$F_{4(4)}$の実最低次元表現の軌道分解について

(西尾昭宏氏との共同研究)

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Orbit decomposition of Jordan matrix algebras of order three under the automorphism groups

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* **Abstract.** The orbit decomposition is given under the automorphism group on the real split Jordan algebra of all hermitian matrices of order three corresponding to any real split composition algebra, or the automorphism group on the complexification, explicitly, in terms of the cross product of H. Freudenthal and the characteristic polynomial.

0. Introduction.

Let $\mathcal{J}'$ be a split exceptional simple Jordan algebra over a field $\mathbb{F}$ of characteristic not two, that is, the set of all hermitian matrices of order three whose elements are split octonions over $\mathbb{F}$ with the Jordan product. And let G' be the automorphism group of $\mathcal{J}'$. N. Jacobson [16, p.389, Theorem 10] found that $X, Y \in \mathcal{J}'$ are in the same G'-orbit if and only if X, Y admit the same minimal polynomial and the same generic minimal polynomial, by imbedding a generating subalgebra with the identity element E in terms of the Jordan product into a special Jordan algebra. When $\mathbb{F} = \mathbb{R}$, the field of all real numbers, some elements of $\mathcal{J}'$ are not diagonalizable under the action of $G' = F_{4(4)}$, since $\mathcal{J}'$ admits a G'-invariant non-defnite $\mathbb{R}$-bilinear form such that the restriction to the subspace of all diagonal elements is positive-definite [19, Theorem 2], although every element of $\mathcal{J}'$ is diagonalizable under the action of a linear group $E_{6(6)}$ containing $F_{4(4)}$ on $\mathcal{J}'$ by [15] (cf. [17]) or under the action of the maximal compact subgroup $Sp(4)/\mathbb{Z}_2$ of $E_{6(6)}$ on $\mathcal{J}'$ given by [22].

This paper presents a concrete orbit decomposition under the automorphism group on a real split Jordan algebra of all hermitian matrices of order

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(1/2)

(記号)。 $F=\mathbb{R}$ の時、

$\tilde{K} = \mathbb{R},\ \mathbb{C},\ \mathbb{H},\ \mathbb{O};\ \mathbb{C}',\ \mathbb{H}',\ \mathbb{O}'.$

。 $F=\mathbb{C}=\mathbb{R}+\sqrt{-1}\,\mathbb{R}$ の時、

$\tilde{K} = \mathbb{R}^C,\ \mathbb{C}^C,\ \mathbb{H}^C,\ \mathbb{O}^C.$ ($\cong \mathbb{C}\otimes_{\mathbb{R}}$(上の$\tilde{K}$))

ここで、 $\mathbb{O} = \underbrace{\mathbb{R}1\oplus\mathbb{R}e_1\oplus\mathbb{R}e_2\oplus\mathbb{R}e_3}_{\mathbb{H}}\oplus\underbrace{\mathbb{R}e_4\oplus\mathbb{R}e_5\oplus\mathbb{R}e_6\oplus\mathbb{R}e_7}_{\mathbb{H}e_4}$

$\mathbb{C} = \mathbb{R}1\oplus\mathbb{R}e_1$

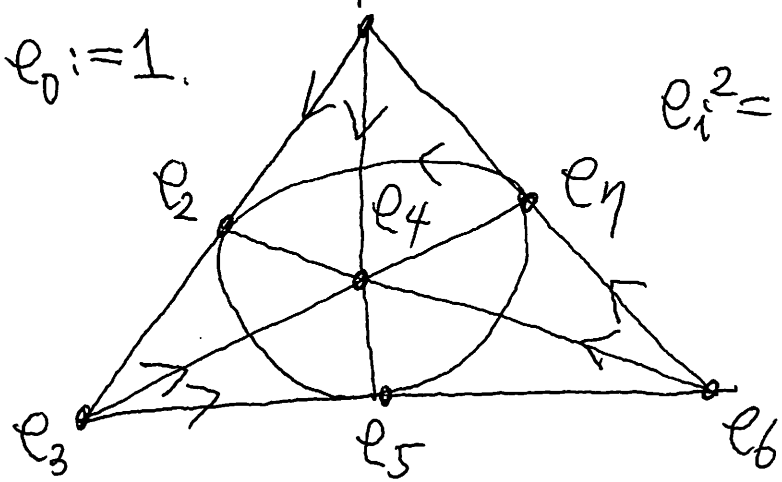
$\mathbb{O}' = \mathbb{H}\oplus\mathbb{H}\sqrt{-1}\,e_4$

