- Discrete Dynamical System and **Differential Equation Dynamical System**
（離散力学系と**差分方程式力学系**）
- Invariants and Level Set
（不変量と等位面）
- Variety of Periodic Points(VPP) and **Invariant Variety of Periodic Points(IVPP)**
（周期点代数多様体と**不変周期点代数多様体**）
- **IVPP Theorem** and Integrable/Non Integrable Conjecture
（**IVPP 定理**と可積分/非可積分予想）

Discrete Dynamical System

Discrete Dynamical System（離散力学系）：

$$F^{(T)} : \boldsymbol{x}^t \mapsto \boldsymbol{x}^{t+T}, \quad \boldsymbol{x}^t, \boldsymbol{x}^{t+T} \in \mathbb{C}^d, \quad T \in \mathbb{Z}$$



Difference Equation (DE) （差分方程式）:

$$\tilde{F}_i(\boldsymbol{x}^t, \boldsymbol{x}^{t+1}) = 0, \quad \boldsymbol{x}^t, \boldsymbol{x}^{t+1} \in \mathbb{C}^d, \quad i = 1, \ldots, d$$

Map/Difference Equation and Indeterminate Points (IDP)（不定点）:

Solve DE about variable $\boldsymbol{x}^{t+1}$ $\Rightarrow$ Map :

$$\boldsymbol{x}^{t+1} = F(\boldsymbol{x}^t), \quad \boldsymbol{x}^t \in \mathbb{C}^d \setminus \mathrm{IDP}(\tilde{F})$$

$\mathrm{IDP}(\tilde{F})$ is set of **indeterminate points (IDP)** of DE $\tilde{F}$ by implicit function theorem :

$$\mathrm{IDP}(\tilde{F}) := \left\{ \boldsymbol{x} \in \mathbb{C}^d \;\middle|\; \partial_{\boldsymbol{x}^{t+1}} \tilde{F}_i(\boldsymbol{x}^t, \boldsymbol{x}^{t+1}) = 0,\ i = 1, \ldots, d \right\}$$

* We consider a map as “**rational** map”（有理写像）, and a DE as “**algebraic** DE(ADE)”（代数差分方程式）[1,2].

[1] W. Michael, *“Algebraic difference equations”*, Lecture notes, http://www.algebra.rwth-aachen.de/en/Mitarbeiter/Wibmer/AlgebraicDifferenceEquations.pdf, 2013.
[2] R. M. Cohn, *“Difference algebra”* , Interscience Publishers John Wiley and Sons, New York-London-Sydeny, 1965.

The Time Evolution of The DE $\tilde{F}$:

$$\tilde{F}_i(\boldsymbol{x}^t, \boldsymbol{x}^{t+1}) = 0, \quad \tilde{F}_i(\boldsymbol{x}^{t+1}, \boldsymbol{x}^{t+2}) = 0, \quad i = 1, \dots d$$

$$\Downarrow \quad \text{Elimination of } \boldsymbol{x}^{t+1}$$

$$\tilde{F}_i^{(2)}(\boldsymbol{x}^t, \boldsymbol{x}^{t+2}) = 0, \quad \boldsymbol{x}^t, \boldsymbol{x}^{t+2} \in \mathbb{C}^d, \quad i = 1, \dots, d$$

In this manner, we can decide the time evolution of the difference equation $\tilde{F}$ at any time.

* We can use the method of the **Gröbner basis** for an ADE.

Integrability v.s. Invariants

2 dimensional Maps :

$$F_{\mathrm{2dM\ddot{o}b}} : \boldsymbol{x}^t := (x_1^t, x_2^t) \mapsto \boldsymbol{x}^{t+1} := (x_1^{t+1}, x_2^{t+1}) = \left(x_1^t \frac{1 - x_2^t}{1 - x_1^t}, x_2^t \frac{1 - x_1^t}{1 - x_2^t} \right)$$

$$F_{\mathrm{2dlog}} : \boldsymbol{x}^t \mapsto \boldsymbol{x}^{t+1} = \left((x_1^t)^2 x_2^t (1 - x_1^t), \frac{1}{x_1^t (1 - x_1^t)} \right)$$

Invariant（不変量）：

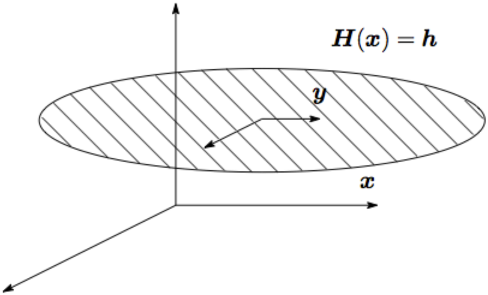
$$H(\boldsymbol{x}^{t+1}(\boldsymbol{x}^t)) = H(\boldsymbol{x}^t) = x_1^t x_2^t$$

$\Downarrow$ Elimination of x_2 using the invariant, $y := x_1$

Maps on Level Set（等位面） $H(\boldsymbol{x}) = h$:

Möbius Map :	$(F_{\mathrm{2dM\ddot{o}b}})_h : y^t \mapsto y^{t+1} = \frac{y^t - h}{1 - y^t}$,	**Integrable!**
logistic Map :	$(F_{\mathrm{2dlog}})_h : y^t \mapsto y^{t+1} = h y^t (1 - y^t)$,	**NonIntegrable!**

Invariants and Level Set



where $\boldsymbol{y}$ is dep. variable on level set（等位面） $\boldsymbol{H}(\boldsymbol{x}) = \boldsymbol{h}$.

