可微分写像の特異ファイバーと同境群

Takahiro YAMAMOTO

Department of Mathematics, **T**okyo **G**akugei **U**niversity

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第**25**回　沼津改め　静　岡　研　究　会

———幾何，数理物理，そして量子論———

静岡大学理学部Ｃ棟３０９号室

In this talk, each mfds N, Q and each maps $N \to Q$ are smooth of class C^∞ unless otherwise stated.

§ 0 My interest

Q For a given mfd N, **STUDY** N by using maps $f\colon N \to Q$, especiallly by using **singularity of stable maps** $f\colon N \to Q$!!

A C^∞ map $f\colon N \to Q$ is C^∞ stable (or C^0 stable)
$\overset{\text{def}}{\Leftrightarrow}$ $\exists N(f) \overset{\text{open}}{\subset} C^\infty(N, Q)$: a nbd of f s.t. $\forall g \in N(f)$, $\exists \Phi\colon N \to N$ and $\Psi\colon Q \to Q$: diffeo.s (resp. homeo.s) which make the following diagram commutative:

$$\begin{array}{ccc} N & \xrightarrow{\Phi} & N \\ f\downarrow & & \downarrow g \\ Q & \xrightarrow{\Psi} & Q, \end{array}$$

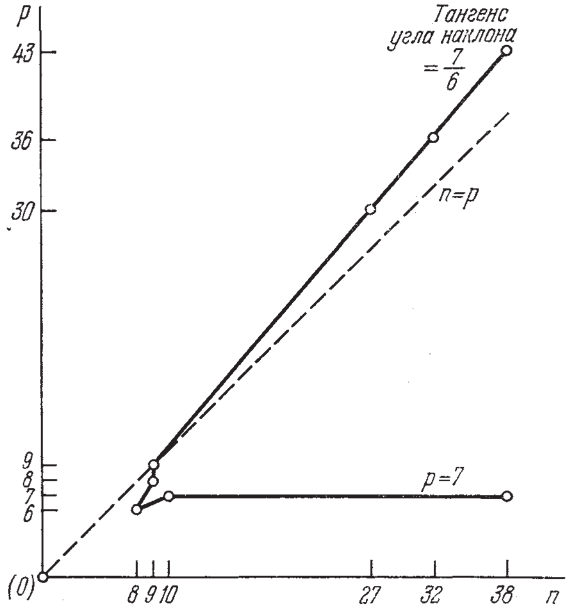
where $C^\infty(N, Q)$ is equipped with **the Whitney C^∞ topology**.

Mather '71

N^n: a cpt. n-mfd, $\quad Q^p$ be a p-mfd.

$$\{f\colon N^n \to Q^p : \text{a } C^\infty \text{ stable map}\} \overset{\text{dense}}{\subset} C^\infty(N^n, Q^p)$$

if the pair (n,p) is in the NICE RANGE in the sense of Mather.



If the dim pair (n,p) is in the nice range, then each map $f\colon M^n \to N^p$ is approximated by a stable map.

For a C^∞ map $f\colon N \to Q$,

$S(f) := \{p \in M \mid \text{rank} df_p < \text{dim} Q\}$: the set of singul. pt.s of f,

$f(S(f))$: the set of singul. values of f.

Singular pt.s (or singular values) of stable maps $N \to Q$ relate to topology of N and Q.

R.Thom

For a stable map $f\colon N^n \to \mathbb{R}^2$, $(n \geq 2)$ of closed n-mfd, we have

$$\chi(N) \equiv \#\text{cusps}(f) \pmod 2,$$

where $p \in N^n$ is a cusp point if the map-germ (f, p) is Right-Left equiv. to the form

$$(x_1, x_2^3 + x_1x_2 \pm x_3^2 \pm \cdots \pm x_n^2).$$

Pignoni, Kamenosono-Y

For a stable map $f\colon N^2 \to S^2$ of a closed surface, we have

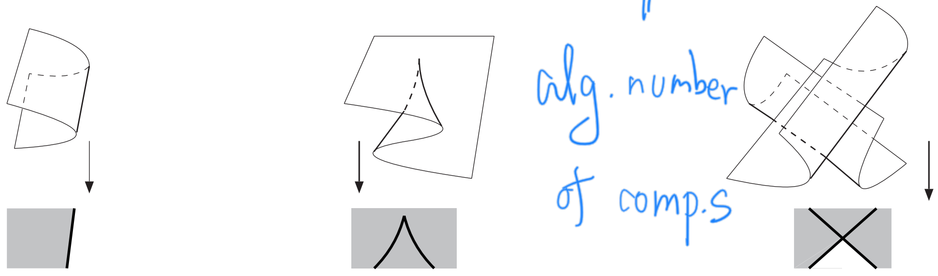
$$g = \varepsilon\left((N^+ - N^-) + \frac{c(f)}{2} + (1 + i^+ - i^-) - m(f)\right),$$

where g is the genus of N.

alg. number of Nodes ($N^+ - N^-$)

alg. number of comp.s ($i^+ - i^-$)

$m(f)$: $\min\{\#f^{-1}(q) \mid q \in S^2 : \text{reg. value}\}$



a　　　b　　　c

Saeki-Y

For a stable map $f\colon N^4 \to \mathbb{R}^3$ of an oriented 4-mfd, we have

$$\sigma(N^4) = ||\mathrm{III}^8(f)|| \in \mathbb{Z}, \qquad \mathrm{III}^8$$

where $||\mathrm{III}^8(f)||$ denotes the algebraic number of **singular fibers** of type III^8.

For $f\colon N \to Q$ and $q \in Q$, the *fiber* over q is the map germ

$$f\colon (N, f^{-1}(q)), \to (Q, q)$$

along the level set $f^{-1}(q)$. In particular, the fiber over q is called

- a *regular fiber* if $q \in Q$ is a regular value of f and $f|_\partial$,
- a *singular fiber* otherwise.

Fibers over $q_i \in Q$, $(i = 0, 1)$, are C^0 eq.