$\mathbb{H}' = \mathbb{C}\oplus\mathbb{C}\sqrt{-1}\,e_4$

$\mathbb{C}' = \mathbb{R}1\oplus\mathbb{R}\sqrt{-1}\,e_4$

$e_0 := 1.$

$e_i^2 = -1.$



- $\overline{} : \mathbb{O}^C \to \mathbb{O}^C ; x_0 e_0 + \sum_{i=1}^{7} x_i e_i \longmapsto x_0 e_0 - \sum_{i=1}^{7} x_i e_i$

  C-linear conjugation.

- $\tilde{\mathfrak{J}} := H(3, \tilde{K}) = \{ X \in M(3, \tilde{K}) \mid {}^t\overline{X} = X \}$

  $= \left\{ \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} \;\middle|\; \xi_i \in F,\ x_i \in \tilde{K}\ (i=1,2,3) \right\}.$

- $\tilde{\mathfrak{J}} \ni X, Y \Rightarrow \tilde{\mathfrak{J}} \ni X \circ Y := \frac{1}{2}(XY + YX)$: Jordan積

- $\tilde{G} := \mathrm{Aut}(\tilde{\mathfrak{J}}) = \{ \alpha \in GL_F(\tilde{\mathfrak{J}}) \mid \alpha(X \circ Y) = (\alpha X) \circ (\alpha Y) \}$:

  F-algebra $\tilde{\mathfrak{J}}$ の 自己同型群

$\tilde{K}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\tilde{G}$	A_1	A_2	C_3	$F_4 = F_{4(-52)}$

compact 型

(H. Freudenthal: Oktaven, Ausnahmegruppen und Oktavengeometrie, 1951, new revised 1960)

$\tilde{K}$		$\mathbb{C}'$	$\mathbb{H}'$	$\mathbb{O}'$
$\tilde{G}$		A_2'	C_3'	$F_4' = F_{4(4)}$

split 型

$\tilde{K}$	$\mathbb{R}^C$	$\mathbb{C}^C$	$\mathbb{H}^C$	$\mathbb{O}^C$
$\tilde{G}$	A_1^C	A_2^C	C_3^C	F_4^C

complex 型

(H. Freudenthal: Lie Groups in the foundations of Geometry, p. 165 参照)

定理 (N. Jacobson: Structure and Rep. of Jordan Alg., AMS, 1968)

$\tilde{K} = \mathfrak{O}'$ の時、$X, Y \in \tilde{\mathfrak{J}} = H_3(\mathfrak{O}')$ について、

$\tilde{G}\cdot X = \tilde{G}\cdot Y \Leftrightarrow \chi_X(\lambda) = \chi_Y(\lambda)$: minimal polynomial
& $\varphi_X(\lambda) = \varphi_Y(\lambda)$: generic minimal polynomial.

以上は、軌道分解を行わないで、得られていた。

(目的) ① 軌道分解 (以上の結果の拡張と精密化)
② $\tilde{G}$ に関する一般的結果の整理 (Simple Proof)

(記号) ◦ $x = x_0 + \sum_i x_i e_i$, $y = y_0 + \sum_i y_i e_i \in \tilde{K}$

$\Rightarrow (x|y) := x_0 y_0 + \sum_i x_i y_i \in F$.

◦ $X = \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}$, $Y = \begin{pmatrix} \eta_1 & y_3 & \bar{y}_2 \\ \bar{y}_3 & \eta_2 & y_1 \\ y_2 & \bar{y}_1 & \eta_3 \end{pmatrix}$, $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \tilde{J}$

$\Rightarrow \mathrm{tr}(X) := \xi_1 + \xi_2 + \xi_3 \in F$

$(X|Y) := \mathrm{tr}(X \circ Y) = \sum_{i=1}^{3} \{\xi_i \eta_i + 2(x_i|y_i)\} \in F$