Fixed Points（不動点）：

- Map :

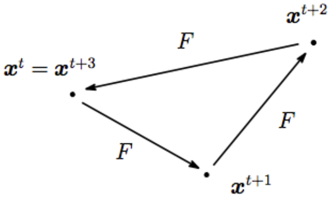
$$\mathrm{FP}(F) := \left\{ \boldsymbol{x} \in \mathbb{C}^d \;\middle|\; \boldsymbol{x} = F(\boldsymbol{x}) \right\}$$

- DE :

$$\mathrm{FP}(\tilde{F}) := \left\{ \boldsymbol{x} \in \mathbb{C}^d \;\middle|\; \tilde{F}_i(\boldsymbol{x}, \boldsymbol{x}) = 0,\ i = 1, \dots, d \right\}$$

Periodic Points（周期点）：

$$\mathrm{Period}(F, n) := \mathrm{FP}(F^{(n)}) \setminus \left(\bigcup_{n|m} \mathrm{FP}(F^{(m)}) \right)$$



Invariant Variety of Periodic Points

2 dimensional Möbius map

- Rational Map（有理写像）：

$$F_{2\text{dMöb}} : \boldsymbol{x}^t \mapsto \boldsymbol{x}^t = \left(\frac{N_1(\boldsymbol{x}^t)}{D_1(\boldsymbol{x}^t)}, \frac{N_2(\boldsymbol{x}^t)}{D_2(\boldsymbol{x}^t)} \right) = \left(x_1^t \frac{1 - x_2^t}{1 - x_1^t}, x_2^t \frac{1 - x_1^t}{1 - x_2^t} \right)$$

- **Algebraic** Difference Equation(ADE)（**代数**差分方程式）：

$$(\tilde{F}_{2\text{dMöb}})_i(\boldsymbol{x}^t, \boldsymbol{x}^{t+1}) = x_i^{t+1}(1 - x_i^t) - x^t(1 - x_{i+1}^t), \quad i \in \mathbb{Z}/2\mathbb{Z}$$

- Invariant（不変量）：

$$H(\boldsymbol{x}^t) = x_1^t x_2^t$$

- Fixed Points and **Periodic Points**（不動点と**周期点**）：

$$\mathrm{FP}(F_{2\text{dMöb}}) = \left\{ \boldsymbol{x} \in \mathbb{C}^2 \;\middle|\; x_1 - x_2 = 0 \right\}$$

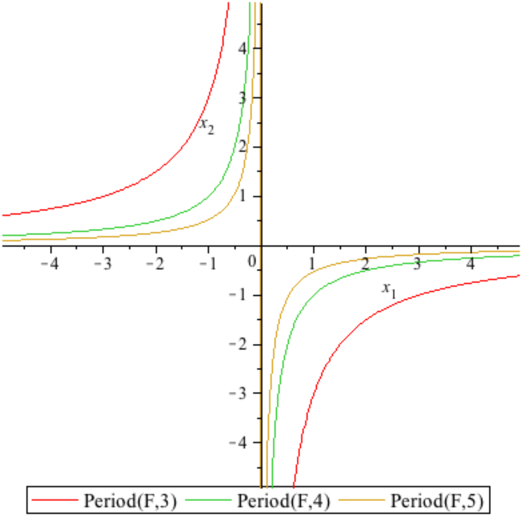
$$\mathrm{Period}(F_{2\text{dMöb}}, 2) = \emptyset$$

$$\mathrm{Period}(F_{2\text{dMöb}}, 3) = \left\{ \boldsymbol{x} \in \mathbb{C}^2 \;\middle|\; x_1 x_2 + 3 = H(\boldsymbol{x}) + 3 = 0 \right\}$$

$$\mathrm{Period}(F_{2\text{dMöb}}, 4) = \left\{ \boldsymbol{x} \in \mathbb{C}^2 \;\middle|\; x_1 x_2 + 1 = H(\boldsymbol{x}) + 1 = 0 \right\}$$

etc.

Invariant Variety of Periodic Points



(Algebraic) Variety of Periodic Points

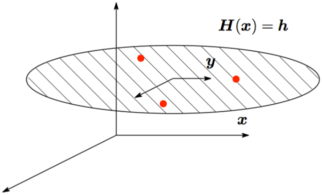


Figure: UC-VPP

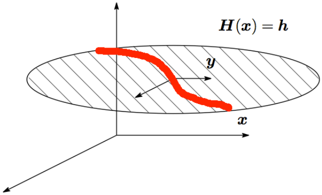


Figure: C-VPP

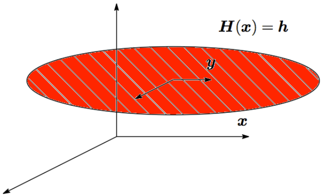


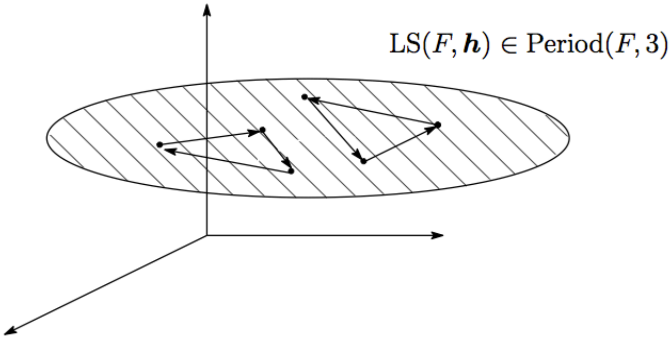
Figure: FC-VPP

Invariant Variety of Periodic Points

A FC-VPP is given by only information of the invariants, thus it is also called an **Invariant Variety of Periodic Points (IVPP)（不変周期点代数多様体）**.

