冪零 Lie 群上の左不変計量に関する測地流の完全積分可能性について

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Wolfram Bauer 氏 (Universität Hannover) との共同研究に基づく

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§1. Introduction

Backgrounds 1

The free rigid body is an example of completely integrable geodesic flows on Lie groups w. r. t. left-invariant metrics.

It is nothing but the geodesic flow on $SO(3)$ w. r. t. a left-invariant Riemannian metric.

Classical result: this dynamics can be solved by quadrature.

Modern formulation: a Hamiltonian system on $T^*SO(3)$ admitting three functionally independent first integrals (constants of motion). $\rightsquigarrow$ Completely integrable.

Backgrounds 2

Completely integrable systems on symplectic manifolds:
(M, ω): sympl. mfd (i.e. ω: non-degenerate closed 2-form).
For $H \in \mathcal{C}^\infty(M)$: Hamiltonian, the Hamiltonian vector field Ξ_H is defined through

$$\iota_{\Xi_H}\omega = -\mathrm{d}H.$$

Assume $\dim M = 2n$. The Hamiltonian system is called *completely integrable* in the sense of Liouville, if there exist n functionally independent functions $F_1, \cdots, F_{n-1}, F_n (= H)$ which Poisson commute:

$$\{F_i, F_j\} = 0, \qquad (i, j = 1, \cdots, n).$$

Here, $\{F, G\} = \omega\left(\Xi_F, \Xi_G\right) = \Xi_F\left(G\right)$.

Backgrounds 3

Theorem (Liouville-Arnol'd)

Assume that the Hamiltonian system (M, ω, H) is completely integrable, whose first integrals are $F_1, \cdots, F_{n-1}, F_n(= H)$. Set $F = (F_1, \cdots, F_n) : M \to \mathbb{R}^n$. Then, for $\mu \in \mathrm{Im} F$: regular value, $F^{-1}(\mu)$ is a torus, if it is compact and connected. Further, F is a torus bundle around this fibre and on each fibre the induced flow is linear on the torus.

Short history

Researches on the complete integrability of geodesic flow on Lie groups w.r.t. a left-invariant metric.

- $SO(3)$: Euler, Poinsot, Jacobi, $\cdots$.
- $SO(n)$: Mishchenko, Dikii, Manakov, Ratiu (1970's, 1980's).
- semi-simple Lie groups (complex, normal real form, compact real form, their intersection): Mishchenko-Fomenko (1978).
- $U(n)$: Iwai (2004), Ratiu-T (2015).

Nilpotent Lie group case

- Existence of a complete set of Poisson commuting functions on the dual to nilpotent Lie algebras: Vergne (1972).
- Heisenberg groups (explicit construction of first integrals): Kocsard-Ovando-Reggiani (2016).

Problem:

How about more general nilpotent Lie groups?

§2. Geodesic flow on step-two nilpotent Lie groups

Sympl. form on the tangent bundle to Lie groups

G: Lie group, $\mathfrak{g}$: Lie algebra. Consider the left-trivializations:

$$TG \supset T_g G \ni X_g \mapsto (g, \mathrm{d}L_{g^{-1}} X_g) \in G \times \mathfrak{g},$$
$$T^*G \supset T_g^* G \ni \alpha_g \mapsto \left(g, L_g^* \alpha_g\right) \in G \times \mathfrak{g}^*.$$

Given $\langle \cdot, \cdot \rangle$: left-invariant metric on G (or equivalently metric on $\mathfrak{g}$), we can identify $T^*G \cong G \times \mathfrak{g}^*$ with $TG \cong G \times \mathfrak{g}$. Then, the canonical symplectic form on T^*G induces the one Ω on $TG \cong G \times \mathfrak{g}$, which is described as

$$\Omega\,(g, X)\,((U, V)\,,(U', V')) = \langle U, V' \rangle - \langle V, U' \rangle - \langle X, [U, U'] \rangle\,,$$

where $(g, X) \in G \times \mathfrak{g} \cong TG$,
$(U, V)\,,(U', V') \in \mathfrak{g} \times \mathfrak{g} \cong T_{(g,X)}\,(G \times \mathfrak{g})$.