$X \times Y := X \circ Y - \frac{1}{2}\{\mathrm{tr}(X)Y + \mathrm{tr}(Y)X\} + \frac{1}{2}\{\mathrm{tr}(X)\mathrm{tr}(Y) - (X|Y)\}E$

$\in \tilde{J}$: クロス積 (H. Freudenthal による定義)

では 定理

$$\circ\ X = \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix}$$

$$\Rightarrow\ X^{\times 2} := X \times X$$

$$= \begin{pmatrix} \xi_2\xi_3 - x_1\overline{x_1} & \overline{x_1x_2} - \xi_3 x_3 & x_3x_1 - \xi_2\overline{x_2} \\ x_1x_2 - \xi_3\overline{x_3} & \xi_3\xi_1 - x_2\overline{x_2} & \overline{x_2x_3} - \xi_1 x_1 \\ \overline{x_3x_1} - \xi_2 x_2 & x_2x_3 - \xi_1\overline{x_1} & \xi_1\xi_2 - x_3\overline{x_2} \end{pmatrix}$$

$\circ\ X, Y, Z \in \widetilde{\mathfrak{J}} \Rightarrow (X|Y|Z) := (X|Y\times Z)$

(H. Freudenthal による定義では，右辺を左辺で定義)

$\circ\ X \in \widetilde{\mathfrak{J}} \Rightarrow \det(X) := \frac{1}{3}(X|X|X)$

(H. Freudenthal による定義では，右辺を左辺で定義)

<u>定義</u>：$\Phi_X(\lambda) := \det(\lambda E - X)$ ：Xの特性多項式．

<u>補題</u>：$\Phi_X(\lambda) = \lambda^3 - \mathrm{tr}(X)\lambda^2 + \mathrm{tr}(X^{\times 2})\lambda - \det(X)$;

$\mathrm{tr}(X^{\times 2}) := \mathrm{tr}(X\times X)$,

$\mathrm{tr}(X\times Y) = \frac{1}{2}\{\mathrm{tr}(X)\mathrm{tr}(Y) - (X|Y)\}$．

(N. Jacobson (1968) は，補題の右辺に当たる式を，直接 $\varphi_X(\lambda)$ と定義している)

$$\circ\ X = \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix}$$

$$\begin{aligned}
\Longrightarrow \det X &= \xi_1\xi_2\xi_3 + (x_1x_2)x_3 + \overline{(x_1x_2)x_3} - \xi_1 x_1\overline{x_1} - \xi_2 x_2\overline{x_2} - \xi_3 x_3\overline{x_3} \\
&= \xi_1\xi_2\xi_3 + x_1(x_2x_3) + \overline{x_1(x_2x_3)} - \xi_1 x_1\overline{x_1} - \xi_2 x_2\overline{x_2} - \xi_3 x_3\overline{x_3} \\
&= \xi_1\xi_2\xi_3 + x_2(x_3x_1) + \overline{x_2(x_3x_1)} - \xi_1 x_1\overline{x_1} - \xi_2 x_2\overline{x_2} - \xi_3 x_3\overline{x_3} \\
&= \xi_1\xi_2\xi_3 + x_3(x_1x_2) + \overline{x_3(x_1x_2)} - \xi_1 x_1\overline{x_1} - \xi_2 x_2\overline{x_2} - \xi_3 x_3\overline{x_3} \\
&= \xi_1\xi_2\xi_3 + (x_2x_3)x_1 + \overline{(x_2x_3)x_1} - \xi_1 x_1\overline{x_1} - \xi_2 x_2\overline{x_2} - \xi_3 x_3\overline{x_3} \\
&= \xi_1\xi_2\xi_3 + (x_3x_1)x_2 + \overline{(x_3x_1)x_2} - \xi_1 x_1\overline{x_1} - \xi_2 x_2\overline{x_2} - \xi_3 x_3\overline{x_3}
\end{aligned}$$

(横田一郎：例外型単純リー群(現代数学社) p.2, p.40 参照)

命題 (1) (O. Shukuzawa, I. Yokota: J. Fac. Sci. Shinshu Univ. 14-1 (1979))

$$\tilde{G} = \{\alpha \in \tilde{G} \mid tr(\alpha X) = tr(X),\ \alpha E = E\}$$