IVPP :

$$\mathrm{Period}(F, n) = \left\{ \boldsymbol{x} \in \mathbb{C}^d \,\middle|\, \gamma_{l_n}^{(n)}(\boldsymbol{H}(\boldsymbol{x})) = 0,\ l_n = 1, \ldots, L_n \leq d - p \right\}$$

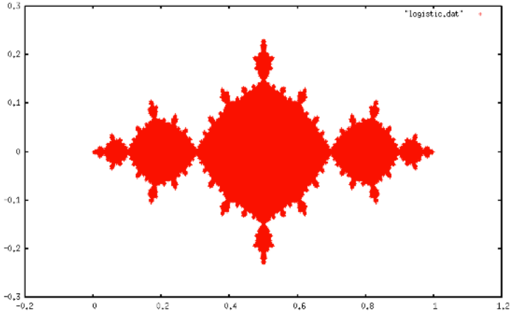


Non Integrable System and Julia Set

Julia Set :

In general, a non integrable system F has a **Julia set** $J(F)$.

$$J(F) := \overline{\bigcup_n \{\text{“repelling''}\ \text{periodic points of period}\ n\}} \subset \overline{\bigcup_n \text{Period}(F,n)}$$



Julia set is a fractal set, i.e. Julia set has non integer dimension.

IVPP Theorem（IVPP 定理）[3] :

Let F be a d dimensional map/DE with p invariants. If $p \geq d/2$, an IVPP and a U-VPP of **any** period do not exist in one map, simultaneously.

$\Downarrow$ Conjecture

IVPP/Julia Set Conjecture :

If a map F has an IVPP/Julia set then the map F can not have a Julia set/IVPP.

$\Downarrow$ Corollary

Integrable/Non Integrable Conjecture（可積分/非可積分予想） :

If a map F has an IVPP/Julia set then the map F is Integrable/Non Integrable.

[3] S. Saito and N. Saitoh, “ *Invariant varieties of periodic points* ” in Mathematical Physics Research Developments, 2008 Nova Science Publishers, Inc., Capt.3 pp 85-139, 2008.

2. IVPP Theorem and Intersections of VPPs

- Axiom and IVPP Theorem
  （公理と IVPP 定理）
- **“Non generic points”** and **Intersections** of VPPs
  （**“一般的でない点”** と周期点代数多様体達の**交差**）

Axiom and IVPP Theorem

Axiom :

VPPs of different periodicity have **no intersection on "generic points"**
（"一般的な点" の上では異なる周期点が同時に存在しない）.

$\Downarrow$

IVPP Theorem :

Let $\tilde{F}$ be a d dimensional ADE with p invariants. If there exists $n \geq 2$ such that $p \geq J_n$, and a VPP of period n is an IVPP, then a VPP of period m is not an U-VPP for any $m \geq 2$.

What are the "non generic points", ***i.e.*** **intersections of VPPs?**
（交差はどこにあるのか？）

3 dimensional Lotka-Volterra (3dLV) Map :

- Map :

$$(F_{3\mathrm{dLV}})_i : \boldsymbol{x}^t \to x_i^{t+1} := x_i^t \frac{D_{i-1}(\boldsymbol{x}^t)}{D_i(\boldsymbol{x}^t)} = x_i^t \frac{1 - x_{i+1}^t + x_{i+1}^t x_{i+2}^t}{1 - x_{i+1}^t + x_{i+2}^t x_i^t}, \quad i \in \mathbb{Z}/3\mathbb{Z}$$

- Invariants :

$$f := H_1(\boldsymbol{x}^t) = x_1^t x_2^t x_3^t - (1 - x_1^t)(1 - x_2^t)(1 - x_3^t)$$
$$g := H_2(\boldsymbol{x}^t) = 1 + (1 - x_1^t)(1 - x_2^t)(1 - x_3^t)$$

- IVPPs :

$$\gamma^{(2)}(f, g) = g$$
$$\gamma^{(3)}(f, g) = f^2 + fg + g^2$$
$$\gamma^{(4)}(f, g) = f^3 + (1 - g)(f + g)^3$$

etc.