$\overset{\text{def}}{\Leftrightarrow}$ $\exists U_i \subset Q_i$: a nbd of $q_i \in Q$, $(i = 0, 1)$,

$\exists\ \Phi$: $(f_0^{-1}(U_0), f_0^{-1}(r_0)) \to (f_1^{-1}(U_1), f_1^{-1}(q_1))$: homeo preserving ∂

$\exists\ \varphi$: $U_0 \to U_1$: homeo with $\varphi(q_0) = q_1$

s.t. which make the following diagram commutative:

$$\begin{array}{ccc} (f_0^{-1}(U_0), f_0^{-1}(q_0)) & \xrightarrow{\Phi} & (f_1^{-1}(U_1), f^{-1}(q_1)) \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ (U_0, q_0) & \xrightarrow{\varphi} & (U_1, q_1). \end{array}$$

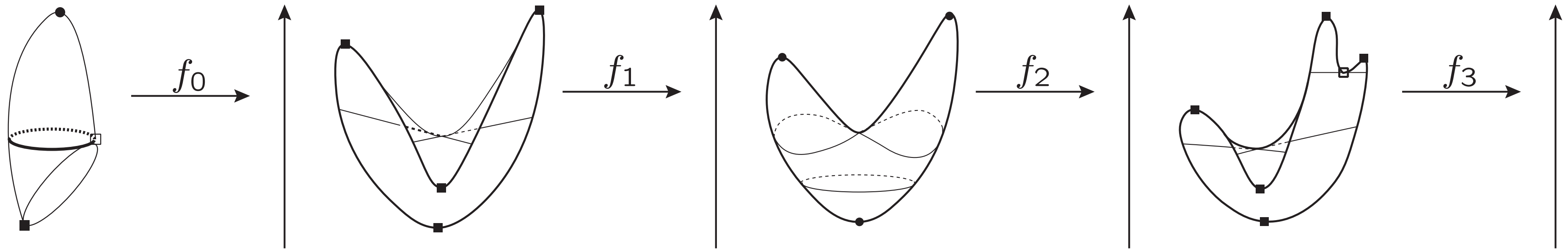
§ 1 Cobordism relations among Morse ft.s

N: a cpt n-mfd with ∂,

$f\colon N \to \mathbb{R}$ is a Morse ft

$\overset{\text{def}}{\Longleftrightarrow}$ $\begin{cases} f \text{ has } \textbf{NO} \text{ critaical pt.s on a nbd of } \partial \text{ and,} \\ \text{critical pt.s of } f \text{ and } f|_\partial \text{ are all } \textbf{NON-degenerate}. \end{cases}$

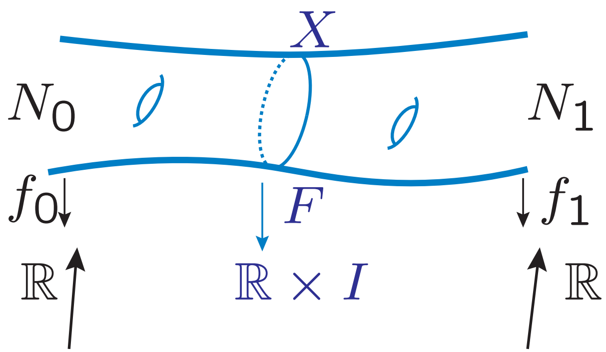
$\overset{\text{iff}}{\Longleftrightarrow}$ $\begin{cases} \forall p \in N, \text{ map germ } (f,p) \text{ is equiv. to one of the following ft.s:} \\ f = x_1 \quad \text{or} \quad f = \pm x_1^2 \pm \cdots \pm x_n^2 \qquad \text{if } p \in \mathrm{Int}N, \\ f = x_1 \quad \text{or} \quad f = \pm x_1^2 \pm \cdots x_{n-1}^2 \pm x_n \quad \text{if } p \in \partial N, \\ \text{where } \mathrm{Int}N \leftrightarrow \{x_n > 0\},\ \partial \leftrightarrow \{x_n = 0\} \text{ for coord. } (x_1, \dots, x_n). \end{cases}$



! $\{f\colon N\to\mathbb{R} : \text{Morse ft.}\} \overset{\text{open, dense}}{\subset} C^\infty(N,\mathbb{R})$.

Q (1) For N, classify Morse ft.s $f\colon N\to\mathbb{R}$.
(2) Study topology of the space $b\mathcal{N}(n) = \{f\colon N\to\mathbb{R} : \text{Morse ft}\}$.

Today, we study cobordism relations on $b\mathcal{N}(n)$:



Hitory of cobordism theory:

Thom '54: Cobordism grp.s of embeddings

Wells '66: Cobordism grp.s of immersions

Rimayni and Szűcz '98: Cobordism grp.s of maps of singularities $N^n \to Q^{n+k}$

Saeki, Ikegami, Kalmar: Cobordism grp.s of Morse ft.s on closed mfd.s.

$\mathcal{N}_n$: Codordism grp of Morse ft.s on un-oriented and closed n-mfd.s

Kalmar '05 $\mathcal{N}_2 \cong \mathbb{Z}_2 \oplus \mathbb{Z}$,
Ikegami '04 $\mathcal{N}_n \cong \mathfrak{N}_n \oplus \mathbb{Z}^{\lfloor n/2 \rfloor}$,
where $\mathfrak{N}_n$ denotes the un-oriented cobordism grp of closed n-mfd.s.

$\mathcal{M}_n$: Codordism grp of Morse ft.s on oriented and closed n-mfd.s

Ikegami-Saeki '03 $\mathcal{M}_2 \cong \mathbb{Z}$
Ikegami '04 $\mathcal{M}_n \cong \begin{cases} \Omega_n \oplus \mathbb{Z}^{\lfloor n/2 \rfloor} \oplus \mathbb{Z}_2 & n \equiv 1 \pmod 4, \\ \Omega_n \oplus \mathbb{Z}^{\lfloor n/2 \rfloor} & \text{otherwise}, \end{cases}$
where Ω_n denotes the oriented cobordism grp of closed n-mfd.s

! In the case of mfd.s with ∂, n-dim, ($n \geq 1$), un-oriented Cobordism grp is trivial. Namely any n-mfd with ∂ is null-cobordant:

N_i: cpt n-mfd possibly with ∂

$N_0 \overset{\text{Cob}}{\sim} N_1 \overset{\text{def}}{\Leftrightarrow} \exists X$:a cpt $(n+1)$-mfd possibly with corners

$\exists F\colon X \to \mathbb{R} \times [0,1]$: a C^∞ map

s.t. (1) $N_1, Q \overset{\text{submfd.s}}{\subset} \partial X$: cod 1, $\quad \partial X = N_0 \cup Q \cup N_1$,

$N_0 \cap N_1 = \emptyset$, $\partial Q = (N_0 \cap Q) \cup (N_1 \cap Q) \quad \partial X = N_0 \cup Q \cup N_1$

(2) X has corners along ∂Q

For N: a mfd with bdry ∂, put $X = N \times I$ and $Q = N \times \{1\} \cup \partial N \times I$. It implies that $N \overset{\text{Cob}}{\sim} \emptyset$.

! Un-oriented cobordism grp $\mathfrak{N}_n$ of n-mfd.s is the following:

$\mathfrak{N}_0 \cong \mathbb{Z}_2, \quad \mathfrak{N}_1 \cong 0, \quad \mathfrak{N}_2 \cong \mathbb{Z}_2, \quad \mathfrak{N}_3 \cong 0, \quad \mathfrak{N}_4 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2, \ \ldots$

Prop. **(Stable maps $N^n \to Q^2$)**

A map $f\colon N^{n\geq 3} \to Q^2$ is stable $\overset{\text{iff}}{\Leftrightarrow}$ f satisfies the followings :

(1) (Local conditions) In the following, for $p \in \partial$, we use local coord. $(x_1, \ldots, x_n)$ around p s.t. Int$N \leftrightarrow \{x_n > 0\}$ and $\partial \leftrightarrow \{x_n = 0\}$.