(2) (N. Jacobson, J. Reine Angew Math. 201 (1959), 178–195; Lemma 1)

$$\tilde{G} = \{\alpha \in GL_F(\tilde{J}) \mid \det(\alpha X) = \det(X),\ \alpha E = E\}.$$

(for a general ch(F) by a general theory of Jordan Algebras)

(3) $$\tilde{G} = \{\alpha \in GL_F(\tilde{J}) \mid \Phi_{\alpha X}(\lambda) = \Phi_X(\lambda)\}$$

$$= \{\alpha \in GL_F(\tilde{J}) \mid \det(\alpha X) = \det(X),\ (\alpha X \mid \alpha Y) = (X \mid Y)\}$$

$$= \{\alpha \in GL_F(\tilde{J}) \mid \alpha(X \times Y) = (\alpha X) \times (\alpha Y)\}$$

命題の証明の準備

V : F-alg. by xy (積) の時, $\forall x \in V$ について,

$L_x : V \to V ; y \longmapsto xy$ (Left multiplication) とおく。

$Aut(V) := \{ \alpha \in GL_F(V) \mid \alpha(xy) = (\alpha x)(\alpha y) \}$ とおく。

補題2 (1) $\alpha \in Aut(V) \Rightarrow \begin{cases} \mathrm{trace}(L_{(\alpha x)}) = \mathrm{trace}(L_x) \\ \det(L_{(\alpha x)}) = \det(L_x) \end{cases}$

($\because$ $L_{(\alpha x)} = \alpha L_x \alpha^{-1}$.)

(2) $ex = xe = x$ for all $x \in V \Rightarrow \alpha e = e$ ($\because$ $(\alpha e)(\alpha x) = \underset{= \alpha x}{(\alpha x)(\alpha e)}$)

(3) $L_X^{\circ} : \tilde{J} \to \tilde{J} ; Y \longmapsto X \circ Y$, $L_X^{\times} : \tilde{J} \to \tilde{J} ; Y \longmapsto X \times Y$.

$\Rightarrow \mathrm{trace}(L_X^{\circ}) = (\dim_F \tilde{K} + 1)\, tr(X)$, $\mathrm{trace}(L_X^{\times}) = \frac{-\dim_F \tilde{K}}{2}\, tr(X)$.

($\because$ 具体的基底で表現行列を計算できる)

命題の証明

(1) $\tilde{G} \subset \{\alpha \in GL_F(\tilde{J}) \mid tr(\alpha X) \underset{\text{補題2(1)\&(3)}}{=} tr(X),\ \alpha E \underset{\text{補題2(2)}}{=} E\}$

を示せばよい。

(2) $\tilde{G} \subset \{\alpha \in GL_F(\tilde{J}) \mid \det(\alpha X) \overset{\text{補題2(1)\&(3)}}{=} \det(X),\ \alpha E \overset{\text{補題2(2)}}{=} E\}$ (∵ $\det X =$ tr, $X^{\circ 2}$, Eの組合せ)

$\subset \{\alpha \in GL_F(\tilde{J}) \mid \Phi_{\alpha X}(\lambda) = \Phi_X(\lambda)\}$ (∵ $\Phi_X(\lambda) := \det(\lambda E - X)$)

$\subset \{\alpha \in GL_F(\tilde{J}) \mid \det(\alpha X) = \det(X),\ (\alpha X \mid \alpha Y) = (X \mid Y)\}$ (∵ 補題)

$\subset \{\alpha \in GL_F(\tilde{J}) \mid \alpha(X \times Y) = (\alpha X) \times (\alpha Y)\}$ (∵ $\det X = (X \mid X \times X)$, $X \times Y$; 対称的故, $(X \mid Y \times Z)$を保存する。$(X \mid Y)$: non-deg.)

$\subset \tilde{G}$ (∵ $tr(\alpha X) = tr(X)$: 補題2(1)&(3).
$tr((\alpha X) \times (\alpha Y)) = tr(X \times Y) = \frac{1}{2}(tr(X)tr(Y) - (X \mid Y))$: 補題.
∴ $(\alpha X \mid \alpha Y) = (X \mid Y)$.
∴ $(X \circ Y \mid Z) = (X \times Y \mid Z) + \frac{1}{2}\{tr(X)(Y \mid Z) + tr(Y)(X \mid Z) - (tr(X)tr(Y) - (X \mid Y))tr(Z)\}$
はα-不変. ∴ $\alpha \in \tilde{G}$). これより, (3)と(2)を得る. //

以上, ②.