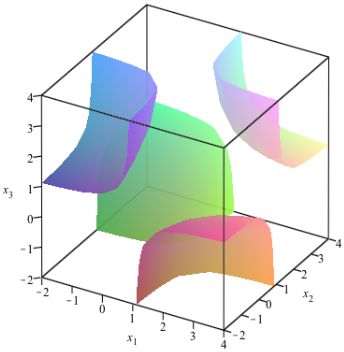


Figure: IVPP of period 2 of 3dLV map

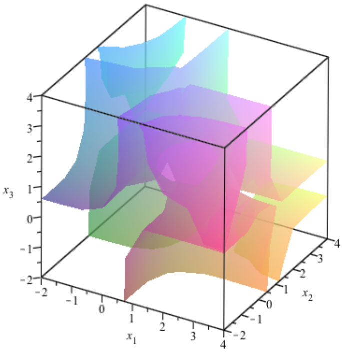


Figure: IVPP of period 4 of 3dLV map

Intersections of IVPPs

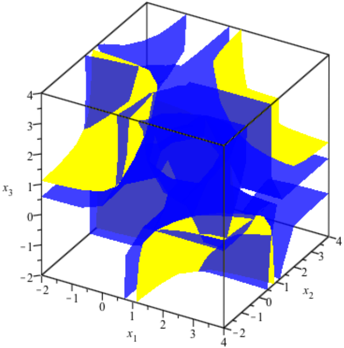


Figure: Intersections of IVPPs of period 2,4 of 3dLV map

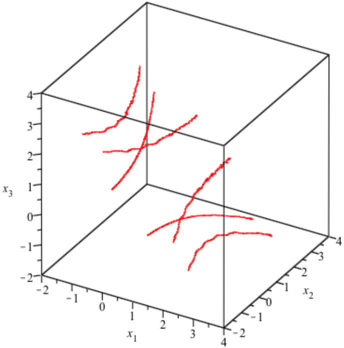


Figure: Intersections of IVPPs of period 2,4 of 3dLV map

Intersections of IVPPs

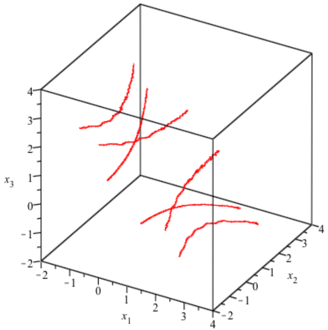


Figure: Intersections of IVPPs of period 2,4 of 3dLV map

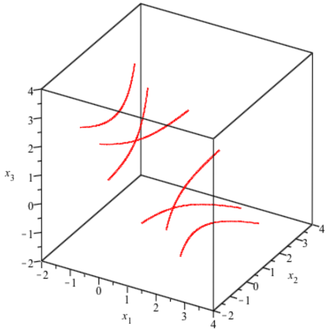


Figure: IDP of 3dLV map

Indeterminate Points (IDP)（不定点集合）：

$$\mathrm{IDP}(F_{3\mathrm{dLV}}) := \left\{\, \boldsymbol{x} \in \mathbb{C}^3 \;\middle|\; D_i(\boldsymbol{x}) = 0, i = 1,2,3 \,\right\} := \left\{ \left(1 - \frac{1}{t}, \frac{1}{1-t}, t\right) \in \mathbb{C}^3 \;\middle|\; t \in \mathbb{C} \right\}$$

3. Invariant/Parameter Duality

- Recurrence Equation
  （再帰方程式）
- ADE on Level Set and IVPP/RE Duality
  （等位面上の代数差分方程式と IVPP/再帰方程式双対性）

Recurrence Equation

Recurrence Equation（再帰方程式）[4]：

$$x^{t+1} = \frac{1 + x^t}{x^{t-1}} :$$

$$x^0 = a, \quad x^1 = b,$$

$$x^2 = \frac{1+b}{a} \to x^3 = \frac{1+a+b}{ab} \to x^4 = \frac{1+a}{b} \to x^5 = a$$

[4] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics* (Addison-Wesley), 1994.

ADE on Level Set and IVPP/RE Duality

- (New) Invariants :

$$r := H_1'(\boldsymbol{x}^t) = x_1^t x_2^t x_3^t, \quad s := H_2'(\boldsymbol{x}^t) = (1 - x_1^t)(1 - x_2^t)(1 - x_3^t)$$

- ADE on Level Set ($y := x_1$ and eliminate x_2) :

$$\begin{aligned}(\tilde{F}_{3\mathrm{dLV}})_h(y^t, y^{t+1}) &= (1+r)(y^t)^2(y^{t+1})^2 - (1-s+2r)(y^{t+1}(y^t)^2 + (y^{t+1})^2 y^t) \\ &\quad +(r-s)((y^{t+1})^2 + (y^t)^2) + (1+3r+rs+s^2)y^t y^{t+1} \\ &\quad -r(1+s)(y^t + y^{t+1})\end{aligned}$$

- REs ($r = 0$ and eliminate s) :
  - 2 periodic RE

$$y^t y^{t+1} - (y^t + y^{t+1}) = 0$$

  - 3 periodic RE

$$(y^t)^2(y^{t+1})^2 + ((y^t)^2 + (y^{t+1})^2) - (y^t)^2 y^{t+1} - 2y^t(y^{t+1})^2 + y^t y^{t+1} = 0$$

  - 4 periodic RE

$$(y^t)^2(y^{t+1})^2 + ((y^t)^2 + (y^{t+1})^2) - 2y^t(y^{t+1}) = 0$$

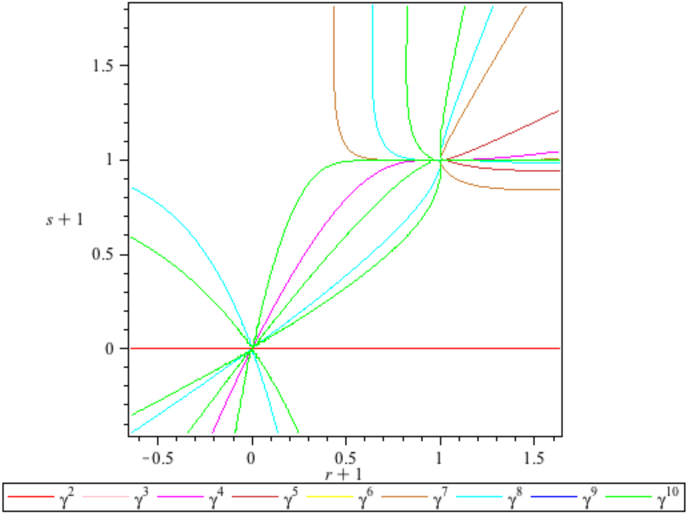


Figure: IVPPs of 3dLV Map **in Parameter Space**（係数空間上の 3LV 写像の IVPP 達）

4. Derivation of IVPPs from Singularity Confinement

- **Singularity Confinement(SC)**
  （特異点閉じ込め）
- Algorithm of derivation of IVPPs from SC
  （特異点閉じ込めに依る IVPP の導出アルゴリズム）

