Examples of solvmanifolds without LCK structures

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1. Introduction

Definition 1.
G : simply-connected solvable Lie group.
Γ : lattice, that is, discrete co-compact subgroup of G.
$\Longrightarrow \Gamma\backslash G$: solvmanifold.
(G : nilpotent Lie group $\Longrightarrow \Gamma\backslash G$: nilmanifold)

Theorem 2. [Hasegawa '06]
A solvmanifold admitting a Kähler structure is a finite quotient of a complex torus which has the structure of a complex torus bundle over a complex torus.

Definition 3.

(M, g, J) : Hermitian manifold.

Ω : the fundamental 2-form $(\Omega(X, Y) = g(X, JY))$.

(M, g, J) : locally conformal Kähler (LCK)

$\underset{\text{def}}{\Longleftrightarrow}$ ${}^{\exists}\omega$: closed 1-form such that $d\Omega = \omega \wedge \Omega$.

(We call ω Lee form.)

Remark 4.

If $\omega = df$, then $(M, e^{-f}g, J)$ is Kähler.

Definition 5.

(M, g, J) : LCK manifold.

(M, g, J) : Vaisman manifold

$\underset{\text{def}}{\Longleftrightarrow}$ Lee form ω is parallel with respect to g.

Definition 6.
M : manifold,
α : closed 1-form on M.
$d_\alpha : A^p(M) \to A^{p+1}(M)$

$$d_\alpha \beta := \alpha \wedge \beta + d\beta \qquad (d_\alpha^2 = 0).$$

We call β α-closed (α-exact), if $d_\alpha \beta = 0$ ($\beta = d_\alpha \gamma$).

Similarly, we can define the new differential operator on a Lie algebra.

(M, g, J) : LCK manifold.
$\iff d\Omega = \omega \wedge \Omega$ (ω : closed 1-form).
$\iff 0 = -\omega \wedge \Omega + d\Omega$.
$\iff 0 = d_{-\omega}\Omega$, that is, Ω: $-\omega$-closed.

Theorem 7. [León-López-Marrero-Padrón, '03]
(M, g) : compact Riemannian manifold
α : parallel 1-from with respect to g
$\Rightarrow$ any α-closed form is α-exact.

The fundamental 2-form Ω of a Vaisman manifold is $-\omega$-exact:

$$\Omega = d_{-\omega}\eta = -\omega \wedge \eta + d\eta.$$

Remark 8. [S. '15]
On solvmanifolds, the inverse is hold.

Examples

- Hopf surface (Vaisman '79) : Vaisman manifold

- Inoue surfaces (Tricerri '82) : non-Vaisman manifold

- nilmanifold $S^1 \times \Gamma\backslash H$ where, H is a Hisenberg Lie group (Fernandez etal '86) : Vaisman manifold

- Oeljeklaus-Toma manifold (Oeljeklaus-Toma, '05) : non-Vaisman manifold

LCK nilmanifold $\Gamma\backslash G$

Theorem 9. [S. '07]
$\Gamma\backslash G$ has a LCK structure (g, J) such that J is left-invariant,
$\Rightarrow G = \mathbb{R} \times H$, where H is a Heisenberg Lie group.

Theorem 10. [Bazzoni. '17]
$\Gamma\backslash G$ has a Vaisman structure (g, J),
$\Rightarrow G = \mathbb{R} \times H$, where H is a Heisenberg Lie group.

In this talk, we consider the following solvable Lie group:

$$G_n = \left\{ \left(t, \begin{pmatrix} x_i \\ y_i \end{pmatrix}, z \right) : t, x_i, y_i, z \in \mathbb{R}, i = 1, \cdots, n \right\},$$

where a structure group on G_n is defined by

$$\begin{aligned} &\left(t, \begin{pmatrix} x_i \\ y_i \end{pmatrix}, z \right) \cdot \left(t', \begin{pmatrix} x'_i \\ y'_i \end{pmatrix}, z' \right) \\ &= \left(t + t', \begin{pmatrix} e^{a_i t} x'_i + x_i \\ e^{-a_i t} y'_i + y_i \end{pmatrix}, z' + \frac{1}{2} \sum_{i=1}^{n} (-e^{a_i t} y_i x'_i + e^{-a_i t} x_i y'_i) + z \right) \end{aligned}$$

and $a_i \in \mathbb{Z} - \{0\}$.