以下, ①について, 記述する。

(記号) $X \in \mathfrak{J}$, $\Phi_X(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)$

($\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} = \mathbb{R}^C = \mathbb{R}+\sqrt{-1}\,\mathbb{R}$) の時,

$\Lambda_X := \{\lambda_1, \lambda_2, \lambda_3\}$: 特性根全体(Xの)

$V_X := \{aX^{\times 2} + bX + cE \mid a, b, c \in \mathbb{F}\}$: minimal subspace

$\nu_X := \dim_{\mathbb{F}} V_X \in \{1, 2, 3\}$ とおく。

$\Longrightarrow$ V_Xは, EとXで生成されるクロス積 subalgebra.

(∵) $E \times X = \frac{1}{2}(\mathrm{tr}(X)E - X)$, $(X^{\times 2})^{\times 2} = (\det X)X$,

$X^{\times 2} \times X = -\frac{1}{2}\{\mathrm{tr}(X)X^{\times 2} + \mathrm{tr}(X^{\times 2})X - (\mathrm{tr}(X^{\times 2})\mathrm{tr}(X) - \det(X))E\}$ //

(11)

<u>定理</u> $\mathbb{F}=\mathbb{C}$の時, $\tilde{K}=\mathbb{R}^C, \mathbb{C}^C, \mathbb{H}^C, \mathbb{O}^C$;
$\mathbb{F}=\mathbb{R}$の時, $\tilde{K}=\mathbb{C}, \mathbb{H}, \mathbb{O}$ とおく.

$\tilde{G}'=\tilde{G}$ or $\tilde{G}_0$ とする。(連結成分)

(1) $X, Y \in \tilde{\mathfrak{J}}$について, $\tilde{G}'\cdot X=\tilde{G}'\cdot Y \Leftrightarrow v_X=v_Y \;\&\; \Lambda_X=\Lambda_Y$

(2) $\mathbb{F}=\mathbb{C}$の時, $(\tilde{G}', \tilde{\mathfrak{J}})$の軌道の代表元は次表の通り:

$\#\Lambda_X \backslash v_X$	3	2	1
3	$\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)$	(無)	(無)
2	$\mathrm{diag}(\lambda_1,\lambda_2,\lambda_2)+M_1$	$\mathrm{diag}(\lambda_1,\lambda_2,\lambda_2)$	(無)
1	λ_1E+M_{23}	λ_1E+M_1	λ_1E

ここで, $M_{23}:=\begin{pmatrix}0 & 1 & \sqrt{-1}\\ 1 & 0 & 0\\ \sqrt{-1} & 0 & 0\end{pmatrix}$, $M_1:=\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & \sqrt{-1}\\ 0 & \sqrt{-1} & -1\end{pmatrix}=M_{23}^{\times 2}$.

(3) $F=\mathbb{R}$ の時, $(\hat{G},\hat{J})$ の軌道の代表元は次表の通り。

$\#\Lambda_X$ \ v_X	3	2	1
3	$\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)$; $\lambda_1>\lambda_2>\lambda_3$. or $\mathrm{diag}(\lambda_1,p,p)+\begin{pmatrix}0&0&0\\0&0&q\sqrt{-1}e_4\\0&-q\sqrt{-1}e_4&0\end{pmatrix}$	(無)	(無)
2	$\mathrm{diag}(\lambda_1,\lambda_2,\lambda_2)+M_1'$	$\mathrm{diag}(\lambda_1,\lambda_2,\lambda_2)$	(無)
1	$\lambda_1 E+M_{2/3}$	$\lambda_1 E+M_\gamma$	$\lambda_1 E$

ここで, $M_{2/3}:=\begin{pmatrix}0&1&\sqrt{-1}e_4\\1&0&0\\-\sqrt{-1}e_4&0&0\end{pmatrix}$, $M_\gamma:=\begin{pmatrix}0&0&0\\0&1&\sqrt{-1}e_4\\0&-\sqrt{-1}e_4&-1\end{pmatrix}=M_{2/3}^{\times 2}$

