LCK structures and Vaisman structures on solvmanifolds

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This talk based on

- Example of the six-dimensional LCK solvmanifold, Complex Manifolds, **4** (2017), 37-42.

- Remarks of LCK structures on Inoue surface, preprint.

1. Introduction

Definition 1.
G : simply-connected solvable Lie group.
Γ : lattice, that is, discrete co-compact subgroup of G.
$\Longrightarrow \Gamma\backslash G$: solvmanifold.
(G : nilpotent Lie group $\Longrightarrow \Gamma\backslash G$: nilmanifold)

Theorem 2. [Hasegawa '06]
A solvmanifold admitting a Kähler structure is a finite quotient of a complex torus which has the structure of a complex torus bundle over a complex torus.

Definition 3.

(M, g, J) : Hermitian manifold.

Ω : the fundamental 2-form ($\Omega(X, Y) = g(X, JY)$).

(M, g, J) : locally conformal Kähler (LCK)

$\underset{\text{def}}{\Longleftrightarrow}$ ${}^{\exists}\omega$: closed 1-form such that $d\Omega = \omega \wedge \Omega$.

(We call ω Lee form.)

Remark 4.

If $\omega = df$, then $(M, e^{-f}g, J)$ is Kähler.

Definition 5.

(M, g, J) : LCK manifold.

(M, g, J) : Vaisman manifold

$\underset{\text{def}}{\Longleftrightarrow}$ Lee form ω is parallel with respect to g.

Definition 6.
M : manifold,
α : closed 1-form on M.
$d_\alpha : A^p(M) \to A^{p+1}(M)$

$$d_\alpha \beta := \alpha \wedge \beta + d\beta \qquad (d_\alpha^2 = 0).$$

We call β α-closed (α-exact), if $d_\alpha \beta = 0$ ($\beta = d_\alpha \gamma$).

Similarly, we can define the new differential operator on a Lie algebra.

(M, g, J) : LCK manifold.
$\Longleftrightarrow$ $d\Omega = \omega \wedge \Omega$ (ω : closed 1-form).
$\Longleftrightarrow$ $0 = -\omega \wedge \Omega + d\Omega$.
$\Longleftrightarrow$ $0 = d_{-\omega}\Omega$, that is, Ω: $-\omega$-closed.

Theorem 7. [León-López-Marrero-Padrón, '03]
(M, g) : compact Riemannian manifold
α : parellel 1-from with respect to g
$\Rightarrow$ any α-closed form is α-exact.

The fundamental 2-form Ω of a Vaisman manifold is $-\omega$-exact:

$$\Omega = d_{-\omega}\eta = -\omega \wedge \eta + d\eta.$$

Remark 8. [S. '15]
On solvmanifolds, the inverse is hold.

Examples

- Hopf surface (Vaisman '79) : Vaisman manifold
- Inoue surfaces (Tricerri '82) : non-Vaisman manifold
- nilmanifold $S^1 \times \Gamma \backslash H$ where, H is a Hisenberg Lie group (Fernandez etal '86) : Vaisman manifold
- Oeljeklaus-Toma manifold (Oeljeklaus-Toma, '05) : non-Vaisman manifold

Theorem 9. [S. '07]
G : nilpotent Lie group with a left-invariant complex structure J.
$(\Gamma\backslash G, J)$ has a LCK structure,
$\Rightarrow G = \mathbb{R} \times H$, where H is a Heisenberg Lie group.

Main Theorem.
There exits LCK solvmanifolds without Vaisman structures.

2. Preliminary

$(M = \Gamma\backslash G, g, J)$: LCK solvmanifold with Lee form ω such that

1. J is left-invariant.
2. $^{\exists}$ left-invariant closed 1-form ω_0 s.t $\omega_0 - \omega = df$.

Theorem 10. [Belgun '00]
For $X, Y \in \mathfrak{g}$,

$$\langle X, Y\rangle := \int_M e^f g(X, Y) d\mu$$

$\Longrightarrow (\mathfrak{g}, \langle\ ,\ \rangle, J)$: LCK solvable Lie algebra with Lee form ω_0.

