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Quantum versus classical integrability in Calogero–Moser systems

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Abstract

Calogero–Moser systems are classical and quantum integrable multiparticle dynamics defined for any root system Δ. The *quantum* Calogero systems having $1/q^2$ potential and a confining q^2 potential and the Sutherland systems with $1/\sin^2 q$ potentials have 'integer' energy spectra characterized by the root system Δ. Various quantities of the corresponding *classical* systems, e.g. minimum energy, frequencies of small oscillations, the eigenvalues of the classical Lax pair matrices etc, at the equilibrium point of the potential are investigated analytically as well as numerically for all root systems. To our surprise, most of these classical data are also 'integers', or they appear to be 'quantized'. To be more precise, these quantities are polynomials of the coupling constant(s) with integer coefficients. The close relationship between quantum and classical integrability in Calogero–Moser systems deserves fuller analytical treatment, which would lead to better understanding of these systems and of integrable systems in general.

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1. Introduction

The contrast and resemblance between classical and quantum mechanics and/or field theory has been a good source of stimulus for theoretical physicists since the inception of quantum theory at the beginning of the twentieth century. In spite of the well-publicized differences such as the instability (stability) of the hydrogen atom in classical (quantum) mechanics, the photoelectric effect and tunnelling effects, classical and quantum mechanics share many common theoretical structures (in particular, the canonical formalism) and under certain circumstances provide (almost) the same predictions, as exemplified by the correspondence principle and Ehrenfest's theorem.

In this paper, we discuss issues related to the *quantum* and *classical integrability* in Calogero–Moser systems [1–3], having a rational potential with harmonic confining force (the Calogero systems) and/or a trigonometric potential (the Sutherland systems). This is a part of a program for establishing a *quantum Liouville theorem* on completely integrable systems. As is well known, a classical Hamiltonian system with finitely many degrees of freedom can be transformed into action-angle variables by *quadrature* if a complete set of involutive independent conserved quantities can be obtained. It is a good challenge to formulate a quantum counterpart of the 'transformation into the action-angle variables by quadrature'. Calogero–Moser systems are expected to provide the best materials in this quest. They are known to be integrable at both quantum and classical levels, and the integrability is deeply related to the invariance of the Hamiltonian with respect to a finite (Coxeter, Weyl) reflection group G_Δ based on the root system Δ.

Calogero–Moser systems for any root systems were formulated by Olshanetsky and Perelomov [4], who provided Lax pairs for the systems based on the classical root systems, i.e. the A, B, C, D and BC type root systems. A universal *classical* Lax pair applicable to all the Calogero–Moser systems based on any root systems including the E_8 and the non-crystallographic root systems was derived by Bordner–Corrigan–Sasaki [5] which unified various types of Lax pairs known at that time [6, 7]. A universal *quantum* Lax pair applicable to all the Calogero–Moser systems based on any root systems and for degenerate potentials was derived by Bordner–Manton–Sasaki [8] which provided the basic tools for the present paper. These universal classical and quantum Lax pairs are very closely related to each other and also to the Dunkl operators [9, 8], another well-known tool for quantum systems. For quantum systems, universal formulae for the discrete spectra and the ground state wavefunctions as well as the proof of lower triangularity of the Hamiltonian and the creation–annihilation operator formalism etc have been obtained by Khastgir–Pocklington–Sasaki [10] based on the universal quantum Lax pair. In this respect, the works of Heckman and Opdam [11, 12] offer a different approach based on Dunkl operators.

The quantum Calogero and Sutherland systems have 'integer' energy eigenvalues characterized by the root system Δ. Various quantities of the corresponding *classical* systems, for example minimum energy, frequencies of small oscillations, the eigenvalues of the classical Lax pair matrices etc, at the equilibrium point of the potential are investigated in the present paper. Some of these problems were tackled by Calogero and his collaborators [13–15], about a quarter of a century ago. They showed, mainly for the A-type theories, that the eigenvalues of Lax matrices at equilibrium are 'integers', and that the equilibrium positions are related to zeros of classical polynomials (Hermite, Laguerre etc). The present paper provides systematic answers, both analytical and numerical, to these old problems and presents new results, thanks to the universal Lax pair [5, 8], which are applicable to all root systems. To our surprise, most of the classical data are 'integers', and appear to be 'quantized'.