(1a) For $p \in \mathrm{Int}N$, (f, p) is right-left equivalent to one of

$$(x, \ldots, x_n) \mapsto \begin{cases} (x_1, x_2) & p\text{: regular pt.,} \\ (x_1, \pm x_2^2 \cdots \pm x_n^2) & p\text{: fold pt.,} \\ (x_1, x_2^3 + x_1 x_2 \pm x_3^2 \pm \cdots \pm x_n^2) & p\text{: cusp pt.} \end{cases}$$

(1b) For $p \in \partial$, (f, p) is right-left equivalent to one of

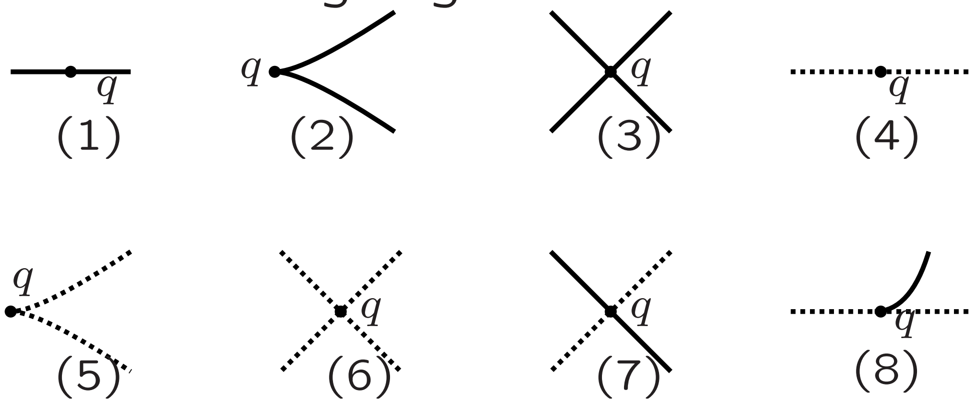
$$(x_1, \ldots, x_n) \mapsto \begin{cases} (x_1, x_2) & p\text{: regular pt. of } f|_\partial \\ (x_1, \pm x_2^2 \pm \cdots \pm x_{n-1}^2 \pm x_n) & p\text{: } \partial\text{-fold pt.,} \\ (x_1, x_2^3 + x_1 x_2 \pm x_3^2 \pm \cdots \pm x_{n-1}^2 \pm x_n) & p\text{: } \partial\text{-cusp pt.,} \\ (x_1, \pm x_2^2 + \cdots \pm x_n^2 + x_1 x_n) & p\text{: } B_2 \text{ pt.} \end{cases}$$

Proposition. **(Stable maps $N^n \to Q^2$ Conti.)**

(Global conditions) For each $q \in f(S(f)) \cup f(S(f|_N))$, the multi-germ

$$(f|_{S(f)\cup S(f|_\partial)}, f^{-1}(q) \cap (S(f) \cup S(f|_\partial)))$$

is right-left equivalent to one of the eight multi-germs whose images are depicted in the following Figure:



where **solid line** $\leftrightarrow f(S(f))$ and **dotted line** $\leftrightarrow f(S(f|_N))$.

A stable map $f\colon N \to Q$ is a stable fold map

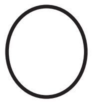
$\overset{\text{def}}{\Leftrightarrow}$ f has **NO cusps** and **NO bdry cusps**.

A stable map $f\colon N \to Q^2$ is admissible

$\overset{\text{def}}{\Leftrightarrow}$ f is **submersion** on a nbd of $\partial \overset{\text{iff}}{\Leftrightarrow} f$ has **NO B_2 pt.s**.

Fibers of stable maps $N^3 \to Q^2$ of 3-mfd.s with ∂:

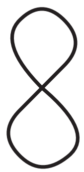
$\kappa = 0$

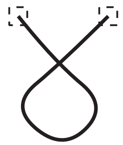
$\widetilde{\mathrm{b0}}^{0}$ 

$\widetilde{\mathrm{b0}}^{1}$ 

$\kappa = 1$

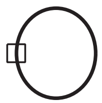
$\widetilde{\mathrm{bI}}^{2}$ 

$\widetilde{\mathrm{bI}}^{3}$ 

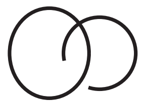
$\widetilde{\mathrm{bI}}^{4}$ 

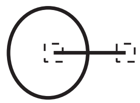
$\widetilde{\mathrm{bI}}^{5}$ 

$\widetilde{\mathrm{bI}}^{6}$ 

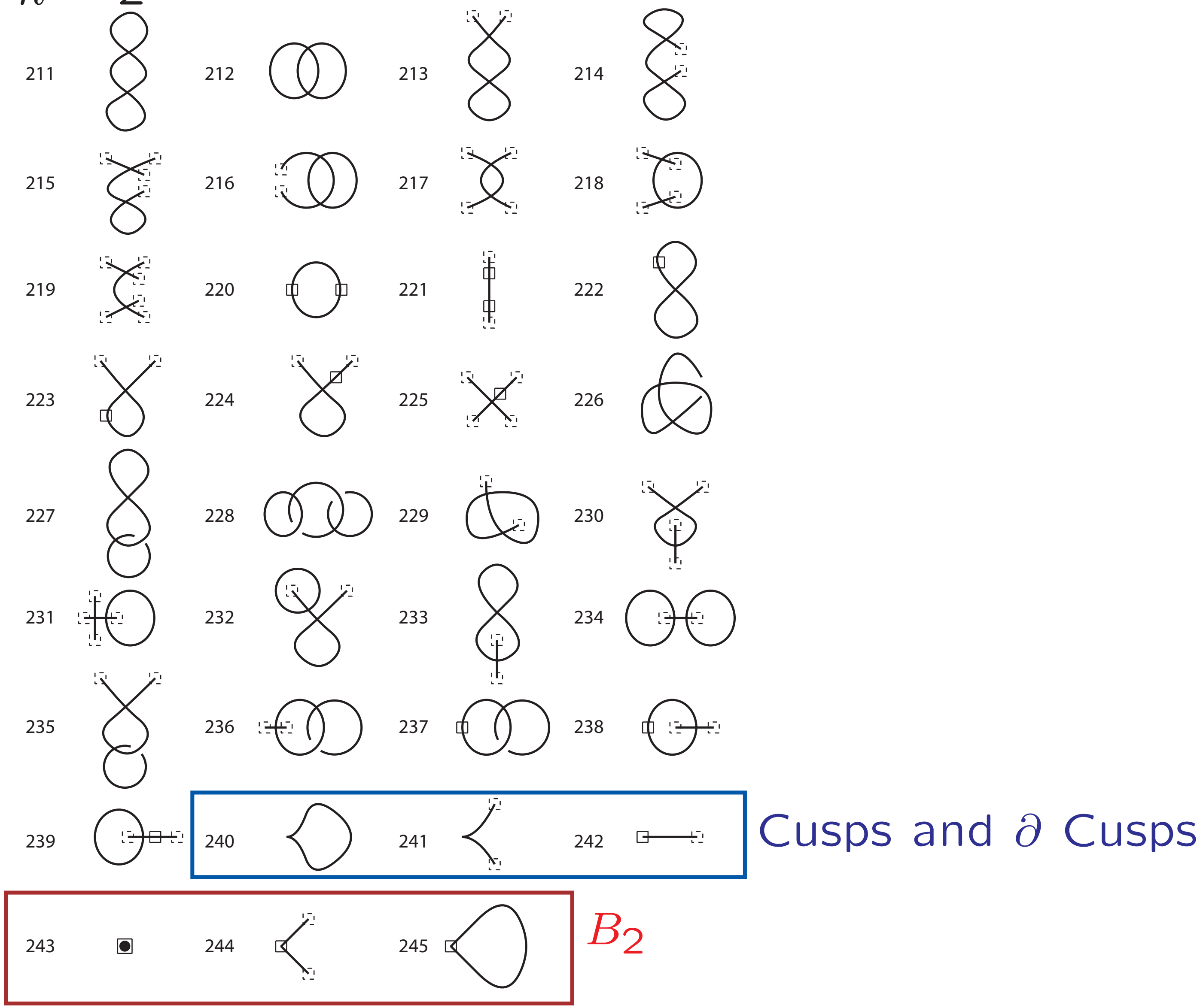
$\widetilde{\mathrm{bI}}^{7}$ 

$\widetilde{\mathrm{bI}}^{8}$ 

$\widetilde{\mathrm{bI}}^{9}$ 

$\widetilde{\mathrm{bI}}^{10}$ 

$\kappa = 2$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Cusps and ∂ Cusps
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B_2

N_i: cpt n-mfd possibly with ∂ ($n \geq 2$) , $f_i\colon N_i \to \mathbb{R}$: Morse ft.s, ($i = 0, 1$)

$f_0 \overset{\text{ad-stable Cob}}{\sim} f_1$ (or $f_0 \overset{\text{fold-Cob}}{\sim} f_1$, $f_0 \overset{\text{ad fold-Cob}}{\sim} f_1$)

$\overset{\text{def}}{\Leftrightarrow}$ $\exists X$:a cpt $(n+1)$-mfd possibly with corners

$\exists F\colon X \to \mathbb{R} \times [0,1]$: a C^∞ map

s.t. (1) $N_1, Q \overset{\text{submfd.s}}{\subset} \partial X$: cod 1, $\partial X = N_0 \cup Q \cup N_1$,

$N_0 \cap N_1 = \emptyset$, $\partial Q = (N_0 \cap Q) \cup (N_1 \cap Q)$ $\partial X = N_0 \cup Q \cup N_1$

(2) X has corners along ∂Q

(3) $F|_{N_0 \times [0,\varepsilon)} = f_0 \times \mathrm{id}_{[0,\varepsilon)}$, $F|_{N_1 \times (1-\varepsilon,1]} = f_1 \times \mathrm{id}_{(1-\varepsilon,1]}$

(4) $F^{-1}(\mathbb{R} \times \{i\}) = N_i$, and

$F|_{X \setminus (N_0 \cup N_1)}$ is a proper admissible stable map

(or stable fold map, admissible stable fold map).

! (1) Cobordism relations $\overset{\text{ad-stable Cob}}{\sim}$, $\overset{\text{fold-Cob}}{\sim}$ and $\overset{\text{ad fold-Cob}}{\sim}$ are equiv. relations.

(2) $b\mathcal{N}(n)/\overset{*\text{-Cob}}{\sim}$ forms an additive grp under the disjoint union.

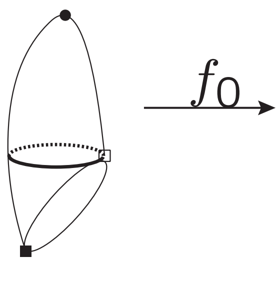
Cf. $[0] = [f : \emptyset \to \mathbb{R}]$, $[-f : N \to \mathbb{R}, x \mapsto -f(x)] = -[f]$

Denote $b\mathfrak{N}_n := b\mathcal{N}(n)/\overset{\text{ad-stable Cob}}{\sim}$, $\quad b\mathfrak{F}_n := b\mathcal{N}(n)/\overset{\text{fold-Cob}}{\sim}$ and

$\mathfrak{AF}_n := b\mathcal{N}(n)/\overset{\text{ad-fold Cob}}{\sim}$.

Theorem [Saeki-Y '16]

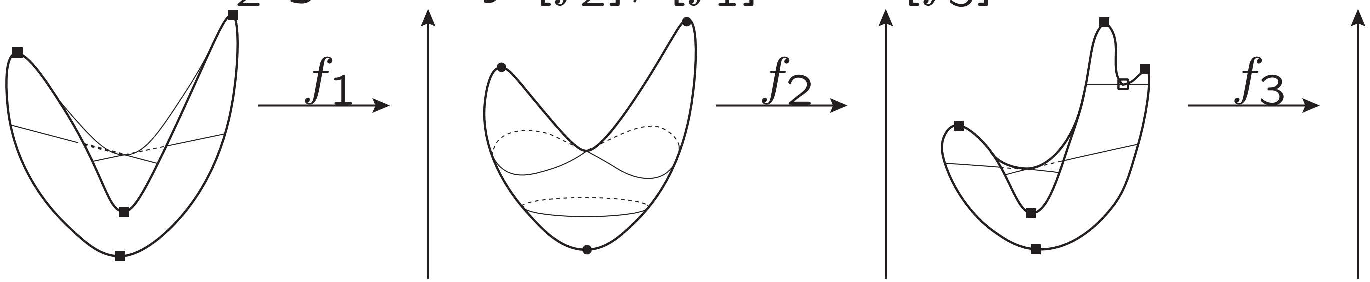
$b\mathfrak{N}_2 \cong \mathbb{Z}_2$ gene. by $[f_0]$:



Theorem [Y '16]

$b\mathfrak{F}_2 \cong \mathbb{Z}_2$ gene. by $[f_1]$.

$\mathfrak{AF}_2 \cong \mathbb{Z} \cong \mathbb{Z} \cong \mathbb{Z}_2$ gene. by $[f_2]$, $[f_1]$ and $[f_3]$.