T. Yumibayashi, S. Saito, Y. Wakimoto, *Phys. Lett. A.*, **378 2014.**

Singularity Confinement(SC)（特異点閉じ込め）[5]：

- 3dLV Map：

$$F_{3\mathrm{dLV}}: \boldsymbol{x}^t \mapsto \boldsymbol{x}^{t+1} = \left(x_1^t \frac{1 - x_2^t + x_2^t x_3^t}{1 - x_3^t + x_3^t x_1^t},\ x_2^t \frac{1 - x_3^t + x_3^t x_1^t}{1 - x_1^t + x_1^t x_2^t},\ x_3^t \frac{1 - x_1^t + x_1^t x_2^t}{1 - x_2^t + x_2^t x_3^t} \right)$$

- Initial Point (example)：

$$\boldsymbol{x}^0 = \left(x_1^0, x_2^0, \frac{1}{1 - x_1^0} \right) \in \mathrm{SP}(F_{3\mathrm{dLV}}, 1) \subset \mathrm{SP}(F_{3\mathrm{dLV}})$$

which satisfies

$$D_1(\boldsymbol{x}^0) = 0$$

- Sequence of an iterative mapping：

$$\boldsymbol{x}^0 \to (\infty, 0, 1) \to (1, 0, \infty) \to \left(\frac{1}{1 - x_1^0}, x_2^0, x_1^0 \right) \in \mathrm{SP}\left(F_{3\mathrm{dLV}}^{(-1)}, 3 \right) \subset \mathrm{SP}\left(F_{3\mathrm{dLV}}^{(-1)} \right)$$

hence $N_{sc} = 3$.

[5] B. Grammaticos, A. Ramani, and V. Papageorgiou, *Phys. Rev. Lett.* **67**, 1825, 1991.

Algorithm

Algorithm :

- Write the initial point $\boldsymbol{x}^0$ by the invariants（発散点を不変量で係数付ける）：

$$D_1(\boldsymbol{x}) = 0, \quad H(\boldsymbol{x}) = h \quad \Rightarrow \quad \boldsymbol{x}^0 = (1, h) \in \mathrm{SP}(F_{2d\mathrm{M\ddot{o}b}}, 1).$$

- Compute $F^{(n)}(\boldsymbol{x}^0)$, $n \geq N_{sc}$ iteratively（写像する）：

$$\boldsymbol{x}^0 \to (\infty, 0) \to (-1, -h) \to \left(-\frac{1+h}{2}, -\frac{2r}{1+h}\right)$$

$$\to \left(-\frac{1+3h}{3+h}, -\frac{h(3+h)}{1+3h}\right) \to \left(-\frac{1+6h+h^2}{4(1+h)}, -\frac{4h(1+h)}{1+6h+h^2}\right) \to \cdots$$

- $D_1^{(n+1)}(\boldsymbol{x}^0) = 0$ gives IVPP of period n（$n+1$ 回写像が発散する点が IVPP！）：

$$\begin{aligned} \gamma^{(3)} &= 3 + h \\ \gamma^{(4)} &= 1 + h \\ \gamma^{(5)} &= 5 + 10h + h^2 \\ &\text{etc.} \end{aligned}$$

5. SC and "Projective Resolution" of "Triangulated Category"

- Hirota-Miwa Equation
  （広田-三輪方程式）
- String/Soliton Correspondence
  （弦/ソリトン対応）
- Transformation of Function and **Gauge Symmetry**
  （函数変換と**ゲージ対称性**）
- SC and **"Projective Resolution"**
  （特異点閉じ込めと **"射影分解"**）
- **Localization**
  （**局所化**）

S. Saito, T. Yumibayashi, and Y. Wakimoto, Prog. Theor. Exp. Phys. 2014, 023A08

Hirota-Miwa Equation(HM eq.)（広田三輪方程式）[5,6]：

$$a_{14}a_{23}\tau_{14}(p)\tau_{23}(p) - a_{24}a_{13}\tau_{24}(p)\tau_{13}(p) + a_{34}a_{12}\tau_{34}(p)\tau_{12}(p) = 0$$

$$\tau(p) \in \mathbb{C}, \quad p = (p_1, p_2, p_3, p_4) \in \mathbb{C}^4, \quad a_{ij} = -a_{ji} \in \mathbb{C}$$

In above and hereafter we use the abbreviations, such as

$$\tau_j(p) := \mathrm{D}_j\tau(p) = \tau(p + \delta_j), \quad \tau_{ij}(p) := \mathrm{D}_i\mathrm{D}_j\tau(p) = \tau(p + \delta_i + \delta_j),$$

$$\delta_j := (\delta_{1j}, \delta_{2j}, \delta_{3j}, \delta_{4j})$$

with the Kronecker symbol δ_{ij}.