Ω_0 : the fundamental 2-form of $(\langle\ ,\ \rangle, J)$

Remark 11. [S. '12]
$(M = \Gamma\backslash G, g, J)$: Vaisman solvmanifold $(\Omega = d_{-\omega}\eta)$
$\Longrightarrow \Omega_0 = d_{-\omega_0}\eta_0$

Theorem 12. [Hasegawa, '05]
A complex structure on 4-dimensional solvmanifolds is left-invariant.

Theorem 13. [Vaisman, '82]
The first Betti number b_1 of a Vaisman manifold is odd.

Theorem 14. [Kasuya, '12]
Let $G = \mathbb{R}^n \ltimes \mathbb{R}^m$.
If $\dim[\mathfrak{g}, \mathfrak{g}] > \dim \mathfrak{g}/2$, then $(\Gamma \backslash G, J)$ has no Vaisman structures.

Theorem 15. [S. '17]
$(\Gamma \backslash G, J)$: Vaisman completely solvable solvmanifold
$\Rightarrow G = \mathbb{R} \times H$, where H is a Heisenberg Lie group.

3. Examples

Example 1. [Inoue surface S^0]

$$G_1 = \left\{ \begin{pmatrix} \alpha^t & 0 & 0 & x \\ 0 & \beta^t & 0 & z \\ 0 & 0 & \overline{\beta}^t & \overline{z} \\ 0 & 0 & 0 & 1 \end{pmatrix} : t, x \in \mathbb{R}, z \in \mathbb{C} \right\},$$

where $\alpha, \beta, \overline{\beta}$ are eigenvalues of a unimodular matrix $B \in \mathrm{SL}(3, \mathbb{Z})$ such that $\beta \neq \overline{\beta}$.

Remark 16.
Inoue surface S^0 is given by $\mathbb{H} \times \mathbb{C}$:

$$(x + \sqrt{-1}\alpha^t, z) \cdot (x' + \sqrt{-1}\alpha^{t'}, z') = (\alpha^t x' + x + \sqrt{-1}\alpha^{t+t'}, \beta^t z' + z)$$

LCK structure on $\Gamma_1 \backslash G_1$ (Tricerri '82)

$\{\varphi, \mu, \nu_i\}$ is a base of $\mathfrak{g}_1^*$ as follows:

$$\begin{aligned} d\varphi &= 0, \\ d\mu &= -\varphi \wedge \mu, \\ d\nu_1 &= \tfrac{1}{2}\varphi \wedge \nu_1 + c\varphi \wedge \nu_2, \\ d\nu_2 &= \tfrac{1}{2}\varphi \wedge \nu_2 - c\varphi \wedge \nu_1, \end{aligned}$$

· The complex structure J such that
$w = \varphi + \sqrt{-1}\mu$ and $w_2 = \nu_1 + \sqrt{-1}\nu_2$ are $(1,0)$-form.

· The 2-form Ω such that $\Omega = \sqrt{-1}(\omega_1 \wedge \bar{\omega}_1 + \omega_2 \wedge \bar{\omega}_2)$.

· Lee form $\omega_1 + \bar{\omega}_1$ is not parallel.

- $\Gamma_1 \backslash G_1$ has a LCK structure (Tricerri '82)

- $b_1(\Gamma_1 \backslash G_1) = 1$.

- $\Gamma_1 \backslash G_1$ has no Vaisman structures,
  because $G_1 = \mathbb{R} \ltimes_{\varphi} \mathbb{R}^3$ (Kasuya, '12).

Example 2. [O-T manifold of type (2,1), '05]

$$G_2 = \left\{ \begin{pmatrix} \alpha_1^{t_1}\alpha_1'^{t_2} & 0 & 0 & 0 & x_1 \\ 0 & \alpha_2^{t_1}\alpha_2'^{t_2} & 0 & 0 & x_2 \\ 0 & 0 & \beta^{t_1}\beta'^{t_2} & 0 & z \\ 0 & 0 & 0 & \bar{\beta}^{t_1}\bar{\beta}'^{t_2} & \bar{z} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : t_i, x_i \in \mathbb{R}, z \in \mathbb{C} \right\},$$

where $\alpha_1, \alpha_2, \beta, \bar{\beta}$ are roots of the polynomial $f_1(x) = x^4 - 2x^3 - 2x^2 + x + 1$, and $\alpha_i' = \alpha_i^{-1} + \alpha_i^{-2} (i = 1, 2), \beta' = \beta^{-1} + \beta^{-2}$.