$[G_n, G_n]$: $(2n+1)$-dimensional Heisenberg Lie group

Construction of solvmanifold $\Gamma_n \backslash G_n$

$B = \begin{pmatrix} 0 & -1 \\ 1 & k \end{pmatrix}$ be a unimodular matrix with distinct positive eigenvalues λ, λ^{-1}

$\exists P = \begin{pmatrix} 1 & \lambda \\ 1 & \lambda^{-1} \end{pmatrix}$ such that $PBP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$

$\varphi : \mathbb{R}^{2n+2} \to G_n$ diffeomorphism

$$\varphi(A) = \left((\log \lambda)t, 2P \begin{pmatrix} x_i \\ y_i \end{pmatrix}, |P|\, z \right)$$

$\Rightarrow$ We can prove that $\varphi(\mathbb{Z}^{2n+2})$ is a lattice Γ_n on G_n

Main Theorem .

Let J_n be a left-invariant complex structure on $\Gamma_n \backslash G_n$.

- In the case of $n = 1$,
  $(\Gamma_1 \backslash G_1, J_1)$ has a LCK structure, but has no Vaisman structures.
- In the case of $n \geq 2$,
  $(\Gamma_n \backslash G_n, J_n)$ has no LCK structures.

Remark 11.

On a LCK structure, in absence of a complex structure,
it is said to be *locally conformal symplectic (LCS)*.
The solvmanifold $\Gamma_n \backslash G_n$ has LCS structures.

2. Preliminary

Definition 12.
G : simply-connected solvable Lie group.
G : completely solvable
$\underset{\text{def}}{\Longleftrightarrow}$ For ${}^{\forall}X \in \mathfrak{g}$, $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ has only real eigenvalues,
where $\mathfrak{g}$: Lie algebra of G.

Theorem 13. [Hattori '60]
$H^*_{\mathrm{DR}}(\Gamma\backslash G) \cong H^*(\mathfrak{g})$, where $\mathfrak{g}$: Lie algebra of G.

Note that G_n is completely solvable.

$(M=\Gamma\backslash G, g, J)$: LCK solvmanifold with Lee form ω such that

1. J is left-invariant.

2. $^{\exists}$ left-invariant closed 1-form ω_0 s.t $\omega_0 - \omega = df$.

Theorem 14. [Belgun '00]
For $X, Y \in \mathfrak{g}$,

$$\langle X, Y\rangle := \int_M e^f g(X,Y) d\mu$$

$\Longrightarrow (\mathfrak{g}, \langle\ ,\ \rangle, J)$: LCK solvable Lie algebra with Lee form ω_0.

Ω_0 : the fundamental 2-form of $(\langle\ ,\ \rangle, J)$

Remark 15. [S. '12]
$(M=\Gamma\backslash G, g, J)$: Vaisman solvmanifold ($\Omega = d_{-\omega}\eta$)
$\Longrightarrow \Omega_0 = d_{-\omega_0}\eta_0$

Definition 16.

J_1, J_2 : complex structure on $\mathfrak{g}$

J_1, J_2 are equivalent

$\underset{\text{def}}{\Longleftrightarrow}$ ${}^{\exists}F \in \mathrm{Aut}(\mathfrak{g})$ such that $J_1 \circ F = F \circ J_2$

(g, J_1) : Hermitian structure $\Longrightarrow$ $(F^* g, J_2)$: Hermitian structure

Proposition 17. [Ugarte '07]

(g, J_1) : LCK structure $\Longrightarrow$ $(F^* g, J_2)$: LCK structure

3. In the case of $n = 1$

$\mathfrak{g}_1 = \text{span}\{A, X, Y, Z : [A, X] = X, [A, Y] = -Y, [X, Y] = Z\}$

$\{\theta, \alpha, \beta, \gamma\}$ is a dual base of $\{A, X, Y, Z\}$:

$$\begin{aligned} &d\theta = 0, \\ &d\alpha = -\theta \wedge \alpha, d\beta = \theta \wedge \beta, \\ &d\gamma = -\alpha \wedge \beta \end{aligned}$$

LCK structure on $\Gamma_1 \backslash G_1$ (Inoue surface S^+)
(Tricerri '82, Fernández etal '89, Kamishima '01)

- $\langle\ ,\ \rangle$: left-inv. metric s.t $\{A, X, Y, Z\}$ is an orthonormal frame.
- J : left-inv. complex structure s.t $JA = Y, JZ = X$.