The present paper is organized as follows. In section 2 we recapitulate the basic ingredients of the Calogero–Moser systems and the solution mechanisms, the reflection operators and the root systems, the quantum and classical Hamiltonian and potentials (section 2.1), the discrete spectra (section 2.2), classical Lax pairs (section 2.3) in order to introduce notation. In section 3 the properties of the classical equilibrium point and its uniqueness, its representation in terms of the Lax pairs, are discussed. The importance of the *pre-potential* W, which is the logarithm of the ground state wavefunction (2.6), is stressed. The formulation of the spin exchange models [16–19], by 'freezing the dynamical freedom at the equilibrium point' [20] is explained. Their definition is also based on a root system Δ and a set of vectors $\mathcal{R}$. The uniqueness of the equilibrium point and the minimality of the classical potential as well as the maximality of the pre-potential are proved universally. The explanation of the highly organized nature of the energy spectra of the spin exchange models [16, 21] in terms of the Lax pairs at equilibrium is one of the motivations of the present paper. Sections 4 and 5 contain the main results—the classical data of the Calogero systems (section 4) and of the Sutherland systems (section 5). In section 4.1 we show that the minimum energies are 'integer valued'. A general 'virial theorem' is derived based on the classical potential and the pre-potential. In section 4.2 the determination of the classical equilibrium points is discussed. For A_r and B_r the equilibrium points are known to be given by the zeros of the Hermite and Laguerre polynomials [13, 14]. For the other root systems, the equilibrium points are determined numerically. In section 4.3 the Lax pair matrices (L and M) at the equilibrium points are shown to satisfy classical versions of the creation–annihilation operator relations. As a consequence, the eigenvalues of the M matrix at equilibrium are shown to be equally spaced. The eigenvalue–multiplicity relation of the M matrix at equilibrium is shown to be the same as the height–multiplicity relation of the chosen set of vectors $\mathcal{R}$. The eigenvalues of L^+L^- at equilibrium are also evaluated. In section 5.1 the minimum energy of the Sutherland system is shown to be 'quantized' since it is identical with the ground state energy of the quantum system. The equilibrium position of the A_r Sutherland system is known to be 'equally spaced' (5.14). We show in section 5.2.2 that the equilibrium positions of BC_r and D_r Sutherland systems are given as zeros of Jacobi polynomials, which is a new analytical result. The equivalence to the classical problem of maximizing the van der Mond determinant is also noted. The Jacobi polynomials are known to reduce to simple trigonometric (Chebyshev etc) polynomials for three specific values of α and β (5.35), in which the zeros are again 'equally spaced'. We show that these three cases are utilized for the spin exchange models based on BC_r root system by Bernard–Pasquier–Serban [18]. The eigenvalues of the L_K (5.32) and M matrices at the equilibrium are all 'integer valued'. In particular, the eigenvalue–multiplicity relation of the L_K matrix at equilibrium is shown to be the same as the height–multiplicity relation of the chosen set of vectors $\mathcal{R}$. In this case, the 'height' is determined by the 'deformed Weyl vector' ϱ (2.10) in contrast to the ordinary Weyl vector δ (2.11) which determines the height–multiplicity relation for the M matrix in Calogero system discussed in section 4. The final section is devoted to comments and discussion. In the appendix, we discuss a remarkable constant matrix K (2.40) which plays an important role in many parts of Calogero–Moser theory. It is a non-negative matrix with integer elements only. For any root system Δ and set of vectors $\mathcal{R}$ its eigenvalues are all 'integers' with multiplicities. The eigenvectors of the K matrix span representation spaces of the Weyl group whose dimensions are the multiplicities of the corresponding eigenvalues.

2. Calogero–Moser systems

In this section, we briefly summarize the *quantum* and *classical* Calogero–Moser systems along with as much of the appropriate notation and background as is necessary for the main body of the paper. A Calogero–Moser model is a Hamiltonian system associated with a root system Δ of rank r. This is a set of vectors in $\mathbf{R}^r$ invariant under reflections in the hyperplane perpendicular to each vector in Δ:

$$\Delta \ni s_\alpha(\beta) = \beta - (\alpha^\vee \cdot \beta)\alpha \qquad \alpha^\vee = \frac{2\alpha}{\alpha^2} \quad \alpha, \beta \in \Delta. \tag{2.1}$$

The set of reflections $\{s_\alpha, \alpha \in \Delta\}$ generates a finite reflection group G_Δ, known as a Coxeter (or Weyl) group. For detailed and unified treatment of Calogero–Moser models based on various root systems and various potentials, we refer to [8, 10].

The dynamical variables of the Calogero–Moser model are the coordinates $\{q_j\}$ and their canonically conjugate momenta $\{p_j\}$, with the canonical commutation (Poisson bracket) relations (throughout this paper we put $\hbar = 1$):

$$\left.\begin{array}{ll}(Q)\colon [q_j, p_k] = \mathrm{i}\delta_{jk} & [q_j, q_k] = [p_j, p_k] = 0 \\ (C)\colon \{q_j, p_k\} = \delta_{jk} & \{q_j, q_k\} = \{p_j, p_k\} = 0\end{array}\right\} \qquad j, k = 1, \ldots, r.$$

These will be denoted by vectors in $\mathbf{R}^r$,

$$q = (q_1, \ldots, q_r) \qquad p = (p_1, \ldots, p_r).$$

In quantum theory, the momentum operator p_j acts as a differential operator:

$$p_j = -\mathrm{i}\frac{\partial}{\partial q_j} \qquad j = 1, \ldots, r.$$

2.1. Hamiltonians and potentials

We will concentrate on those cases in which bound states occur, meaning those with discrete spectra. In other words, we deal with the *rational potential with harmonic confining force* (to be called Calogero systems [1] for short) and *trigonometric potential* (to be referred to as the Sutherland systems [2]):

$$(Q)\colon \mathcal{H}_Q = \frac{1}{2}p^2 + V_Q \qquad V_Q = \begin{cases} \dfrac{\omega^2}{2}q^2 + \dfrac{1}{2}\displaystyle\sum_{\rho\in\Delta_+} \dfrac{g_\rho(g_\rho - 1)\rho^2}{(\rho\cdot q)^2} \\ \dfrac{1}{2}\displaystyle\sum_{\rho\in\Delta_+} \dfrac{g_\rho(g_\rho - 1)\rho^2}{\sin^2(\rho\cdot q)} \end{cases} \tag{2.2}$$