Poisson bracket and Hamiltonian vector field

The corresponding Poisson bracket $\{\cdot,\cdot\}$ on $TG \cong G \times \mathfrak{g}$ is given as

$$\{F, G\}(g, X) = \langle V', U\rangle - \langle U', V\rangle - \langle X, [V', V]\rangle,$$

if $\mathrm{grad} F(g, X) = (U, V), \mathrm{grad} G(g, X) = (U', V')$.
Hamiltonian vector field Ξ_F for the Hamiltonian F is written as

$$\Xi_F(g, X) = (V, (\mathrm{ad}_V)^* X - U).$$

The Hamiltonian of the geodesic flow is $H(g, X) = \frac{1}{2}\langle X, X\rangle$ whose Hamiltonian vector field is

$$\Xi_H(g, X) = (X, (\mathrm{ad}_X)^* X).$$

Step-two nilpotent Lie groups 1

Assume that G is a step-two nilpotent Lie group.
I.e., $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{z}$, where $\mathfrak{z} \subset \mathfrak{g}$ is the centre of $\mathfrak{g}$.
W.r.t. the metric $\langle \cdot, \cdot \rangle$, the complement to $\mathfrak{z}$ is denoted by $\mathfrak{v} \subset \mathfrak{g}$; $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$: the orthogonal decomposition.

For any $Z \in \mathfrak{z}$, consider the endomorphism $j(Z) : \mathfrak{v} \to \mathfrak{v}$ defined through

$$\langle j(Z)Y, Y' \rangle = \langle Z, [Y, Y'] \rangle ,$$

$Y, Y' \in \mathfrak{v}$.
Note that $j(Z)$ is skew-symmetric w.r.t. $\langle \cdot, \cdot \rangle|_{\mathfrak{v}}$.

Step-two nilpotent Lie groups 2

The Hamiltonian vector field for the geodesic flow is written as

$$\Xi_H(p, Y) = (Y, j(Y_{\mathfrak{z}})\, Y_{\mathfrak{v}}), \quad (p, Y) \in G \times \mathfrak{g} \cong TG.$$

Here, $Y = Y_{\mathfrak{v}} + Y_{\mathfrak{z}}$ is the decomposition according to the orthogonal decomposition $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$.

Problem

Prove the complete integrability of this Hamiltonian system.

In the following, a set of Poisson commuting functions is found explicitly.

§3. Complete integrability of geodesic flow on step-two nilpotent Lie groups

First integrals obtained from maps $\alpha : \mathfrak{z} \to \mathfrak{v}$

Assume that G is simply connected, so that the exponential mapping $\exp : \mathfrak{g} \to G$ is invertible. $\log : G \to \mathfrak{g}$: the inverse.

For an arbitrary differentiable mapping $\alpha : \mathfrak{z} \to \mathfrak{v}$, set

$$F_\alpha(p, Y) := \langle \alpha(Y_\mathfrak{z}), j(Y_\mathfrak{z}) \log p - Y_\mathfrak{v} \rangle, \quad (p, Y) \in G \times \mathfrak{g} \cong TG.$$

Proposition

For $\alpha, \beta : \mathfrak{z} \to \mathfrak{v}$: differentiable and for $F \in C^\infty(TG)$: left-invariant,

1. $\{F_\alpha, F_\beta\}(p, Y) = \langle Y_\mathfrak{z}, [\alpha(Y_\mathfrak{z}), \beta(Y_\mathfrak{z})] \rangle = \langle j(Y_\mathfrak{z})(\alpha(Y_\mathfrak{z})), \beta(Y_\mathfrak{z}) \rangle$,
2. $\{F_\alpha, F\} = 0$. *In particular,* $\{F_\alpha, H\} = 0$.

Left-invariant first integrals

For arbitrary left-invariant functions $F, G \in \mathcal{C}^\infty(TG)$, we have

$$\{F, G\}(p, Y) = -\langle Y_{\mathfrak{z}}, [V'_{\mathfrak{v}}, V_{\mathfrak{v}}]\rangle = \langle j(Y_{\mathfrak{z}})(V_{\mathfrak{v}}), V'_{\mathfrak{v}}\rangle,$$

where $\operatorname{grad} F(p, Y) = (0, V), \operatorname{grad} G(p, Y) = (0, V')$.

As to the Hamiltonian $H(p, Y) = \frac{1}{2}\langle Y, Y\rangle$, we have

$$\{F, H\}(p, Y) = \langle j(Y_{\mathfrak{z}})(V_{\mathfrak{v}}), Y_{\mathfrak{v}}\rangle = \langle V_{\mathfrak{v}}, j(Y_{\mathfrak{z}})(Y_{\mathfrak{v}})\rangle.$$

A left-invariant function $F(p, Y) = F(Y_{\mathfrak{z}})$, depending only on the central component $Y_{\mathfrak{z}}$ of Y, is a first integral of the geodesic flow.

H type Lie group

Now, assume that $\mathfrak{g}$ is an H type (due to Kaplan (1980's)), i.e. defined by a Clifford representation as follows:

Consider the Clifford algebra $C\ell\left(\mathfrak{z}, \langle\cdot,\cdot\rangle|_{\mathfrak{z}}\right)$ and assume that $(\mathfrak{v}, \langle\cdot,\cdot\rangle|_{\mathfrak{v}})$ is a Clifford representation of it.