$\Rightarrow$ We can constructa lattice Γ_2 on G_2 (S. '17)

Remark 17.
O-T manifold is given by $\mathbb{H}^2 \times \mathbb{C}$:

$$(x_1 + \sqrt{-1}\alpha_1{}^{t_1}\alpha_1'{}^{t_2}, x_2 + \sqrt{-1}\alpha_2{}^{t_1}\alpha_2'{}^{t_2}, z) \cdot (x_1' + \sqrt{-1}\alpha_1{}^{t_1'}\alpha_1'{}^{t_2'}, x_2' + \sqrt{-1}\alpha_2{}^{t_1'}\alpha_2'{}^{t_2'}, z')$$
$$= (\alpha_1{}^{t_1}\alpha_1{}^{t_2}x_1' + x_1 + \sqrt{-1}\alpha_1{}^{t_1+t_1'}\alpha_1'{}^{t_2+t_2'}, \alpha_2{}^{t_1}\alpha_2'{}^{t_2}x_2' + x_2 + \sqrt{-1}\alpha_2{}^{t_1+t_1'}\alpha_2'{}^{t_2+t_2'}, \beta^{t_1}\beta'^{t_2}z' + z)$$

LCK structure on $\Gamma_2 \backslash G_2$ (Oeljeklaus-Toma, '05, Kasuya, '13)
$\{\varphi_i, \mu_i, \nu_i\}$ is a base of $\mathfrak{g}_2^*$ as follows:

$$
\begin{aligned}
&d\varphi_1 = 0, d\varphi_2 = 0,\\
&d\mu_1 = -\varphi_1 \wedge \mu_1,\ \ d\mu_2 = -\varphi_2 \wedge \mu_2,\\
&d\nu_1 = \tfrac{1}{2}(\varphi_1 + \varphi_2) \wedge \nu_1 + (c_1\varphi_1 + c_2\varphi_2) \wedge \nu_2,\\
&d\nu_2 = \tfrac{1}{2}(\varphi_1 + \varphi_2) \wedge \nu_2 - (c_1\varphi_1 + c_2\varphi_2) \wedge \nu_1,
\end{aligned}
$$

· The complex structure J such that
$w_i = \varphi_i + \sqrt{-1}\mu_i$ for $i = 1, 2$ and $w_3 = \nu_1 + \sqrt{-1}\nu_2$ are $(1,0)$-form.
· The 2-form Ω such that
$\Omega = \sqrt{-1}(2(\omega_1 \wedge \bar{\omega}_1 + \omega_2 \wedge \bar{\omega}_2) + \omega_1 \wedge \bar{\omega}_2 + \omega_2 \wedge \bar{\omega}_1 + \omega_3 \wedge \bar{\omega}_3)$.
· Lee form $\omega_1 + \bar{\omega}_1 + \omega_2 + \bar{\omega}_2$ is not parallel.

- The solvmanifold $\Gamma_2 \backslash G_2$ has a LCK structure.

- The solvmanifold $\Gamma_2 \backslash G_2$ has no Vaisman structures, because

  - $b_1(\Gamma_2 \backslash G_2) = 2$ (Vaisman '82).

  - $G_2 = \mathbb{R}^2 \ltimes_\varphi \mathbb{R}^4$, where J is left-inv. (Kasuya, '12)

Example 3. [Inoue surface S^{+}(**Main example**)]

$$G_3 = \left\{ \begin{pmatrix} 1 & -e^t y & e^{-t}x & z \\ 0 & e^t & 0 & x \\ 0 & 0 & e^{-t} & y \\ 0 & 0 & 0 & 1 \end{pmatrix} : t, x, y, z \in \mathbb{R} \right\}.$$

$$\mathfrak{g}_3 = \mathrm{span}\{A, X, Y, Z\}$$
$$[A, X] = X, [A, Y] = -Y, [X, Y] = Z$$

Remark 18.

