球面の共形的ツォル変形
Conformal Zoll deformations on the sphere

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g_0: the standard $C_{2\pi}$-metric on S^n.
g_λ: a one-parameter family of $C_{2\pi}$-metrics on S^n $\quad\gamma_\lambda(s)$: a geodesic, $\gamma_\lambda(s+2\pi)=\gamma_\lambda(s)$.

$$\gamma(s,\lambda)=\gamma_\lambda(s),\quad \delta_\lambda(s)=\frac{\partial\gamma}{\partial\lambda}(s,\lambda),\quad \dot\gamma_\lambda(s)=\frac{d\gamma_\lambda}{ds}(s),\quad h_\lambda=\frac{\partial g_\lambda}{\partial\lambda}.$$

$$\nabla^\lambda_{\dot\gamma_\lambda}\dot\gamma_\lambda=0\Rightarrow \nabla^\lambda_{\dot\gamma_\lambda}\nabla^\lambda_{\dot\gamma_\lambda}\delta_\lambda+R^\lambda(\delta_\lambda,\dot\gamma_\lambda)\dot\gamma_\lambda+\beta_\lambda(\dot\gamma_\lambda,\dot\gamma_\lambda)=0$$

$$g_\lambda(\beta_\lambda(X,Y),Z)=\frac{1}{2}\Big\{(\nabla^\lambda_X h_\lambda)(Y,Z)+(\nabla^\lambda_Y h_\lambda)(X,Z)-(\nabla^\lambda_Z h_\lambda)(X,Y)\Big\}$$

$$g_\lambda(\dot\gamma_\lambda,\dot\gamma_\lambda)=1\Rightarrow h_\lambda(\dot\gamma_\lambda,\dot\gamma_\lambda)+2\frac{d}{ds}g_\lambda(\delta_\lambda,\dot\gamma_\lambda)=0$$

$$\delta_\lambda(0)=0\Rightarrow g_\lambda(\delta_\lambda,\dot\gamma_\lambda)(s)=-\frac{1}{2}\int_0^s h_\lambda(\dot\gamma_\lambda,\dot\gamma_\lambda)(u)\,du$$

$$\delta_\lambda(2\pi)=0\Rightarrow\int_0^{2\pi}h_\lambda(\dot\gamma_\lambda,\dot\gamma_\lambda)(s)\,ds=0$$

一方, g_λ: a one-parameter family of metrics on S^n, $\quad g=g_0$: the standard $C_{2\pi}$-metric on S^n.
$h=h_0,\quad \gamma=\gamma_0,\quad \delta=\delta_0.$

$$\int_0^{2\pi}h(\dot\gamma,\dot\gamma)(s)\,ds=0\Rightarrow\delta(s+2\pi)=\delta(s)$$

$g_\lambda = \exp(\rho_\lambda) g_0$: a one-parameter family of $C_{2\pi}$-metrics on S^n, $\quad \rho_0 = 0$.

$h_\lambda = \dfrac{\partial \rho_\lambda}{\partial \lambda} \exp(\rho_\lambda) g_0, \quad f = \left.\dfrac{\partial \rho_\lambda}{\partial \lambda}\right|_{\lambda=0}, \quad h_0 = f g_0.$

Theorem (Funk)

$$\forall \text{ geodesic } \gamma, \quad \int_0^{2\pi} f(\gamma(s))\, ds = 0 \Leftrightarrow f \text{ is an odd function on } S^n$$

$\overrightarrow{e_1} = \gamma(0), \quad \overrightarrow{e_2} = \dot\gamma(0), \quad \overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}, \ldots, \overrightarrow{e_{n+1}}$: an OrthoNormal Basis of R^{n+1} $(\supset S^n)$.