If, for any $Z \in \mathfrak{z}$, the skew-symmetric operator $j(Z) : \mathfrak{v} \to \mathfrak{v}$ defined through $\langle j(Z)X, Y\rangle = \langle Z, [X, Y]\rangle$, $X, Y \in \mathfrak{v}$, satisfies

$$\langle j(Z)X, j(Z)Y\rangle = \langle Z, Z\rangle\langle X, Y\rangle,$$

then the step-two nilpotent Lie algebra $\mathfrak{g} = \mathfrak{v} + \mathfrak{z}$ is called of H type.

First integrals for H type Lie group 1

On an H type nilpotent Lie algebra $\mathfrak{g}$, the operator $j(Z)$ is a skew-symmetric matrix with $j(Z)^2 = -\langle Z, Z\rangle \mathrm{id}_{\mathfrak{v}}$. In this case $\dim \mathfrak{v} (=: 2m)$ is even.
There exists an orthonormal basis $v_1(Z), \cdots, v_{2m}(Z) \in \mathfrak{v}$ s.t.

$$\langle v_i, j(Z) v_j\rangle = \begin{cases} \sqrt{\langle Z, Z\rangle}, & \text{if } i+m=j, \\ -\sqrt{\langle Z, Z\rangle}, & \text{if } i=j+m, \\ 0, & \text{otherwise.} \end{cases}$$

Then, for $F_{v_1}, \cdots, F_{v_{2n}}$, we have

$$\{F_{v_i}, F_{v_i}\}(p, Y) = \begin{cases} \sqrt{\langle Y_{\mathfrak{z}}, Y_{\mathfrak{z}}\rangle}, & \text{if } i+m=j, \\ -\sqrt{\langle Y_{\mathfrak{z}}, Y_{\mathfrak{z}}\rangle}, & \text{if } i=j+m, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, $\{F_{v_i}, F_{v_j}\} = 0$, $i, j = 1, \cdots, m$.

First integrals for H type Lie group 2

For $Y = Y_{\mathfrak{v}} + Y_{\mathfrak{z}} \in \mathfrak{v} + \mathfrak{z} = \mathfrak{g}$, using the same basis $v_1(Y_{\mathfrak{z}}), \cdots, v_{2m}(Y_{\mathfrak{z}}) \in \mathfrak{v}$, we write

$$Y = y_1 v_1(Y_{\mathfrak{z}}) + \cdots + y_{2m} v_{2m}(Y_{\mathfrak{z}}) + Y_{\mathfrak{z}}.$$

Then, $g_i(p, Y) = y_i^2 + y_{i+m}^2$, $i = 1, \cdots, m$, satisfy $\mathrm{grad} g_i(p, Y) = (0, y_i v_i + y_{i+m} v_{i+m})$ and hence $\{g_i, g_j\} = 0$, $i, j = 1, \cdots, m$.

Further, writing $Y_{\mathfrak{z}} = y'_1 z_1 + \cdots + y'_k z_k$ in a basis $z_1, \cdots, z_k$ of $\mathfrak{z}$, the functions $h_l = y'_l$, $l = 1, \cdots, k$, Poisson commute with all the other functions on $G \times \mathfrak{g} \cong TG$.

Complete integrability for H type Lie groups

We can show that $H = \frac{1}{2}\left(g_1 + \cdots + g_m + h_1^2 + \cdots + h_k^2\right)$.

Theorem (Bauer-T.)

The functions $g_1, \cdots, g_m, h_1, \cdots, h_k, F_{v_1}, \cdots, F_{v_m}$ form a complete set of Poisson commuting functions and hence the geodesic flow $(G \times \mathfrak{g} \cong TG, \Omega, H)$ is completely integrable.

Remark

These first integrals are generalizations of those obtain by Kocsard-Ovando-Reggiani in Heisenberg group case.

Corollary

Set $H' := \frac{1}{2}(g_1 + \cdots + g_m)$. The sub-Riemannian geodesic flow $(G \times \mathfrak{g} \cong TG, \Omega, H')$ is completely integrable.

Example

Take $Z_1, Z_2 \in \mathbb{R}^{2,0}$: orthonormal w.r.t. the inner product $\langle \cdot, \cdot \rangle_{2,0}$.
Let $C\ell_{2,0}$ be the Clifford algebra generated by Z_1, Z_2 with $Z_i^2 = -1$, $i = 1, 2$, $Z_1 Z_2 + Z_2 Z_1 = 0$.
Take $v \in \mathbb{R}^{4,0}$ with $\langle v, v \rangle_{4,0} = 1$ and set

$$X_1 = v, X_2 = j(Z_1) j(Z_2) v, X_3 = j(Z_1) v, X_4 = j(Z_2) v,$$

where $j : C\ell_{2,0} \to \mathbb{R}^{4,0}$: representation of $C\ell_{2,0}$.
X_1, X_2, X_3, X_4: orthonormal basis of $\mathbb{R}^{4,0}$.