$$(C)\colon \mathcal{H}_C = \frac{1}{2}p^2 + V_C \qquad V_C = \begin{cases} \dfrac{\omega^2}{2}q^2 + \dfrac{1}{2}\displaystyle\sum_{\rho\in\Delta_+} \dfrac{g_\rho^2\rho^2}{(\rho\cdot q)^2} \\ \dfrac{1}{2}\displaystyle\sum_{\rho\in\Delta_+} \dfrac{g_\rho^2\rho^2}{\sin^2(\rho\cdot q)}. \end{cases} \tag{2.3}$$

In these formulae, Δ_+ is the set of positive roots and g_ρ are real *positive* coupling constants which are defined on orbits of the corresponding Coxeter group, i.e. they are identical for roots in the same orbit. For crystallographic root systems, there is one coupling constant $g_\rho = g$ for all roots in simply laced models, and there are two *independent* coupling constants, $g_\rho = g_L$ for long roots and $g_\rho = g_S$ for short roots in non-simply laced models. Throughout this paper, we put the scale factor in the trigonometric functions to unity for simplicity; instead of the general form $a^2/\sin^2 a(\rho\cdot q)$, we use $1/\sin^2(\rho\cdot q)$. We also adopt the convention that long roots have squared length $\rho_L^2 = 2$, unless otherwise stated.

The Sutherland systems are integrable, both at the classical and quantum levels, for the crystallographic root systems, that is those associated with simple Lie algebras: $\{A_r, r \geqslant 1\}$,[3] $\{B_r, r \geqslant 2\}$, $\{C_r, r \geqslant 2\}$, $\{D_r, r \geqslant 4\}$, E_6, E_7, E_8, F_4 and G_2 and the so-called $\{BC_r, r \geqslant 2\}$. On the other hand, the Calogero systems are integrable for any root systems, crystallographic and non-crystallographic. The latter are H_3, H_4, whose Coxeter groups are the symmetry groups of the icosahedron and four-dimensional 600-cell, respectively, and $\{I_2(m), m \geqslant 4\}$ whose Coxeter group is the dihedral group of order $2m$.

These potentials, classical and quantum, both rational and trigonometric, have a hard repulsive singularity $\sim 1/(\rho\cdot q)^2$ near the reflection hyperplane $H_\rho = \{q \in \mathbf{R}^r, \rho\cdot q = 0\}$.

[3] For A_r models, it is customary to introduce one more degree of freedom, q_{r+1} and p_{r+1}, and embed all of the roots in $\mathbf{R}^{r+1}$.

The strength of the singularity is given by the coupling constant $g_\rho(g_\rho - 1)(Q)$, $\left(g_\rho^2(C)\right)$, which is *independent* of the choice of the normalization of the roots. This repulsive potential is classically (quantum mechanically, $g_\rho > 1$) insurmountable. Thus the motion is always confined within one Weyl chamber both at the classical and quantum levels. This feature allows us without loss of generality to constrain the configuration space to the principal Weyl chamber (Π is the set of simple roots):

$$PW = \{q \in \mathbf{R}^r | \rho \cdot q > 0, \rho \in \Pi\}. \tag{2.4}$$

In the case of the trigonometric potential, due to the periodicity of the potential the configuration space is further limited to the principal Weyl alcove

$$PW_T = \{q \in \mathbf{R}^r | \rho \cdot q > 0, \rho \in \Pi, \rho_h \cdot q < \pi\} \tag{2.5}$$

where ρ_h is the highest root.

The potentials of the quantum and classical systems are expressed neatly in terms of a *pre-potential W* which is defined through a ground state wavefunction ϕ_0 of the quantum Hamiltonian H_Q (2.2). Since ϕ_0 can be chosen real and positive, because it has no nodes, it can be expressed by a real smooth function W, to be called a *pre-potential*, in the principal Weyl chamber (PW) (2.4) or the principal Weyl alcove (PW_T) (2.5) by

$$\phi_0 = \mathrm{e}^W \tag{2.6}$$

$$\mathcal{H}_Q\phi_0 = \mathcal{E}_0\phi_0. \tag{2.7}$$

The pre-potential W and the ground state energy $\mathcal{E}_0$ are expressed entirely in terms of the coupling constants and roots [8, 10]:

$$W = \begin{cases} -\dfrac{\omega}{2}q^2 + \displaystyle\sum_{\rho\in\Delta_+} g_\rho \log \rho \cdot q \\ \displaystyle\sum_{\rho\in\Delta_+} g_\rho \log \sin(\rho \cdot q) \end{cases} \tag{2.8}$$

$$\mathcal{E}_0 = \begin{cases} \omega\left(\dfrac{r}{2} + \displaystyle\sum_{\rho\in\Delta_+} g_\rho\right) \\ 2\varrho^2. \end{cases} \tag{2.9}$$

The *deformed Weyl vector* ϱ is defined by

$$\varrho = \frac{1}{2}\sum_{\rho\in\Delta_+} g_\rho \rho \tag{2.10}$$

which reduces to the *Weyl vector* δ when all the coupling constants are unity:

$$\delta = \frac{1}{2}\sum_{\rho\in\Delta_+} \rho. \tag{2.11}$$

By plugging (2.6) into (2.7) and (2.2), we obtain a simple formula expressing the quantum potential in terms of the pre-potential W [8, 10]:

$$(Q)\colon V_Q = \frac{1}{2}\sum_{j=1}^{r}\left[\left(\frac{\partial W}{\partial q_j}\right)^2 + \frac{\partial^2 W}{\partial q_j^2}\right] + \mathcal{E}_0 \tag{2.12}$$

Table 1. The degrees f_j at which independent Coxeter invariant polynomials exist.