$R = R^0, \quad \beta = \beta_0.$

$$\delta = g_0(\delta, \dot\gamma)\dot\gamma + \delta^\perp = g_0(\delta, \dot\gamma)\dot\gamma + \sum_{i=3}^{n+1} g_0(\delta, \overrightarrow{e_i})\overrightarrow{e_i}$$

$$\begin{aligned} 0 &= g_0(\nabla_{\dot\gamma}\nabla_{\dot\gamma}\delta + R(\delta, \dot\gamma)\dot\gamma + \beta(\dot\gamma, \dot\gamma), \overrightarrow{e_i}) \\ &= \frac{d^2}{ds^2} g_0(\delta, \overrightarrow{e_i}) + g_0(\delta, \overrightarrow{e_i}) + g_0(\beta(\dot\gamma, \dot\gamma), \overrightarrow{e_i}) \\ &= \frac{d^2}{ds^2} g_0(\delta, \overrightarrow{e_i}) + g_0(\delta, \overrightarrow{e_i}) - \frac{1}{2}(\overrightarrow{e_i} f) \qquad (3 \le i \le n+1) \end{aligned}$$

$$\delta(s) = -\frac{1}{2}\left(\int_0^s f(\gamma(u))\, du\right)\dot\gamma(s) + \frac{1}{2}\sum_{i=3}^{n+1}\left(\int_0^s \sin(s-u)(\overrightarrow{e_i} f)(\gamma(u))\, du\right)\overrightarrow{e_i}$$

$$k_\lambda = \frac{\partial h_\lambda}{\partial \lambda} = \left(\frac{\partial^2 \rho_\lambda}{\partial \lambda^2} + \frac{\partial \rho_\lambda}{\partial \lambda}^2 \right) \exp(\rho_\lambda) g_0.$$

$$h_\lambda(\dot\gamma_\lambda, \dot\gamma_\lambda) + \frac{1}{2}\frac{d}{ds} g_\lambda(\delta_\lambda, \dot\gamma_\lambda) = 0$$

$$\Rightarrow k_\lambda(\dot\gamma_\lambda, \dot\gamma_\lambda) - 2g_\lambda(\beta_\lambda(\dot\gamma_\lambda, \dot\gamma_\lambda), \delta_\lambda) + 2\frac{d}{ds}\left(h_\lambda(\delta_\lambda, \dot\gamma_\lambda) + \frac{1}{4}\frac{\partial}{\partial \lambda} g_\lambda(\delta_\lambda, \dot\gamma_\lambda) \right) = 0$$

$$f_2 = \left.\frac{\partial^2 \rho_\lambda}{\partial \lambda^2}\right|_{\lambda=0}, \quad k_0 = (f_2 + f^2) g_0.$$

$$\begin{aligned}\int_0^{2\pi} k_0(\dot\gamma, \dot\gamma)(s)\, ds &= 2\int_0^{2\pi} g_0(\beta(\dot\gamma, \dot\gamma), \delta)(s)\, ds \\ &= \frac{1}{2}\int_0^{2\pi} (f^2)(\gamma(s))\, ds \\ &\quad - \frac{1}{2}\sum_{i=3}^{n+1} \int_0^{2\pi} (\overrightarrow{e_i} f)(\gamma(s)) \left(\int_0^s \sin(s-u)(\overrightarrow{e_i} f)(\gamma(u))\, du \right) ds\end{aligned}$$

$$\sum_{i=3}^{n+1} \int_0^{2\pi} (\overrightarrow{e_i} f)(\gamma(s)) \left(\int_0^s \sin(s-u)(\overrightarrow{e_i} f)(\gamma(u))\, du \right) ds = -\int_0^{2\pi} (2f_2 + f^2)(\gamma(s))\, ds$$

N. B.: $n \geq 3$ では球面上での「ラドン変換」は, 測地線全体のなす空間上の関数全体の上への写像ではない.

$\overrightarrow{e_1},\ \overrightarrow{e_2},\ \overrightarrow{e_3},\ \overrightarrow{e_4}$: ON.