$\Longrightarrow (\langle\ ,\ \rangle, J)$: LCK structure with Lee form θ.

- $\nabla_Y \theta(Y) = -\theta(\nabla_Y Y) = -\langle A, \nabla_Y Y\rangle = -\langle [A, Y], Y\rangle = \langle Y, Y\rangle \neq 0.$

Proposition 18. [cf. Belgun, '00]
$\Gamma_1\backslash G_1$ has no Vaisman structures.

Proof. • A complex structure on a 4-dimensional solvmanifold is left-invariant [Hasegawa, '05].

• A complex structure on $\Gamma_1\backslash G_1$ is equivalent to

$J_0 : J_0A = Y, J_0Y = -A, J_0Z = X, J_0X = -Z$
or
$J_1 : J_1A = Y + Z, J_1Y = -A - X, J_1Z = X, J_1X = -Z$ [Ovando, '04].

• $(\Gamma_1\backslash G_1, J)$ has a Vaisman structure : $\Omega_0 = d_{-k\theta}\eta$
$\Longrightarrow (\Gamma_1\backslash G_1, J_q)$ has a LCK structure : $\Omega_0^q = d_{-k_q\theta}\eta_q$ ($q = 0$ or 1)

$$\langle Z, Z\rangle = d_{-k_q\theta}\eta_q(J_qZ, Z) = (-k_q\theta \wedge \eta_q + d\eta_q)(X, Z) = 0$$

because $X, Z \in [\mathfrak{g}_1, \mathfrak{g}_1]$ and Z is in the center of $\mathfrak{g}_1$.

□

Remark 19.
$(\Gamma_1 \backslash G_1, J_1)$ has no LCK structures.

Remark 20. [Vaisman, '82]
The first Betti number of a Vaisman manifold is odd.

Proposition 21. [Kasuya '12]
If $\mathfrak{g} = \mathbb{R}^n \ltimes \mathbb{R}^m$ and $\dim[\mathfrak{g}, \mathfrak{g}] > \frac{1}{2} \dim \mathfrak{g}$,
then $(\Gamma \backslash G, J)$ has no Vaisman structures.

$\Rightarrow$ Inoue surface S^0, O-T manifolds have no Vaisman structures.

4. In the case of $n \geq 2$

$\mathfrak{g}_n = \mathrm{span}\{A, X_i, Y_i, Z : [A, X_i] = a_i X_i, [A, Y_i] = -a_i Y_i, [X_i, Y_i] = Z\}$

$\{\theta, \alpha_i, \beta_i, \gamma\}$ is a dual base of $\{A, X, Y, Z\}$:

$$\begin{aligned} &d\theta = 0, \\ &d\alpha_i = -a_i\theta \wedge \alpha_i, d\beta_i = a_i\theta \wedge \beta_i, \\ &d\gamma = -\sum_i \alpha_i \wedge \beta_i \end{aligned}$$

Note that

$$[\mathfrak{g}_n, \mathfrak{g}_n] = \mathrm{span}\{X_i, Y_i, Z\}, \text{ span}\{Z\} \text{ is the center of } \mathfrak{g}_n.$$

We assume that $\mathfrak{g}_n$ has a LCK structure $(\langle\ ,\ \rangle, J_n)$ with Ω_0 ($-k\theta$-closed 2-from).

Lemma 22. $\Omega_0(Z, X_i) = \Omega_0(Z, Y_i) = 0$ for each i.

Proof.

$$\begin{aligned} \Omega_0(Z, X_i) &= \Omega_0([X_j, Y_j], X_i) \\ &= -d\Omega_0(X_j, Y_j, X_i) + \Omega_0([X_j, X_i], Y_j) - \Omega_0([Y_j, X_i], X_j) \\ &= k\theta \wedge \Omega_0(X_j, Y_j, X_i) = 0. \end{aligned}$$

□

$\gamma_0 : \mathfrak{g}^* \to \mathfrak{g}$ isomorphism induced by $\langle \ , \ \rangle$.