Δ	$f_j = 1 + e_j$	Δ	$f_j = 1 + e_j$
A_r	$2, 3, 4, \ldots, r+1$	E_8	2, 8, 12, 14, 18, 20, 24, 30
B_r	$2, 4, 6, \ldots, 2r$	F_4	2, 6, 8, 12
C_r	$2, 4, 6, \ldots, 2r$	G_2	2, 6
D_r	$2, 4, \ldots, 2r-2; r$	$I_2(m)$	$2, m$
E_6	2, 5, 6, 8, 9, 12	H_3	2, 6, 10
E_7	2, 6, 8, 10, 12, 14, 18	H_4	2, 12, 20, 30

and similarly,

$$(C):\ V_C = \frac{1}{2}\sum_{j=1}^{r}\left(\frac{\partial W}{\partial q_j}\right)^2 + \tilde{\mathcal{E}}_0 \qquad \tilde{\mathcal{E}}_0 = \begin{cases} \omega\left(\sum_{\rho\in\Delta_+} g_\rho\right) \\ 2\varrho^2. \end{cases} \tag{2.13}$$

In the context of super-symmetric quantum mechanics [8, 22] the quantities $\partial W/\partial q_j$ are called *super-potentials*. In this paper, we will not discuss super-symmetry at all and we stick to our notion of W being a pre-potential. The difference between the *quantum* and *classical* potential is $\frac{1}{2}\sum_{j=1}^{r}\partial^2 W\big/\partial q_j^2$ plus the zero point energy $\omega r/2$, for the rational cases. These are both *quantum corrections*, being of the order $\hbar$. It should be noted that the quantum Hamiltonian (2.2) with the potential (2.12) can be expressed in a 'factorized form'

$$\mathcal{H}_Q = \sum_{j=1}^{r}\left(p_j - \mathrm{i}\frac{\partial W}{\partial q_j}\right)\left(p_j + \mathrm{i}\frac{\partial W}{\partial q_j}\right) + \mathcal{E}_0 = \sum_{j=1}^{r}\left(p_j + \mathrm{i}\frac{\partial W}{\partial q_j}\right)^{\dagger}\left(p_j + \mathrm{i}\frac{\partial W}{\partial q_j}\right) + \mathcal{E}_0 \tag{2.14}$$

which is obviously positive semi-definite apart from the constant term $\mathcal{E}_0$. Therefore it is elementary to verify, thanks to the simple formulae

$$\left(p_j + \mathrm{i}\frac{\partial W}{\partial q_j}\right)\mathrm{e}^W = 0 \qquad j = 1, \ldots, r \tag{2.15}$$

that $\phi_0 = \mathrm{e}^W$ satisfying (2.7) is the lowest energy state.

2.2. *Discrete spectra*

2.2.1. Rational potentials. The discrete spectrum of the Calogero systems is an *integer* times ω plus the ground state energy $\mathcal{E}_0$. In other words, the energy eigenvalue $\mathcal{E}$ depends on the coupling constant g_ρ only via the ground state energy $\mathcal{E}_0$. The *integer* is specified by an r-tuple of non-negative integers $\vec{n} = (n_1, \ldots, n_r)$ by [10]

$$\mathcal{E}_{\vec{n}} = \omega N_{\vec{n}} + \mathcal{E}_0 \qquad N_{\vec{n}} = \sum_{j=1}^{r} n_j f_j \qquad n_j \in \mathbf{Z}_+ \tag{2.16}$$

and the set of integers $\{f_j\}$ are listed in table 1 for each root system Δ. These are the degrees at which independent Coxeter invariant polynomials occur. They are related to the *exponents* e_j of the root system Δ by

$$f_j = 1 + e_j \qquad j = 1, \ldots, r. \tag{2.17}$$

One immediate consequence of the spectra (2.16) is the *periodicity* of motion. Suppose, at time $t = 0$, the system has the wavefunction Ψ_0 then the system returns to the same physical state after $T = 2\pi/\omega$. Let us introduce a complete set of wavefunctions indexed by the r-tuple of non-negative integers $\vec{n}$,

$$\phi_{\vec{n}}$$

and express the initial state Ψ_0 as the linear combination

$$\Psi_0 = \sum_{\vec{n}} a_{\vec{n}} \phi_{\vec{n}}.$$

Then, at time t the wavefunction is given by

$$\Psi(t) = \sum_{\vec{n}} a_{\vec{n}} \phi_{\vec{n}}\, \mathrm{e}^{-\mathrm{i}\mathcal{E}_{\vec{n}} t} = \mathrm{e}^{-\mathrm{i}\mathcal{E}_0 t} \sum_{\vec{n}} a_{\vec{n}} \phi_{\vec{n}}\, \mathrm{e}^{-\mathrm{i}\omega(\sum_{j=1}^{r} n_j f_j)t}.$$