It is exactly solvable system from which infinitely many soliton eqs. can be derived!

[6] R. Hirota, J. *Phys. Soc. Jpn.*, **50** 3787, 1981.
[7] T. Miwa, *Proc. Japan Acad.*, **58A** 9, 1982.

String/Soliton Correspondence（弦/ソリトン対応）：

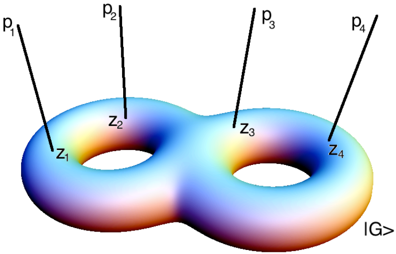
- String Amplitude（弦遷移函数）：

$$\Phi(p,z;G) = \langle 0|V(p_1,z_1)V(p_2,z_2)V(p_3,z_3)V(p_4,z_4)|G\rangle, \quad z=(z_1,z_2,z_3,z_4)\in\mathbb{Z}^4$$

- Vertex Operator（頂点演算子）：

$$V(p_j,z_j) =: \exp(ip_jX(z_j)) :$$

$$\Rightarrow \quad \tau(p) = \frac{\Phi(p,z;G)}{\Phi(p,z;0)}$$



Transformation of Function and Gauge Symmetry

Transformation of Function (HM eq. ⇒ Lotka-Volterra Map) :

$$x_j^t = \frac{\tau_{j+\epsilon+1}^t \tau_{j-\epsilon}^{t+1}}{\tau_{j+1}^t \tau_j^{t+1}}$$

⇓ **Gauge Transformation**

$$x_j^t = \frac{\Phi_{j+\epsilon+1}^t \Phi_{j-\epsilon}^{t+1}}{\Phi_{j+1}^t \Phi_j^{t+1}}$$

⇓

τ function $\sim$ String Amplitude mod Gauge Transformation

SC of 3dLV Map

- Initial Point ;

$$\boldsymbol{x}^0 := \left(\frac{r-s}{r+1}, r\frac{s+1}{r-s}, \frac{r+1}{s+1}\right)$$

- Map :

$$\boldsymbol{x}^0 \to \boldsymbol{x}^1 = (\infty, 0, 1) \to \boldsymbol{x}^2 = (1, 0, \infty) \to \boldsymbol{x}^3 \to \dots$$

where

$$\boldsymbol{x}^3 = \left(\frac{r+1}{s+1}, r\frac{s+1}{r-s}, \frac{r-s}{r+1}\right)$$

$\Downarrow$ On the τ function

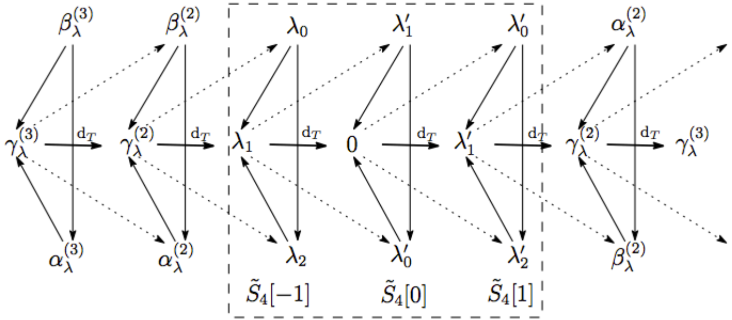
- Initial Points :

$$\boldsymbol{\tau}^{(0)} = (\lambda_1\beta^{(0)}, \lambda_2\gamma^{(0)}, \lambda_0\alpha^{(0)}), \quad \tau^{(1)} = (\lambda_0, \lambda_1, \lambda_2)$$

- Map :

$$\boldsymbol{\tau}^{(0)} = (\lambda_1\beta^{(0)}, \lambda_2\gamma^{(0)}, \lambda_0\alpha^{(0)}) \to (\lambda_0, \lambda_1, \lambda_2) \to (\lambda_2, 0, \lambda_1) \to (\lambda_0', \lambda_1', \lambda_2') \to \dots$$

where λ are part of IDPs.



$$\tilde{S}_4[-1], \tilde{S}_4[0], \tilde{S}_4[1] \subset \Lambda(\infty)$$

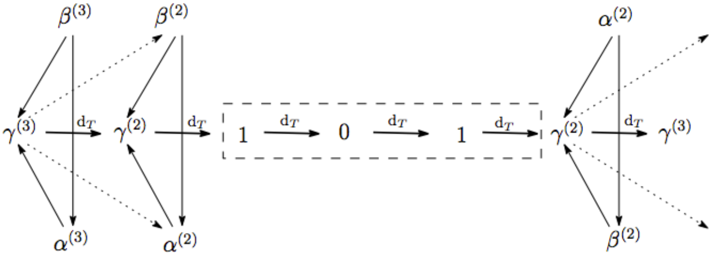
$$\Downarrow$$

$P := \cdots \to \gamma_\lambda^{[3]} \to \gamma_\lambda^{[2]} \to \lambda_1 \to \lambda_0' \to 0$ is a **"projective resolution"（射影分解）** of λ_0'.