§ 2 OutLine of Proof

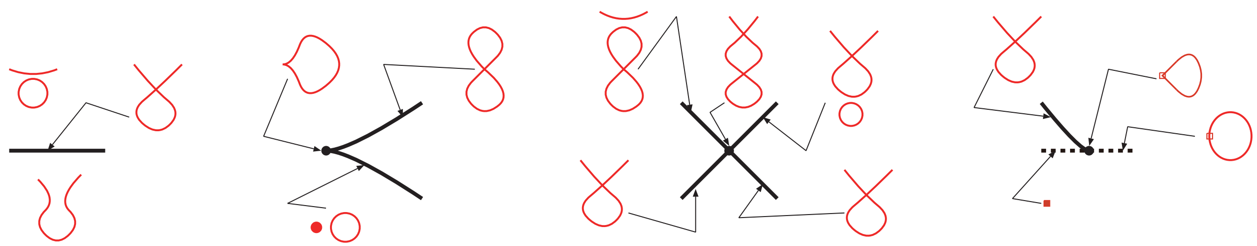
Step 1: Invariants

τ: a class of fibers of proper Thom maps of
codim= ℓ(=dim$Q$ -dim$N$),
ρ: an eq. relation among fibers in τ.

If τ and ρ satisfy **SUITABLE** conditions, we obtain **the univ. cpx of sinlular fibers of τ-maps** of n-mfd into q-mfd

$$\mathcal{C}(\tau(n,q),\rho_{n,q}) = (C^\kappa(\tau(m,n),\rho_{m,n}),\delta_\kappa)_{\kappa\in\mathbb{Z}}$$

$C^\kappa(\tau,\rho)$: the $\mathbb{Z}_2$-vec. sp. spanned by all cod= κ fibers of τ-map $N^n - Q^q$
($C^\kappa = 0$ if $\kappa < 0$ or $\kappa > q$),
δ_k: the cochain map defined by adjacency of fibers:



Prop

$f_i\colon N_i^n \to Q^q$: τ-maps, $(i = 0, 1)$
If $f_0 \sim_{\tau\text{-Cob}} f_1$, then for each $[c] \in H^\kappa(\tau(m, q), \rho_{n,q})$, we have

$$[c(f_0)] = [c(f_1)] \in H_{q-\kappa}(Q^q; \mathbb{Z}_2),$$

where for $c = \sum n_{\mathcal{F}}\mathcal{F}$, $[c(f)]$ denotes the homol. class of $\cup n_{\mathcal{F}}\mathcal{F}(f)$.
Namely, **a cohomology class $[c] \in H^\kappa(\tau(m, q), \rho_{n,q})$ induce τ-cobordism invariant among τ-maps $N^n \to Q^q$** .

! A τ-Cobordism invariant $[c]$ induced by a cohomology class may be trivial.

(1)
$\tau = \mathcal{AS}_{\mathrm{pr}}(3,2)$: the set of fibers of proper **ad. stable maps** $N^3 \to Q^2$,
$\rho = \rho_{3,2}(\mathbf{2})$: C^0 eq relation modulo **TWO** reg fibers.

$$\mathcal{C}(\mathcal{AS}_{\mathrm{pr}}(3,2), \rho_{3,2}(2)) = (C^\kappa(\mathcal{AS}_{\mathrm{pr}}(3,2), \rho_{3,2}(2)), \delta_\kappa)_{\kappa \in \mathbb{Z}}.$$

The cohomology groups of $\mathcal{C}(\mathcal{AS}_{\mathrm{pr}}(3,2), \rho_{3,2}(2))$ are:
$H^0 \cong \mathbb{Z}_2$ generated by $[\widetilde{\mathrm{b0}}]$
$H^1 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, generated by

$$\alpha_1 = [\widetilde{\mathrm{bI}}^2 + \widetilde{\mathrm{bI}}^3 + \widetilde{\mathrm{bI}}^4 + \widetilde{\mathrm{bI}}^5 + \widetilde{\mathrm{bI}}^9 + \widetilde{\mathrm{bI}}^{10}], \qquad \beta_1 = [\widetilde{\mathrm{bI}}^6 + \widetilde{\mathrm{bI}}^7 + \widetilde{\mathrm{bI}}^8],$$
$$\gamma_1 = [\widetilde{\mathrm{bI}}^2_o + \widetilde{\mathrm{bI}}^3_e + \widetilde{\mathrm{bI}}^4_e + \widetilde{\mathrm{bI}}^6_o + \widetilde{\mathrm{bI}}^8_e] = [\widetilde{\mathrm{bI}}^2_e + \widetilde{\mathrm{bI}}^3_o + \widetilde{\mathrm{bI}}^4_o + \widetilde{\mathrm{bI}}^6_e + \widetilde{\mathrm{bI}}^8_o].$$

where $\mathcal{F}_o$ ($\mathcal{F}_e$) denotes the eq. class of the fiber of type $\mathcal{F}$ with odd (resp. even) number of regular fibers and $\mathcal{F} = \mathcal{F}_o + \mathcal{F}_e$.

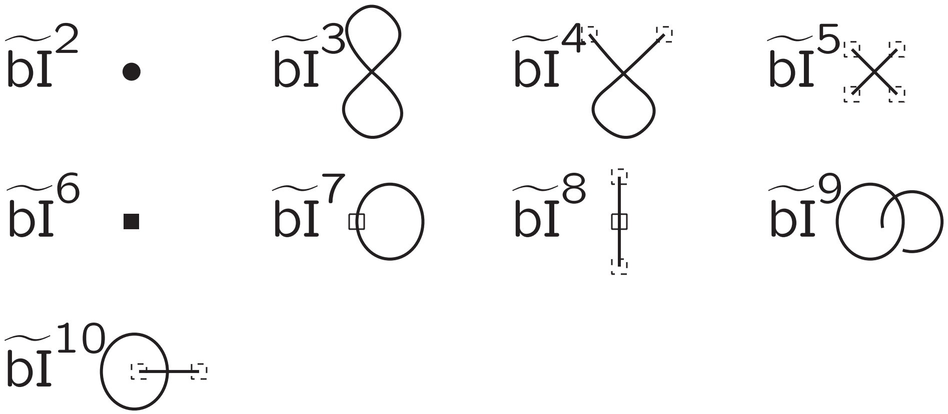
Proposition

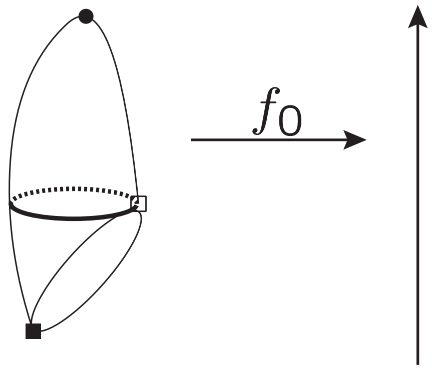
For Morse ft.s $f\colon N^2 \to \mathbb{R}$ of cpt mfd.s possibly with ∂,

(1) $\beta_1(f)$ and $\gamma_1(f)$ are **trivial** $\mathcal{AS}_{pr}$-cobordism invariants.