(註1) $F=\mathbb{R}$; $\widetilde{K}=\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}$の時, $(\widetilde{G},\widehat{\mathfrak{J}})$の代表元は次の通り。

$\#\Lambda_X$ \ ν_X	3	2	1
3	$\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)$; $\lambda_1>\lambda_2>\lambda_3$	(無)	(無)
2	(無)	$\mathrm{diag}(\lambda_1,\lambda_2,\lambda_2)$; $\lambda_1\neq\lambda_2$	(無)
1	(無)	(無)	$\lambda_1 E$

ここで, $X=\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)\Rightarrow X^{\times 2}=\mathrm{diag}(\lambda_2\lambda_3,\lambda_3\lambda_1,\lambda_1\lambda_2)$

$$\begin{vmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_2\lambda_3 & \lambda_3\lambda_1 & \lambda_1\lambda_2 \end{vmatrix}=(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)$$

(註2) $F=C$; $\tilde{K}=\mathbb{C}^C \cong \mathbb{C}'^C$ の時,

$$(\hat{\tilde{G}}_0, \hat{\tilde{J}}) \cong (SU(3,\mathbb{C}'^C), H(3,\mathbb{C}'^C))$$

$$\cong (SL(3,C), M(3,C))$$

(∵)
$$H(3,\mathbb{C}'^C) = \left\{ \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} \middle| \begin{array}{l} \xi_i \in C,\ x_i \in \mathbb{C}'^C = \mathbb{C}' + \sqrt{-1}\,\mathbb{C}' \\ (i=1,2,3) \end{array} \right\}$$

$$= H(3,\mathbb{C}') + \sqrt{-1}\,H(3,\mathbb{C}')$$

$$g : H(3,\mathbb{C}') \xrightarrow{\cong} M(3,\mathbb{R});\ P + \sqrt{-1}e_4 Q \longmapsto P + Q$$

$$\tilde{g} : H(3,\mathbb{C}'^C) \xrightarrow{\cong} M(3,C);\ X + \sqrt{-1}\,Y \longmapsto g(X) + \sqrt{-1}\,g(Y).$$

$$g : SU(3,\mathbb{C}') \xrightarrow{\cong} SL(3,\mathbb{R});\ P + \sqrt{-1}e_4 Q \longmapsto P + Q$$

$$\tilde{g} : SU(3,\mathbb{C}'^C) \xrightarrow{\cong} SL(3,C);\ X + \sqrt{-1}\,Y \longmapsto g(X) + \sqrt{-1}\,g(Y)$$

(横田一郎;古典型単純リー群(現代数学社), p33参照) //

(註3) $(SU(3,\mathbb{C}^C), H(3,\mathbb{C}^C))$の軌道代表元は次の通り。

$\#\Lambda_X$	v_X	$H(3,\mathbb{C}^C)$	$M(3,C)$
3	3	$\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)$	$\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)$
2	3	$\mathrm{diag}(\lambda_1,\lambda_2,\lambda_2)+M_1$	$\mathrm{diag}(\lambda_1,\lambda_2,\lambda_2)+N_1$
1	3	$\lambda_1 E+M_{23}$	$\lambda_1 E+N_2$
2	2	$\mathrm{diag}(\lambda_1,\lambda_2,\lambda_2)$	$\mathrm{diag}(\lambda_1,\lambda_2,\lambda_2)$
1	2	$\lambda_1 E+M_1$	$\lambda_1 E+N_1$
1	1	$\lambda_1 E$	$\lambda_1 E$

ここで、 $N_1:=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}$, $N_2=\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}$.

(註4) S. Ferrara & M. Günaydin:
Orbits of exceptional groups, duality and BPS states in string theory,
Internat. J. Modern Physics, A13 (1998), no.13, 2075-2088.
(hep-th/9708025)
には，$(F_{4(4)}, \mathfrak{J})$：対角化可能 と記述されている。

(註5) T. Miyasaka & I. Yokota:
Constructive diagonalization of an element X of the Jordan algebra $\mathfrak{J}$
by the exceptional group F_4,
Bull. Fac. Edu. & Human Sci. Yamanashi Univ. 2 (2001), 7-10
には，$(F_4', \mathfrak{J}')$：対角化可能でない元の一例が
記述されている。