Localization（局所化）：

3dimensional Map $\sim \mathcal{HM}/\mathcal{S}(\mathcal{N})$:

$$\mathcal{S}(\mathcal{N}) := \left\{ \mathcal{G}[\mu,\nu] \mid \mathcal{G}[\mu,\nu] : \lambda(p) \to \lambda'(p),\ 0 \to \mathcal{G}[\mu,\nu]0 = 0 \right\}$$



6. Transition of Integrable/Non Integrable System

- Deformation of Integrable maps and fate of U-VPPs
  (可積分系の変形と離散周期点の行き先)

S. Saito, N. Saitoh, H. Harada, T. Yumibayashi and Y. Wakimoto, AIP Advances, AIP ID: 003306ADV, 2013.

2 dimensional Möbius Map

Deformation of 2 dimensional Möbius Map :

$$F_{\mathrm{2dMöb}}(x,y) \to (F_{\mathrm{2dMöb}})_a(x,y) = \left(x\frac{1-y}{1-x-a},\ y\frac{1-x}{1-y} \right)$$

$$\Downarrow$$

Fixed Points :

$$(x,y) = (0,0) \text{ and } (-a,0),$$

Period 2 Points :

$$\begin{aligned}
(x_1^2, y_1^2) &= \left(1 - \frac{a}{2} + \sqrt{\frac{a(a^2-4)}{a-4}},\ 1 + \frac{a(2-a)}{4} - \frac{a}{2}\sqrt{\frac{a(a^2-4)}{a-4}} \right) \\
(x_2^2, y_2^2) &= \left(1 - \frac{a}{2} - \sqrt{\frac{a(a^2-4)}{a-4}},\ 1 + \frac{a(2-a)}{4} + \frac{a}{2}\sqrt{\frac{a(a^2-4)}{a-4}} \right)
\end{aligned}$$

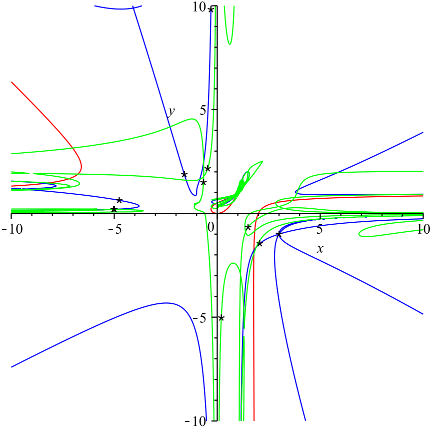


Figure: Paths of points of period two, three and four.

2 dimensional Möbius Map

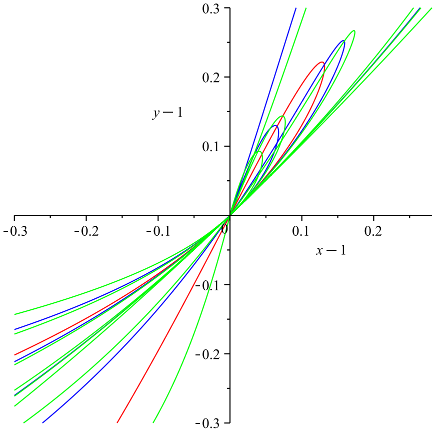


Figure: Details of the paths of period 2,3 and 4.

3dLV Map

Deformation of 3dLV Map :

$$F_{\rm 3dLV}(x,y,z) \quad \to \quad (F_{\rm 3dLV})_{a,b}(x,y,z)$$
$$= \left(x\frac{1-y+yz}{1+a-z+zx}, y\frac{1+b-z+zx}{1-x+xy}, z\frac{1-x+xy}{1-y+yz} \right)$$

3 dimensional Lotka-Volterra Map

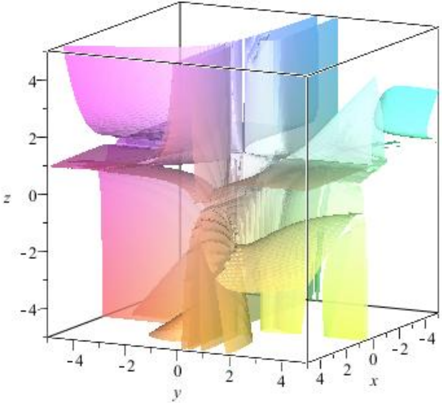


Figure: period 2 surface $K^{(2)} = 0$

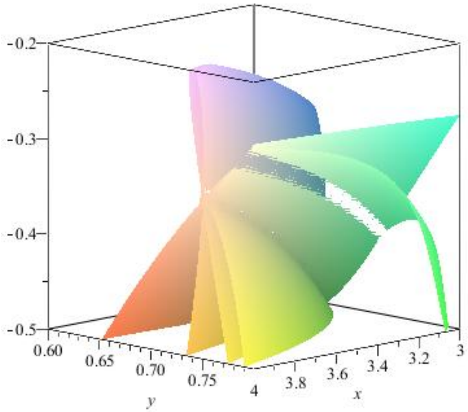


Figure: details near $\mathrm{IDP}(F_{3\mathrm{dLV}})$

7. Conclusion

Conclusion

IVPP Theorem :

IVPP Theorem $\sim$ Integrable/Non Integrable Conjecture

$\Downarrow$ Sufficient condition of integrability

Derivation of IVPPs by SC $\sim$ "Projective resolution" of IDP

$\Downarrow$ Invariant/Parameter duality

Integrable ADE $\sim \bigoplus_n$ Recurrence Equation(RE) of period n

Intersections of IVPPs :

Intersections of IVPPs $\sim$ Problems of IVPP Theorem?