(2) $\alpha_1(f) = \widetilde{\mathrm{bI}}^2(f) + \widetilde{\mathrm{bI}}^3(f) + \widetilde{\mathrm{bI}}^4(f) + \widetilde{\mathrm{bI}}^5(f) + \widetilde{\mathrm{bI}}^9(f) + \widetilde{\mathrm{bI}}^{10}(f)$ is a **NON-tirival** $\mathcal{AS}_{pr}$-cobordism invariant.

$\kappa = 1$





(2)
$\tau = b\mathcal{F}_{\mathrm{pr}}(3,2)$: the set of fibers of proper **stable folds** $N^3 \to Q^2$,
$\rho = \rho_{3,2}(\mathbf{2})$: C^0 eq relation modulo **TWO** reg fibers.

$$\mathcal{C}(b\mathcal{F}_{\mathrm{pr}}(3,2), \rho_{3,2}(2)) = (C^\kappa(b\mathcal{F}_{\mathrm{pr}}(3,2), \rho_{3,2}(2)), \delta_\kappa)_{\kappa \in \mathbb{Z}}.$$

The cohomology groups of $\mathcal{C}(b\mathcal{F}_{\mathrm{pr}}(3,2), \rho_{3,2}(2))$ are:
$H^0 \cong \mathbb{Z}_2$ generated by $[\widetilde{\mathrm{b0}}]$
$H^1 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, generated by

$$\alpha_2 = [\widetilde{\mathrm{bI}}^5 + \widetilde{\mathrm{bI}}^8 + \widetilde{\mathrm{bI}}^9 + \widetilde{\mathrm{bI}}^{10}], \qquad \beta_2 = [\widetilde{\mathrm{bI}}^6 + \widetilde{\mathrm{bI}}^7 + \widetilde{\mathrm{bI}}^8],$$
$$\gamma_2 = [\widetilde{\mathrm{bI}}_o^2 + \widetilde{\mathrm{bI}}_e^3 + \widetilde{\mathrm{bI}}_e^4 + \widetilde{\mathrm{bI}}_o^6 + \widetilde{\mathrm{bI}}_e^8] = [\widetilde{\mathrm{bI}}_e^2 + \widetilde{\mathrm{bI}}_o^3 + \widetilde{\mathrm{bI}}_o^4 + \widetilde{\mathrm{bI}}_e^6 + \widetilde{\mathrm{bI}}_o^8].$$

where $\mathcal{F}_o$ ($\mathcal{F}_e$) denotes the eq. class of the fiber of type $\mathcal{F}$ with odd (resp. even) number of regular fibers and $\mathcal{F} = \mathcal{F}_o + \mathcal{F}_e$.

Proposition

For Morse ft.s $f\colon N^2 \to \mathbb{R}$ of cpt mfd.s possibly with ∂,

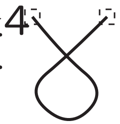
(1) $\beta_2(f)$ and $\gamma_2(f)$ are **trivial** $b\mathcal{F}_{pr}$-cobordism invariants.

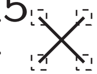
(2) $\alpha_2(f) = \widetilde{\mathrm{bI}}^5(f) + \widetilde{\mathrm{bI}}^8(f) + \widetilde{\mathrm{bI}}^9(f) + \widetilde{\mathrm{bI}}^{10}(f)$ is a **NON-tirival** $b\mathcal{F}_{pr}$-cobordism invariant.

$\kappa = 1$

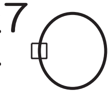
$\widetilde{\mathrm{bI}}^2$ 

$\widetilde{\mathrm{bI}}^3$ 

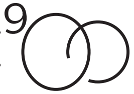
$\widetilde{\mathrm{bI}}^4$ 

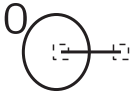
$\widetilde{\mathrm{bI}}^5$ 

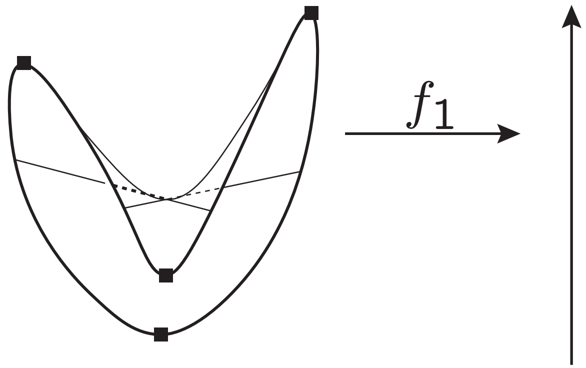
$\widetilde{\mathrm{bI}}^6$ 

$\widetilde{\mathrm{bI}}^7$ 

$\widetilde{\mathrm{bI}}^8$ 

$\widetilde{\mathrm{bI}}^9$ 

$\widetilde{\mathrm{bI}}^{10}$ 



(3)

$\tau = \mathcal{AF}_{\mathrm{pr}}(3,2)$: the set of fibers of proper **ad. stable folds** $N^3 \to Q^2$,
$\rho = \rho_{3,2}(\mathbf{2})$: C^0 eq relation modulo **TWO** reg fibers.

$$\mathcal{C}(\mathcal{AF}_{\mathrm{pr}}(3,2), \rho_{3,2}(2)) = (C^\kappa(\mathcal{AF}_{\mathrm{pr}}(3,2), \rho_{3,2}(2)), \delta_\kappa)_{\kappa\in\mathbb{Z}}.$$

The cohomology groups of $\mathcal{C}(\mathcal{AF}_{\mathrm{pr}}(3,2), \rho_{3,2}(2))$ are:
$H^0 \cong \mathbb{Z}_2$ generated by $[\widetilde{\mathrm{b0}}]$
$H^1 \cong$ ~~$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$~~ $\oplus^5 \mathbb{Z}_2$, generated by

$$\alpha_3 = [\widetilde{\mathrm{bI}}^2], \qquad \beta_3 = [\widetilde{\mathrm{bI}}^6],$$
$$\gamma_3 = [\widetilde{\mathrm{bI}}^2_o + \widetilde{\mathrm{bI}}^3_e + \widetilde{\mathrm{bI}}^4_e + \widetilde{\mathrm{bI}}^6_o + \widetilde{\mathrm{bI}}^8_e] = [\widetilde{\mathrm{bI}}^2_e + \widetilde{\mathrm{bI}}^3_o + \widetilde{\mathrm{bI}}^4_o + \widetilde{\mathrm{bI}}^6_e + \widetilde{\mathrm{bI}}^8_o],$$
$$\zeta_3 = [\widetilde{\mathrm{bI}}^5 + \widetilde{\mathrm{bI}}^8 + \widetilde{\mathrm{bI}}^9 + \widetilde{\mathrm{bI}}^{10}], \qquad \eta_3 = [\widetilde{\mathrm{bI}}^7 + \widetilde{\mathrm{bI}}^8]$$

where $\mathcal{F}_o$ ($\mathcal{F}_e$) denotes the eq. class of the fiber of type $\mathcal{F}$ with odd (resp. even) number of regular fibers and $\mathcal{F} = \mathcal{F}_o + \mathcal{F}_e$.