In other words, we have

$$\Psi(T) = \mathrm{e}^{-\mathrm{i}\mathcal{E}_0 T} \sum_{\vec{n}} a_{\vec{n}} \phi_{\vec{n}}\, \mathrm{e}^{-\mathrm{i}2\pi(\sum_{j=1}^{r} n_j f_j)} = \mathrm{e}^{-\mathrm{i}\mathcal{E}_0 T} \sum_{\vec{n}} a_{\vec{n}} \phi_{\vec{n}} = \mathrm{e}^{-\mathrm{i}\mathcal{E}_0 T} \Psi_0.$$

For some root systems, the quantum state returns to Ψ_0 earlier than $T = 2\pi/\omega$. The corresponding classical theorem, or rather its generalization for the entire hierarchy, is given as proposition III.2 in [23]. It is interesting to note that the $1/(\rho \cdot q)^2$ interactions do not disturb the periodicity of the harmonic potential.

2.2.2. Trigonometric potentials. The discrete spectrum of the Sutherland systems is indexed by a *dominant weight* λ as follows,

$$\mathcal{E}_\lambda = 2(\lambda + \varrho)^2 \tag{2.18}$$

in which ϱ is the deformed Weyl vector (2.10). This spectrum can be interpreted as a 'free' particle energy

$$\mathcal{E} = \tfrac{1}{2} p^2$$

in which the momentum $p \in \mathbf{R}^r$ is simply given by

$$p = 2(\lambda + \varrho).$$

A dominant weight is specified by an r-tuple of non-negative integers $\vec{n} = (n_1, \ldots, n_r)$ by

$$\lambda = \lambda_{\vec{n}} = \sum_{j=1}^{r} n_j \lambda_j \tag{2.19}$$

in which λ_j is the jth *fundamental weight*. We extract explicitly the part of $\mathcal{E}_\lambda$ which depends linearly on $\vec{n}$, and write

$$\mathcal{E}_{\lambda_{\vec{n}}} = 2\left(\lambda_{\vec{n}}^2 + \varrho^2 + 2\sum_{j=1}^{r} n_j \lambda_j \cdot \varrho\right). \tag{2.20}$$

2.3. Classical Lax pairs

The classical equations of motion for the Hamiltonian $\mathcal{H}_C$ are known to be written in a Lax pair form:

$$\dot{q}_j = p_j \qquad \dot{p}_j = -\frac{\partial \mathcal{H}_C}{\partial q_j} \quad \Longleftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} L = [L, M]. \tag{2.21}$$

Table 2. Functions appearing in the Lax pair.

	$V(u)$	$x(u)$	$y(u)$
Rational	$1/u^2$	$1/u$	$-1/u^2$
Trigonometric	$1/\sin^2 u$	$\cot u$	$-1/\sin^2 u$

2.3.1. Universal Lax pair. Here we will summarize the universal formulation applicable to any root system Δ for both the rational ($\omega = 0$ case) and trigonometric potentials [5]. The inclusion of the harmonic confining potential ($\omega \neq 0$) needs a further construction which will be discussed at the end of this section. (For the universal quantum Lax pair, which we will not use in this paper, we refer to [8, 10].) The universal Lax pair operators read

$$L(p,q) = p\cdot\hat{H} + X \qquad X = \mathrm{i}\sum_{\rho\in\Delta_+} g_\rho(\rho\cdot\hat{H})x(\rho\cdot q)\hat{s}_\rho \tag{2.22}$$

$$M(q) = \frac{\mathrm{i}}{2}\sum_{\rho\in\Delta_+} g_\rho\rho^2 y(\rho\cdot q)(\hat{s}_\rho - I) \qquad I : \text{identity operator} \tag{2.23}$$

in which the functions $x(u)$ and $y(u)$ are listed in the table 2. The operators $\hat{H}_j$ and $\hat{s}_\rho$ obey the following commutation relations:

$$[\hat{H}_j, \hat{H}_k] = 0 \tag{2.24}$$

$$[\hat{H}_j, \hat{s}_\alpha] = \alpha_j(\alpha^\vee\cdot\hat{H})\hat{s}_\alpha \tag{2.25}$$

$$\hat{s}_\alpha\hat{s}_\beta\hat{s}_\alpha = \hat{s}_{s_\alpha(\beta)} \qquad \hat{s}_\alpha^2 = 1 \quad \hat{s}_{-\alpha} = \hat{s}_\alpha. \tag{2.26}$$

Let us choose a set of D vectors $\mathcal{R}$,

$$\mathcal{R} = \{\mu^{(1)}, \ldots, \mu^{(D)} | \mu^{(a)} \in \mathbf{R}^r\} \tag{2.27}$$

which form a single orbit of the reflection (Weyl) group G_Δ. That is, any element of $\mathcal{R}$ can be obtained from any other by the action of the reflection (Weyl) group. Let us note that all these vectors have the same length, $(\mu^{(a)})^2 = (\mu^{(b)})^2, a, b = 1, \ldots, D$, which we denote simply as μ^2. They form an over-complete basis[4] of $\mathbf{R}^r$:

$$\sum_{\mu\in\mathcal{R}} \mu_j\mu_k = \delta_{jk}\mu^2 D/r \qquad j,k = 1,\ldots,r. \tag{2.28}$$