$\Downarrow$ Some conjectures and analysis of some examples

Origins of intersections of IVPPs $\sim$ IDP

$\Downarrow$ Integrable/Non Integrable Transition

Fate of Julia set $\Rightarrow$ IDP?

Thank you for your attention!

Derivation of IVPPs from SC "Ansatz"

If $\boldsymbol{x}^0 \in \mathrm{FP}(F)$ is a period n point with $n \geq N_{sc} - 1$, its image $F^{(n+1)}(\boldsymbol{x}^0)$ must be divergent, *i.e.*,

$$\boldsymbol{x}^0 \in \mathrm{SP}(F) \cap \mathrm{Period}(F, n) \quad \Longrightarrow \quad F^{(n+1)}(\boldsymbol{x}^0) \text{ is divergent},$$

hence we get

$$\mathrm{SP}\left(F^{(n+1)}|_{\mathrm{SP}(F)}\right) \subset \mathrm{Period}(F, n)$$

by $F^{(n+1)}(\boldsymbol{x}^0) = F(\boldsymbol{x}^0)$.

Vertex Operator Algebra :

$$V(p,z)V(p',z') = (-1)^{pp'}V(p',z')V(p,z)$$

$$\Downarrow \quad \psi_\pm(z) := V(\pm 1, z)$$

$$\psi_\pm(z)\psi_\pm(z') = -\psi_\pm(z')\psi_\pm(z), \quad \psi_\pm(z)\psi_\mp(z') = -\psi_\mp(z')\psi_\pm(z)$$

$$\Downarrow \quad z = z'$$

$$\psi_+(z)\psi_+(z) = 0, \quad \psi_-(z)\psi_-(z) = 0$$

Hence $\psi_\pm(z)$ are Grassmann fields.

Bäcklund Difference :

- Auto-Bäcklund Transformation $e^{\psi_\pm(z)}$:

$$e^{\psi_\pm(z)} = 1 + \psi_\pm(z) \Rightarrow \psi_\pm(z) = e^{\psi_\pm(z)} - 1$$

- Bäcklund Difference $\hat{\mathrm{D}}_j^\pm$:

$$\hat{\mathrm{D}}_j^\pm \sim \psi_\pm(z_j)$$

Action of Bäcklund Difference

- Partition Function :

$$\Phi_j(p,z;G) := \hat{\mathrm{D}}_j^{\pm}\Phi(p,z;G) :=< 0|\prod_{i=1}^{4} V(p_i,z_i)\psi_{\pm}(z_j)|G >$$

$$\Phi_{ij}(p,z;G) = -\Phi_{ji}(p,z;G) \quad \Rightarrow \quad \mathbf{\Phi_{ii}(p,z;G) = 0}$$

- τ Function :

$$\tau_j(p) := \hat{\mathrm{D}}_j^{\pm}\tau(p) = \tau(p+\delta_j)$$

$$\tau_{ij}(p) = \tau_{ji}(p) \quad \Rightarrow \quad \tau_{\mathbf{ii}}(\mathbf{p}) = \frac{\mathbf{\Phi_{ii}(p,z;G)}}{\mathbf{\Phi_{ii}(p,z;0)}} = \frac{\mathbf{0}}{\mathbf{0}} =?$$

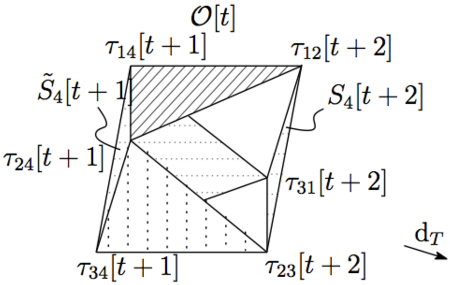
- Triangle :

$$\tilde{S}_4[t+1] := (\tau_{14}[t+1], \tau_{24}[t+1], \tau_{34}[t+1])$$

$$S_4[t+2] := (\tau_{23}[t+2], \tau_{13}[t+2], \tau_{12}[t+2])$$

- Octahedron :

$$O[t] := (\tilde{S}_4[t+1], S_4[t+2])$$



HM eq. and "Triangulated Category" :

- "Triangulated Category" :

$$\mathcal{HM} := (\tau_{ij}, \mathrm{D}(ij, ik), \mathrm{d}_T)$$

- Objects :

$$\tau_{ij} \quad : \quad \tau\text{-functions}$$

- Morphism :

$$\mathrm{D}(ij, ik) := \mathrm{D}_j \mathrm{D}_k^{-1} \quad : \quad \text{Bäcklund Difference}$$

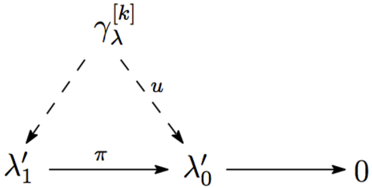
- Shift Functor :

$$\mathrm{d}_T : \mathcal{O}[t] \to \mathcal{O}[t+1]$$

- Octahedron Axiom :

The rule for the flow of information and the connecting condition.

SC and “Projective Resolution”



$\Downarrow$

$\gamma_\lambda^{[k]}$ is a projective object.

$\Downarrow$

$P := \cdots \to \gamma_\lambda^{[4]} \to \gamma_\lambda^{[3]} \to \gamma_\lambda^{[2]} \to \lambda_1 \to \lambda'_0 \to 0$ is a “projective resolution” of λ'_0.