$$\gamma(s) = \overrightarrow{e_1}\cos s + \overrightarrow{e_2}\sin s$$
$$\gamma^+_{pq}(s) = (\overrightarrow{e_1}\cos p + \overrightarrow{e_3}\sin p)\cos s + (\overrightarrow{e_2}\cos q + \overrightarrow{e_4}\sin q)\sin s$$
$$\gamma^-_{pq}(s) = (\overrightarrow{e_1}\cos p + \overrightarrow{e_4}\sin p)\cos s + (\overrightarrow{e_2}\cos q + \overrightarrow{e_3}\sin q)\sin s$$
$$\frac{\partial^2}{\partial p \partial q}\int_0^{2\pi} f(\gamma^+_{p,q}(s))\,ds\bigg|_{p=0,q=0} = \int_0^{2\pi}(\overrightarrow{e_3}\,\overrightarrow{e_4} f)(\gamma(s))\cos s \sin s\, ds$$
$$= \frac{\partial^2}{\partial p \partial q}\int_0^{2\pi} f(\gamma^-_{p,q}(s))\,ds\bigg|_{p=0,q=0}$$

$$\begin{aligned}
f(x_1,x_2,x_3,x_4) = {} & b_{12}(x_1^2x_2 - \tfrac{1}{3}x_2^3) + b_{13}(x_1^2x_3 - \tfrac{1}{3}x_3^3) + b_{14}(x_1^2x_4 - \tfrac{1}{3}x_4^3) \\
& + b_{21}(x_2^2x_1 - \tfrac{1}{3}x_1^3) + b_{23}(x_2^2x_3 - \tfrac{1}{3}x_3^3) + b_{24}(x_2^2x_4 - \tfrac{1}{3}x_4^3) \\
& + b_{31}(x_3^2x_1 - \tfrac{1}{3}x_1^3) + b_{32}(x_3^2x_2 - \tfrac{1}{3}x_2^3) + b_{34}(x_3^2x_4 - \tfrac{1}{3}x_4^3) \\
& + b_{41}(x_4^2x_1 - \tfrac{1}{3}x_1^3) + b_{42}(x_4^2x_2 - \tfrac{1}{3}x_2^3) + b_{43}(x_4^2x_3 - \tfrac{1}{3}x_3^3) \\
& + c_1x_2x_3x_4 + c_2x_3x_4x_1 + c_3x_4x_1x_2 + c_4x_1x_2x_3
\end{aligned}$$

$$F(\gamma) = \sum_{i=3}^{4} \int_0^{2\pi} (\overrightarrow{e_i} f)(\gamma(s)) \left(\int_0^s \sin(s-u)(\overrightarrow{e_i} f)(\gamma(u))\, du \right) ds$$
$$= 2\pi \left(\frac{5}{24} \left((b_{13})^2 + (b_{14})^2 + (b_{23})^2 + (b_{24})^2 \right) + \frac{14}{24} (b_{13}b_{14} + b_{23}b_{24}) - \frac{1}{24} \left((c_3)^2 + (c_4)^2 \right) \right)$$

$$\left.\frac{\partial^2}{\partial p \partial q} F(\gamma_{pq}^{+})\right|_{p=0,q=0} - \left.\frac{\partial^2}{\partial p \partial q} F(\gamma_{pq}^{-})\right|_{p=0,q=0}$$
$$= 2\pi \left(-\frac{1}{3} b_{13}c_3 + \frac{1}{3} b_{14}c_4 + \frac{1}{3} b_{23}c_3 - \frac{1}{3} b_{24}c_4 + \frac{4}{3} b_{31}c_1 - \frac{4}{3} b_{32}c_2 - \frac{4}{3} b_{41}c_1 + \frac{4}{3} b_{42}c_2 \right)$$

$$(b_{31} - b_{41})c_1 - (b_{32} - b_{42})c_2 = 0$$

$h : [-1, 1] \to (-1, 1)$, an odd real function, $h(-1) = h(1) = 0$.

$$(1 + h(\cos\theta))^2 d\theta^2 + \sin^2\theta\, \omega_{n-1}$$

これが Zoll metric. Zoll metric は axi-symmetric, conformal.