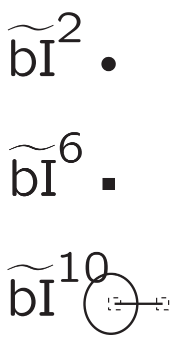
Proposition

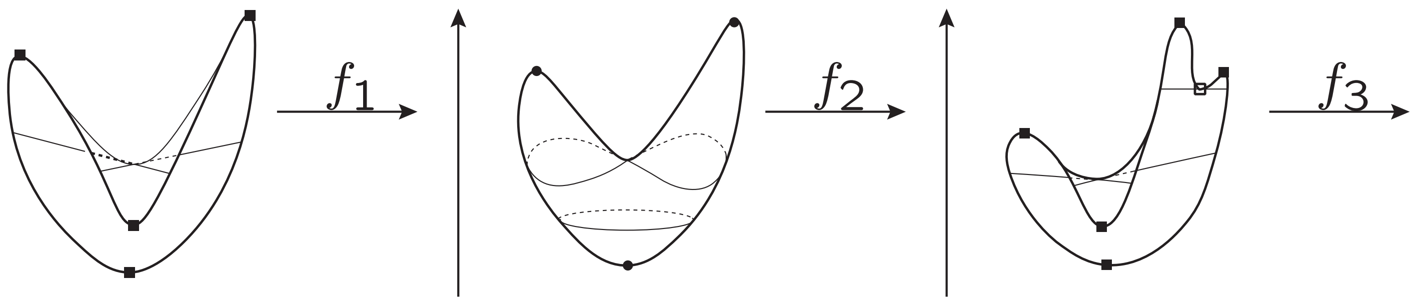
For Morse ft.s $f\colon N^2 \to \mathbb{R}$ of cpt mfd.s possibly with ∂,

(1) $\gamma_3(f)$ and $\eta_3(f)$ are **trivial** $\mathcal{AF}_{pr}$-cobordism invariants.

(2) $\alpha_3(f) = \widetilde{\mathrm{bI}}^2(f)$, $\beta_3(f) = \widetilde{\mathrm{bI}}^6(f)$ and $\zeta_3(f) = \widetilde{\mathrm{bI}}^5(f) + \widetilde{\mathrm{bI}}^8(f) + \widetilde{\mathrm{bI}}^9(f) + \widetilde{\mathrm{bI}}^{10}(f)$ are **NON-tirival** $\mathcal{AF}_{pr}$-cobordism invariants.

$\kappa = 1$





(4)

$\tau = \mathcal{AF}_{\mathrm{pr}}(3,2)$: the set of **Co-oriented** fibers

of proper **ad. stable folds** $N^3 \to Q^2$,

$\rho = \rho_{3,2}(\mathbf{2})$: C^0 eq relation modulo **TWO** reg fibers.

$$\mathcal{CO}(\mathcal{AF}_{\mathrm{pr}}(3,2), \rho_{3,2}(2)) = (CO^{\kappa}(\mathcal{AF}_{\mathrm{pr}}(3,2), \rho_{3,2}(2)), \delta_{\kappa})_{\kappa \in \mathbb{Z}}.$$

The cohomology groups of $\mathcal{CO}(\mathcal{AF}_{\mathrm{pr}}(3,2), \rho_{3,2}(2))$ are:

$H^0 \cong \mathbb{Z}$ generated by $[\widetilde{\mathrm{b}0}]$

$H^1 \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, generated by

$$\alpha_4 = [\widetilde{\mathrm{bI}^2_o} - \widetilde{\mathrm{bI}^2_e}], \qquad \beta_4 = [\widetilde{\mathrm{bI}^6_0} - \widetilde{\mathrm{bI}^6_e}],$$
$$\gamma_4 = [\widetilde{\mathrm{bI}^2_e} + \widetilde{\mathrm{bI}^3_o} + \widetilde{\mathrm{bI}^4_o} + \widetilde{\mathrm{bI}^6_e} + \widetilde{\mathrm{bI}^8_o}] = -[\widetilde{\mathrm{bI}^2_o} + \widetilde{\mathrm{bI}^3_e} + \widetilde{\mathrm{bI}^4_e} + \widetilde{\mathrm{bI}^6_o} + \widetilde{\mathrm{bI}^8_e}],$$

where $\mathcal{F}_o$ ($\mathcal{F}_e$) denotes the eq. class of the fiber of type $\mathcal{F}$ with odd (resp. even) number of regular fibers and $\mathcal{F} = \mathcal{F}_o + \mathcal{F}_e$.

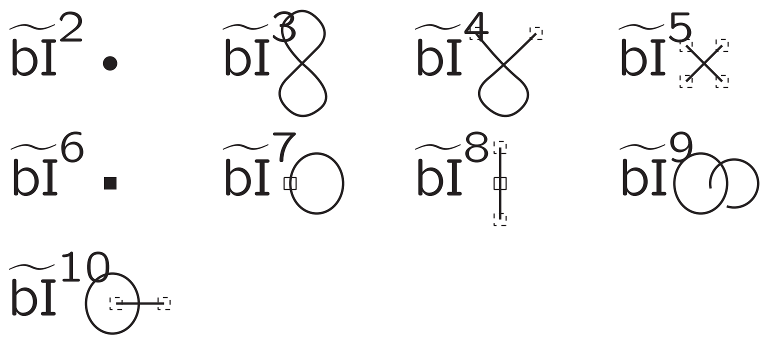
Proposition

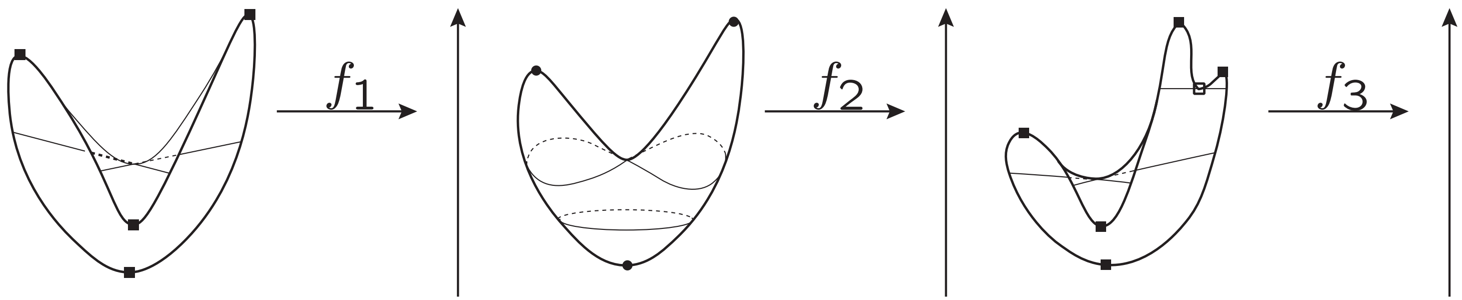
For Morse ft.s $f\colon N^2 \to \mathbb{R}$ of cpt mfd.s possibly with ∂,

(1) $\gamma_4(f)$ is a **trivial** $\mathcal{AF}_{pr}$-cobordism invariant.