Lie algebra $\mathcal{N}_{2,0}$ is defined on $(\mathbb{R}^{4,0} \oplus \mathbb{R}^{2,0}, \langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{4,0} + \langle \cdot, \cdot \rangle_{2,0})$ with the Lie bracket defined through

$$\langle Z, [X, X'] \rangle = \langle j(Z) X, X' \rangle, \quad X, X' \in \mathbb{R}^{4,0}, Z \in \mathbb{R}^{2,0}.$$

Example (continued)

W.r.t. the basis $X_1, X_2, X_3, X_4, j(Z)$, where $Z = z_1 Z_1 + z_2 Z_2 \in \mathbb{R}^{2,0}$, has the matrix representation

$$\begin{pmatrix} 0 & 0 & z_1 & z_2 \\ 0 & 0 & z_2 & -z_1 \\ -z_1 & -z_2 & 0 & 0 \\ -z_2 & z_1 & 0 & 0 \end{pmatrix}.$$

Taking the new basis $X_1, X_2, X_3' := (z_1 X_3 + z_2 X_4)/|Z|, X_4' := (z_2 X_3 - z_1 X_4)/|Z|$, $j(Z)$ is represented by

$$|Z| \begin{pmatrix} 0 & E_2 \\ -E_2 & 0 \end{pmatrix}.$$

Example (continued)

Expand any $Y \in \mathcal{N}_{2,0}$ as

$$Y = y_1 X_1 + y_2 X_2 + y_3' X_3' + y_4' X_4' + y_5 Z_1 + y_6 Z_2.$$

We see that the functions

$$F_{X_1},\ F_{X_2},\ g_1(p, Y) = y_1^2 + {y_3'}^2,\ g_2(p, Y) = y_2^2 + {y_4'}^2,\ y_5,\ y_6$$

are functionally independent and Poisson commuting first integrals of the geodesic flow on the corresponding Lie group.

First integrals via isometry group 1

The isometry group of $(G, \langle \cdot, \cdot \rangle)$ is $K \ltimes G$ where the Lie algebra $\mathfrak{k}$ of K is given as

$$\mathfrak{k} = \{(A, B) \in \mathfrak{so}(\mathfrak{z}, \langle \cdot, \cdot \rangle_{\mathfrak{z}}) \times \mathfrak{so}(\mathfrak{v}, \langle \cdot, \cdot \rangle_{\mathfrak{v}}) | \\ Bj(Z) - j(Z)B = j(AZ), Z \in \mathfrak{z}\}.$$

For $k = (A, B) \in \mathfrak{k}$, the corresponding Killing vector field is

$$X_k^*(p) = \mathrm{d}L_p \left(BW_{\mathfrak{v}} - \frac{1}{2} [W_{\mathfrak{v}}, BW_{\mathfrak{v}}] + AW_{\mathfrak{z}} \right), \quad p \in G.$$

Here, $W = \log p \in \mathfrak{g}$.

First integrals via isometry group 2

In association to the Killing vector field X_k^*, we consider

$$\begin{aligned} f_{X_k^*}(p, Y) &:= \langle Y, \mathrm{d}L_{p^{-1}}(X_k^*) \rangle \\ &= \left\langle Y, BW_{\mathfrak{v}} - \frac{1}{2}[W_{\mathfrak{v}}, BW_{\mathfrak{v}}] + AW_{\mathfrak{z}} \right\rangle . \end{aligned}$$

Proposition

$$\left\{ f_{X_k^*}, H \right\} = 0.$$

First integrals via isometry group 3

The commutation relation with other first integrals:

$$\left\{f_{X_k^*}, g\right\} = \langle Y_{\mathfrak{v}}, B{V'}_{\mathfrak{v}}\rangle + \langle Y_{\mathfrak{z}}, A{V'}_{\mathfrak{z}}\rangle ,$$
$$\left\{f_{X_k^*}, f_{X_{k'}^*}\right\} = f^*_{X_{[k,k']}},$$
$$\left\{f_{X_k^*}, F_\alpha\right\} = F_{\alpha'},$$

where g: left-invariant s.t. $\mathrm{grad} g\,(p, Y) = (0, V')$,
$\alpha'\,(Y_{\mathfrak{z}}) = B\alpha\,(Y_{\mathfrak{z}}) - \dfrac{\partial\alpha}{\partial Y_{\mathfrak{z}}}\,(Y_{\mathfrak{z}}) \cdot (AY_{\mathfrak{z}}),\; Y_{\mathfrak{z}} \in \mathfrak{z}$.