6.7. Electrostatic interpretation of the zeros of the classical polynomials

Stieltjes gave (**4**, **5**; **6**, pp. 75–76; cf., also, Schur **1**) a very interesting derivation of the differential equations of the classical polynomials, which is closely connected with the calculation of the discriminant of these polynomials (cf. §6.71) and can be interpreted as a problem of electrostatic equilibrium.

(1) PROBLEM. *Let p and q be two given positive numbers. If n unit "masses," $n \geqq 2$, at the variable points $x_1, x_2, x_3, \cdots, x_n$ in the interval $[-1, +1]$ and the fixed masses p and q at $+1$ and -1, respectively, are considered, for what position of the points $x_1, x_2, x_3, \cdots, x_n$ does the expression*

$$(6.7.1)\qquad T(x_1, x_2, \cdots, x_n) = T(x) = \prod_{\kappa=1}^{n} (1 - x_\kappa)^p (1 + x_\kappa)^q \prod_{\substack{\nu,\mu=1,2,\cdots,n \\ \nu<\mu}} |x_\nu - x_\mu|$$

become a maximum?

Obviously, $\log (T^{-1})$ can be interpreted as the energy of the system of electrostatic masses just defined. They exert repulsive forces according to the law of logarithmic potential. The maximum position corresponds to the condition of electrostatic equilibrium. A maximum exists because T is a continuous function of $x_1, x_2, \cdots, x_n$ for $-1 \leqq x_\nu \leqq +1$, $\nu = 1, 2, \cdots, n$. It is clear that in the maximum position the x_ν are each different from ± 1 and from one another. In addition, this position is uniquely determined. To show this, let us suppose that (cf. Popoviciu **2**, p. 74)

$$(6.7.2)\qquad \begin{aligned} &+1 > x_1 > x_2 > \cdots > x_n > -1, \\ &+1 > x_1' > x_2' > \cdots > x_n' > -1 \end{aligned}$$

are two positions of this kind; we write

$$(6.7.3)\qquad y_\nu = (x_\nu + x_\nu')/2, \qquad \nu = 1, 2, \cdots, n.$$

Then

$$(6.7.4)\qquad \begin{aligned} |y_\nu - y_\mu| &= \frac{|x_\nu - x_\mu| + |x_\nu' - x_\mu'|}{2} \geqq |x_\nu - x_\mu|^{\frac{1}{2}} |x_\nu' - x_\mu'|^{\frac{1}{2}}, \\ |1 \pm y_\nu| &\geqq |1 \pm x_\nu|^{\frac{1}{2}} |1 \pm x_\nu'|^{\frac{1}{2}}, \end{aligned}$$

so that $T(y) \geqq \{T(x)\}^{\frac{1}{2}}\{T(x')\}^{\frac{1}{2}}$, the equality sign being taken if and only if $x_\nu = x_\nu'$. This establishes the uniqueness.

THEOREM 6.7.1. *Let $p > 0$, $q > 0$, and let $\{x_\nu\}$, $-1 \leqq x_\nu \leqq +1$, be a system of values for which the expression (6.7.1) becomes a maximum. Then the $\{x_\nu\}$ are the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, where $\alpha = 2p - 1$, $\beta = 2q - 1$.*