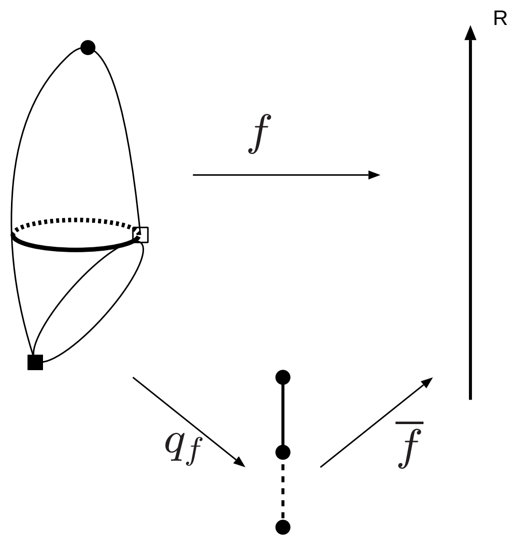
(2) $\alpha_4(f) = \widetilde{\mathrm{bI}}^2_o(f) - \widetilde{\mathrm{bI}}^2_e(f)$, and $\beta_4(f) = \widetilde{\mathrm{bI}}^6_0(f) - \widetilde{\mathrm{bI}}^6_e(f)$ are **NON-tirival** $\mathcal{AF}_{pr}$-cobordism invariants.

$\kappa = 1$



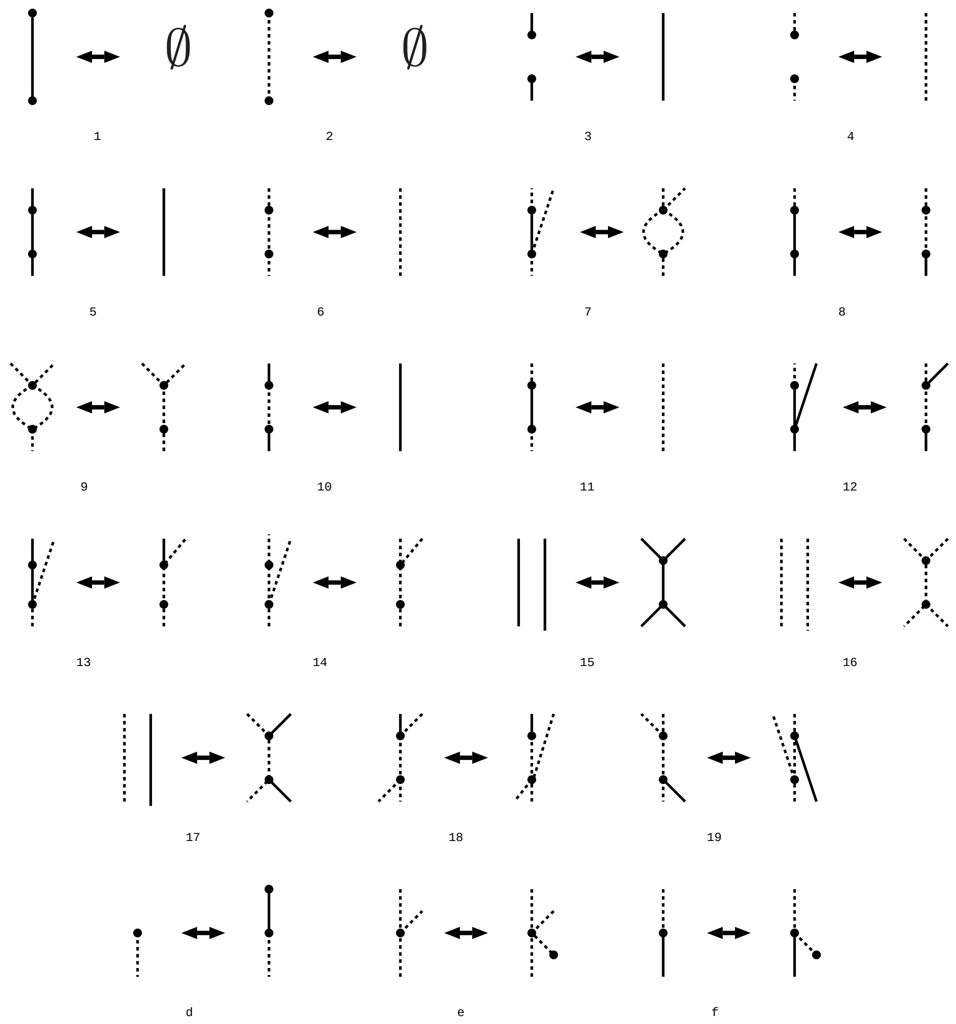


Step 2: Let Cob $= b\mathfrak{N}_2$ or $b\mathfrak{F}_2$, $\mathfrak{A}\mathfrak{F}_2$.
$\rho_{\mathrm{Cob}}\colon \mathrm{Cob} \to \mathcal{R}_{\mathrm{Cob}}$, $[f\colon N \to \mathbb{R}] \mapsto [\overline{f}\colon W_f \to \mathbb{R}]$: iso,
where $\mathcal{R}_{\mathrm{Cob}}$ denote the cobordism group of **labeled Reeb-like ft.s** on labeled Reeb-like graphs.



! For $f\colon N \to \mathbb{R}$ of a surface possibly with ∂ and p_1, $p_2 \in N$,
$p_1 \sim p_2 \overset{\text{def}}{\Leftrightarrow} \exists q \in \mathbb{R}$ s.t. p_1, p_2 are in the same con. comp. of $f^{-1}(q)$.
Then, $W_f := N/\sim$.

Then, calculate $\mathcal{R}_{\mathrm{Cob}}$ by using local moves induced by the Reeb space of stable maps $f\colon N^3 \to Q^2$:



§ FutherWorks

Prop [Y 17]

$n \geq 3$, $b\mathfrak{N}_n \neq 0$, $b\mathfrak{F}_n \neq 0$ and $\mathfrak{A}\mathfrak{F}_n \neq 0$.

Q Study the structure of $b\mathfrak{N}_n$, $b\mathfrak{F}_n$ and $\mathfrak{A}\mathfrak{F}_n$ for $n \geq 3$.