In terms of $\mathcal{R}$, L and M are $D\times D$ matrices whose ingredients $\hat{H}_j$ and $\hat{s}_\rho$ are defined by

$$(\hat{H}_j)_{\mu\nu} = \mu_j\delta_{\mu\nu} \qquad (\hat{s}_\rho)_{\mu\nu} = \delta_{\mu,s_\rho(\nu)} = \delta_{\nu,s_\rho(\mu)}. \tag{2.29}$$

The Lax operators are Coxeter covariant,

$$L(s_\alpha(p), s_\alpha(q)) = \hat{s}_\alpha L(p,q)\hat{s}_\alpha \qquad M(s_\alpha(q)) = \hat{s}_\alpha M(q)\hat{s}_\alpha \tag{2.30}$$

and L (M) is (anti-) Hermitian,

$$L^\dagger = L \qquad M^\dagger = -M \tag{2.31}$$

implying real and pure imaginary eigenvalues of L and M, respectively. For various examples of the sets of vectors $\mathcal{R}$ see the appendix.

[4] The A_r case needs a special attention, since it has one additional degree of freedom due to the embedding (see footnote 3).

2.3.2. Minimal-type Lax pair. A set of weights $\Lambda = \{\mu\}$ is called *minimal* if the following condition is satisfied:

$$\frac{2\rho\cdot\mu}{\rho^2} = 0, \pm 1 \qquad \forall\mu\in\Lambda \quad \text{and} \quad \forall\rho\in\Delta. \tag{2.32}$$

A representation of Lie algebra Δ is called *minimal* if its weights are minimal. All the fundamental representations of the A_r algebras are minimal. The vector, spinor and anti-spinor representations of the D_r algebras are minimal representations. There are three minimal representations belonging to the simply laced exceptional algebras—the **27** and $\overline{\mathbf{27}}$ of E_6 and the **56** of E_7; E_8 has no minimal representations.

When $\mathcal{R}$ is a set of minimal weights Λ, the representation of the operator $\hat{s}_\rho$ simplifies,

$$(\hat{s}_\rho)_{\mu\nu} = \begin{cases}\delta_{\mu-\nu,\rho}\\ \delta_{\mu-\nu,-\rho}\\ \delta_{\mu-\nu,0}\end{cases} \qquad \text{if} \qquad \begin{cases}\rho^\vee\cdot\mu = 1\\ \rho^\vee\cdot\mu = -1\\ \rho^\vee\cdot\mu = 0.\end{cases} \tag{2.33}$$

In this case, a Lax pair with a different functional dependence from the universal Lax pair (2.22) (2.23) is possible for the trigonometric potential systems, which we call a *minimal-type* Lax pair

$$L_m(p,q) = p\cdot\hat{H} + X_m \qquad M_m(q) = D + Y_m. \tag{2.34}$$

The matrix X_m has the same form as before but with a different functional dependence on the coordinates q,

$$X_m = \mathrm{i}\sum_{\rho\in\Delta_+} g_\rho(\rho\cdot\hat{H})x_m(\rho\cdot q)\hat{s}_\rho \qquad x_m(u) = 1/\sin u. \tag{2.35}$$

The matrix Y_m is an off-diagonal matrix,

$$Y_m = \frac{\mathrm{i}}{2}\sum_{\rho\in\Delta} g_\rho\rho^2 y_m(\rho\cdot q)\hat{s}_\rho \qquad y_m(u) = x_m'(u) = -\cos u/\sin^2 u. \tag{2.36}$$

The diagonal matrix D is defined by

$$D_{\mu\nu} = \delta_{\mu,\nu}D_\mu \qquad D_\mu = -\frac{\mathrm{i}}{2}\sum_{\Delta\ni\beta=\mu-\nu} g_\beta\beta^2 z(\beta\cdot q) \quad z(u) = -1/\sin^2 u. \tag{2.37}$$

This type of Lax pair has been known from the early days of Calogero–Moser [4].

2.3.3. Lax pair for Calogero systems. Lax-type representations of the Hamiltonian $\mathcal{H}_C$ (2.3) for the Calogero systems ($\omega \neq 0$) are obtained from the rational Lax pair for the $\omega = 0$ case discussed above. The canonical equations of motion are equivalent to the following Lax equations for $L^\pm$,

$$\frac{\mathrm{d}}{\mathrm{d}t}L^\pm = \{L^\pm, \mathcal{H}_C\} = [L^\pm, M] \pm \mathrm{i}\omega L^\pm \tag{2.38}$$

in which M is the same as before (2.23), and $L^\pm$ and Q are defined by

$$L^\pm = L \pm \mathrm{i}\omega Q \qquad Q = q\cdot\hat{H} \tag{2.39}$$

with L, $\hat{H}$ as earlier (2.22), (2.29). It is easy to see that the classical commutator $[Q, L]$ is a constant matrix (see section 4 of [8] and section II of [24]):

$$QL - LQ = \mathrm{i}K \qquad K \equiv \sum_{\rho\in\Delta_+} g_\rho(\rho\cdot\hat{H})(\rho^\vee\cdot\hat{H})\hat{s}_\rho. \tag{2.40}$$