Corollary 23. $J_n \circ \gamma_0(\theta) \in \mathrm{span}\{Z\}$.

Proof. Since $\langle J_n Z, X_i \rangle = -\Omega_0(Z, X_i) = 0, \langle J_n Z, Y_i \rangle = -\Omega_0(Z, Y_i) = 0$, we have $J_n Z \in \mathrm{span}\{\gamma_0(\theta)\}$, that is, $J_n \circ \gamma_0(\theta) \in \mathrm{span}\{Z\}$. □

$(\mathfrak{g}, \langle\ ,\ \rangle, J)$ LCK solvable Lie algebra with Lee form ω_0.

Proposition 24. [S. '15]
If $\langle [\gamma_0(\omega_0), J \circ \gamma_0(\omega_0)], J \circ \gamma_0(\omega_0) \rangle = 0$,
then $(\langle\ ,\ \rangle, J)$ is a Vaisman structure.

$\Rightarrow$ LCK structure on $(\langle\ ,\ \rangle, J_n)$ on $\mathfrak{g}_n$ is Vaisman,
because $J \circ \gamma_0(\theta) \in \mathrm{span}\{Z\}$.

Proposition 25. [S. '17]
If $\mathfrak{g}$ is completely solvable and $(\langle\ ,\ \rangle, J)$ is a Vaisman structure,
then $\mathfrak{g} = \mathbb{R} \times \mathfrak{h}$, where $\mathfrak{h}$ is a Heisenberg Lie algebra.

$\Rightarrow$ This is a contradiction,
because $\mathfrak{g}_n$ is completely solvable.

Remark 26. A non-degenerate $-\omega$-closed 2-form is called LCS. The solvmanifold $\Gamma_n \backslash G_n$ has LCS structures:

$$\begin{aligned}
\Omega_1 &= d_{-\theta}\gamma = -\theta \wedge \gamma - \sum_i \alpha_i \wedge \beta_i \text{(exact type)} \\
\Omega_2 &= \alpha_k \wedge \alpha_l + d_{(a_k+a_l)\theta}\gamma \\
&= \alpha_k \wedge \alpha_l + (a_k + a_l)\theta \wedge \gamma - \sum_i \alpha_i \wedge \beta_i \text{(non-exact type)}
\end{aligned}$$

Remark 27. If $a_i = 0$ for each i,
then the solvmanifold $\Gamma_n \backslash G_n$ has a LCK structure (Vaisman).

4. Future Work

- Structure Theorem for Vaisman solvmanifolds ?

- Classification of low dimensional LCK solvmanifolds

Proposition 28. [S. '12]
Classification of 4-dimensional LCK solvmanifolds:
Kodaira-Thurston manifold, Inoue surfaces

Proposition 29. [Bock, '16]
Classification of 6-dimensional solvable Lie algebras

Proposition 30. [S. '15]
If Ω_0 is $-\omega_0$-exact, then (Ω, J) is Vaisman.

Since $\mathfrak{g}$ is solvable, $\mathfrak{n} := [\mathfrak{g}, \mathfrak{g}]$ is nilpotent:

$$\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}] \supset \mathfrak{n}^{(1)} = [\mathfrak{n}, \mathfrak{n}] \supset \mathfrak{n}^{(2)} = [\mathfrak{n}, \mathfrak{n}^{(1)}] \supset \cdots \supset \mathfrak{n}^{(r)} \supset \mathfrak{n}^{(r+1)} = 0,$$

where $\mathfrak{n}^{(i+1)} = [\mathfrak{n}, \mathfrak{n}^{i}]$.

Proposition 31.
If $J\mathfrak{n}^{(r)} \subset [\mathfrak{g}, \mathfrak{g}]^{\perp}$, then (Ω, J) is Vaisman,
where $[\mathfrak{g}, \mathfrak{g}]^{\perp}$ is the orthogonal component.

$\Rightarrow$ We can construct 6-dimensional solvmanifolds without LCK structures.