$G_3 = \mathbb{R} \ltimes_{\varphi} H$, where H is a 3-dimensional Heisenberg Lie group.

LCK structure on $\Gamma_3 \backslash G_3$

(Tricerri '82, Andrés-Cordero-Fernández-Mencía, '89)

$\{\varphi, \mu_i, \nu\}$ is a base of $\mathfrak{g}_3^*$ as follows:

$$
\begin{aligned}
& d\varphi = 0, \\
& d\mu_1 = -\varphi \wedge \mu_1, d\mu_2 = \varphi \wedge \mu_2 \\
& d\nu = -\mu_1 \wedge \mu_2
\end{aligned}
$$

· The complex structure J such that

$w_1 = \varphi + \sqrt{-1}\mu_2$ and $w_2 = \nu + \sqrt{-1}\mu_1$ are $(1,0)$-form.

· The 2-form Ω such that $\Omega = \sqrt{-1}(\omega_1 \wedge \bar{\omega}_1 + \omega_2 \wedge \bar{\omega}_2)$.

· Lee form $\omega_1 + \bar{\omega}_1$ is not parallel.

- The solvmanifold $\Gamma_3\backslash G_3$ has a LCK structure.
- $b_1(\Gamma_3\backslash G_3) = 1$.

Theorem 19. [cf. Belgun. '00]
$\Gamma_3\backslash G_3$ has no Vaisman structures.

Proof 1

Definition 20.
G : simply-connected solvable Lie group.
G : completely solvable
$\underset{\text{def}}{\Longleftrightarrow}$ For $^{\forall}X \in \mathfrak{g}$, $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ has only real eigenvalues,
where $\mathfrak{g}$: Lie algebra of G.

Proof. G_3: completely solvable.

Theorem 15. [S. '17]
$(\Gamma\backslash G, J)$: Vaisman completely solvable solvmanifold
$\Rightarrow G = \mathbb{R} \times H$, where H is a Heisenberg Lie group.

$G_3 = \mathbb{R} \ltimes_{\varphi} H$, where H is a 3-dimensional Heisenberg Lie group.
Note that φ is non-trivial. □

Proof 2 A complex structure J_q on $\Gamma\backslash G_3$ is given by

$$J_qA = Y + qZ, J_qY = -A - qX, J_qZ = X, J_qX = -Z,$$

where $q \in \mathbb{R}$ (Hasegawa, 04).

$(\Gamma\backslash G_3, J_q)$ has a Vaisman metric g_q.
$\Longrightarrow$ the fundamental 2-form $\Omega_q = d_{\theta_q}\eta_q = \theta_q \wedge \eta_q + d\eta_q$,
where θ_q is closed.
$\Longrightarrow \langle Z, Z\rangle = \Omega_q(J_qZ, Z) = (\theta_q \wedge \eta_q + d\eta_q)(X, Z) = 0$,
because $X, Z \in [\mathfrak{g}, \mathfrak{g}]$ and Z is a center of $\mathfrak{g}$.

Remark 21.
$(\Gamma_3\backslash G_3, J_q)$ has a LCK structure $\iff$ $q = 0$, i.e., Inoue surface S^+.

Proof. ${}^\exists\Omega_0$: non-degenerate and non-exact $k\varphi$-closed 2-from
$\iff$ $k = \pm 1$.

Case $k = -1$ $\Omega_0^- = a\nu \wedge \mu_2 + d_{-\varphi}\eta_0$

$$\langle Z, Z\rangle = \Omega_0^-(J_qZ, Z) = (a\nu \wedge \mu_2 + d_{-\varphi}\eta_0)(X, Z) = 0$$

Case $k = 1$ $\Omega_0^+ = b\nu \wedge \mu_1 + d_\varphi\eta_0$

$$\begin{aligned} \Omega_0^+(X, Y) &= \Omega_0^+(J_qX, J_qY) \\ d\eta_0(X, Y) &= (b\nu \wedge \mu_1 + d_\varphi\eta_0)(-Z, -A - qX) \\ -\eta_0([X, Y]) = -\eta_0(Z) &= -\eta_0(Z) + bq \\ 0 &= bq \end{aligned}$$

□