We will discuss this interesting matrix K in some detail in the appendix. If we define Hermitian operators $\mathcal{L}_1$ and $\mathcal{L}_2$ by

$$\mathcal{L}_1 = L^+L^- \qquad \mathcal{L}_2 = L^-L^+ \tag{2.41}$$

they satisfy Lax-like equations, and classical conserved quantities are obtained:

$$\dot{\mathcal{L}}_k = [\mathcal{L}_k, M] \qquad \frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Tr}\,\mathcal{L}_k^n = 0 \qquad k = 1, 2. \tag{2.42}$$

This completes the brief summary of Calogero–Moser systems, the quantum and classical Hamiltonians, the discrete spectra and their classical Lax representations.

3. Classical equilibrium and spin exchange models

Here we discuss the properties of the classical potential V_C, the pre-potential W and Lax matrices $L, M, \mathcal{L}_{1,2}$ near the classical equilibrium point:

$$p = 0 \qquad q = \bar{q}. \tag{3.1}$$

For the classical potential the point $\bar{q}$ is characterized as its *minimum* point,

$$\left.\frac{\partial V_C}{\partial q_j}\right|_{\bar{q}} = 0 \qquad j = 1, \ldots, r \tag{3.2}$$

whereas it is a *maximal* point of the pre-potential W and of the ground state wavefunction $\phi_0 = \mathrm{e}^W$:

$$\left.\frac{\partial W}{\partial q_j}\right|_{\bar{q}} = 0 \qquad j = 1, \ldots, r. \tag{3.3}$$

In this connection, it should be noted that condition (2.15) $(p + \mathrm{i}\partial W/\partial q_j)\,\mathrm{e}^W = 0$ is also satisfied classically at this point. In the Lax representation, it is a point at which two Lax matrices commute,

$$0 = [\bar{L}, \bar{M}] \qquad 0 = [\bar{L}_m, \bar{M}_m] \qquad 0 = [\bar{\mathcal{L}}_{(1,2)}, \bar{M}] \tag{3.4}$$

in which $\bar{L} = L(0, \bar{q})$, $\bar{M} = M(\bar{q})$ etc and $\mathrm{d}\bar{L}/\mathrm{d}t = 0$ etc at the equilibrium point. The value of a quantity A at the equilibrium is expressed by $\bar{A}$.

By differentiating (2.13), we obtain

$$\frac{\partial V_C}{\partial q_j} = \sum_{l=1}^{r} \frac{\partial^2 W}{\partial q_j \partial q_l} \frac{\partial W}{\partial q_l}. \tag{3.5}$$

Since $\partial^2 W/\partial q_j \partial q_k$ is *negative definite* everywhere,

$$\frac{\partial^2 W}{\partial q_j \partial q_k} = \begin{cases} -\omega\delta_{jk} - \sum_{\rho\in\Delta_+} g_\rho \dfrac{\rho_j \rho_k}{(\rho\cdot q)^2} \\ -\sum_{\rho\in\Delta_+} g_\rho \dfrac{\rho_j \rho_k}{\sin^2(\rho\cdot q)} \end{cases} \tag{3.6}$$

we find that the equilibrium point of W is a maximum and the two conditions (3.2) and (3.3) are equivalent:

$$\left.\frac{\partial V_C}{\partial q_j}\right|_{\bar{q}} = 0 \qquad j = 1, \ldots, r \quad \Longleftrightarrow \quad \left.\frac{\partial W}{\partial q_j}\right|_{\bar{q}} = 0 \qquad j = 1, \ldots, r. \tag{3.7}$$

By differentiating (3.5) again, we obtain

$$\frac{\partial^2 V_C}{\partial q_j \partial q_k} = \sum_{l=1}^{r} \frac{\partial^2 W}{\partial q_j \partial q_l} \frac{\partial^2 W}{\partial q_l \partial q_k} + \sum_{l=1}^{r} \frac{\partial^3 W}{\partial q_j \partial q_k \partial q_l} \frac{\partial W}{\partial q_l}.$$

Thus at the equilibrium point of the classical potential V_C, the following relation holds:

$$\left.\frac{\partial^2 V_C}{\partial q_j \partial q_k}\right|_{\bar{q}} = \sum_{l=1}^{r} \left.\frac{\partial^2 W}{\partial q_j \partial q_l}\right|_{\bar{q}} \left.\frac{\partial^2 W}{\partial q_l \partial q_k}\right|_{\bar{q}}. \tag{3.8}$$

If we define the following two symmetric $r \times r$ matrices $\tilde{V}$ and $\widetilde{W}$,

$$\tilde{V} = \text{matrix}\left[\left.\frac{\partial^2 V_C}{\partial q_j \partial q_k}\right|_{\bar{q}}\right] \qquad \widetilde{W} = \text{matrix}\left[\left.\frac{\partial^2 W}{\partial q_j \partial q_k}\right|_{\bar{q}}\right] \tag{3.9}$$

we have

$$\tilde{V} = \widetilde{W}^2 \tag{3.10}$$

and

$$\begin{aligned} &\text{eigenvalues}(\tilde{V}) = \left\{w_1^2, \ldots, w_r^2\right\} \\ &\text{eigenvalues}(\widetilde{W}) = \{-w_1, \ldots, -w_r\} \qquad w_j > 0 \quad j = 1, \ldots, r. \end{aligned} \tag{3.11}$$

That is $\tilde{V}$ is *positive definite* and the point $\bar{q}$ is actually a minimal point of V_C.

As mentioned above, the classical potential V_C tends to plus infinity at all the boundaries (including the infinite point in PW) of PW (PW_T). Since it is positive definite (see (2.3)), V_C has at least one equilibrium (minimal) point in PW (PW_T). Next we show that it is unique in PW (PW_T). Suppose there are two classical equilibrium points $\bar{q}^{(1)}$ and $\bar{q}^{(2)}$,

$$\left.\frac{\partial W}{\partial q_j}\right|_{\bar{q}^{(1)}} = \left.\frac{\partial W}{\partial q_j}\right|_{\bar{q}^{(2)}} = 0 \qquad j = 1, \ldots, r$$

then (see (2.13))

$$V_C(\bar{q}^{(1)}) = V_C(\bar{q}^{(2)}) = \tilde{\mathcal{E}}_0.$$

Let us consider a space P of paths of finite length $q(t)$, $0 \leqslant t \leqslant 1$, connecting these two equilibrium points, $q(0) = \bar{q}^{(1)}$ and $q(1) = \bar{q}^{(2)}$. For each path $q(t)$ there is maximum

$$m[q(t)] = \max_{0<t<1} V_C(q(t)).$$

Since $m[q(t)] > \tilde{\mathcal{E}}_0$, there is a minimum of $m[q(t)]$ in the space of paths P:

$$\text{Min} = \min_{q(t)\in P} m[q(t)].$$

Let us denote the extremal path achieving Min by $q_C(t)$ and $q_C(t_M) = \bar{q}_C$ be its maximal point. By definition of $\bar{q}_C$, it is an extremal point of V_C with one negative eigenvalue of $\partial^2 V_C/\partial q_j \partial q_k$ in the direction of $q_C(t)$. However, from (3.11) we know it is impossible. Thus the assumption of two extremal points $\bar{q}^{(1)}$ and $\bar{q}^{(2)}$ is false.

A few remarks are in order. Most of the discussion in this section, except for those depending on the explicit form of W (3.6), are valid in any classical potentials of multiparticle quantum mechanical systems. The dynamics of the pre-potentials W (2.8), or rather that of $-W$, for the rational and trigonometric and hyperbolic potentials has been discussed by Dyson [25] from a different point of view. It was also introduced by Calogero and collaborators [13, 14] in the context of determining the equilibrium but without the connection with the quantum ground state wavefunction.

At the end of this section, let us briefly summarize the basic ingredients of the spin exchange models associated with the Calogero (Sutherland) system based on the root system Δ and with the set of vectors $\mathcal{R}$ [20]. They are defined at the equilibrium points (3.1) of the corresponding classical systems. Here we call each element μ of $\mathcal{R}$ a *site* to which a dynamical degree of freedom called *spin* is attached. The spin takes a finite set of discrete values. In the simplest, and typical case, they are an up ($\uparrow$) and a down ($\downarrow$). The dynamical state of the spin exchange model is represented by a vector ψ_{Spin} which takes values in the tensor product of D copies of a vector space $\mathcal{V}$ whose basis consists of an up ($\uparrow$) and a down ($\downarrow$):

$$\psi_{\mathrm{Spin}} \in \overset{D}{\otimes} \mathcal{V}_\mu. \tag{3.12}$$

The Hamiltonian of the spin exchange model $\mathcal{H}_{\mathrm{Spin}}$ is

$$\mathcal{H}_{\mathrm{Spin}} = \begin{cases} \dfrac{1}{2}\displaystyle\sum_{\rho\in\Delta_+} g_\rho \rho^2 \frac{1}{(\rho\cdot\bar{q})^2}(1-\hat{\mathcal{P}}_\rho) \\ \dfrac{1}{2}\displaystyle\sum_{\rho\in\Delta_+} g_\rho \rho^2 \frac{1}{\sin^2(\rho\cdot\bar{q})}(1-\hat{\mathcal{P}}_\rho) \end{cases} \tag{3.13}$$

in which $\{\hat{\mathcal{P}}_\rho\}$, $\rho \in \Delta_+$ are the dynamical variables called spin exchange operators. The operator $\hat{\mathcal{P}}_\rho$ exchanges the spins of sites μ and $s_\rho(\mu), \forall\mu \in \mathcal{R}$. In terms of the operator-valued Lax pairs

$$L_{\mathrm{Spin}} = \begin{cases} \mathrm{i}\displaystyle\sum_{\rho\in\Delta_+} g_\rho(\rho\cdot\hat{H})\frac{1}{\rho\cdot\bar{q}}\hat{\mathcal{P}}_\rho\hat{s}_\rho \\ \mathrm{i}\displaystyle\sum_{\rho\in\Delta_+} g_\rho(\rho\cdot\hat{H})\cot(\rho\cdot\bar{q})\hat{\mathcal{P}}_\rho\hat{s}_\rho \end{cases} \tag{3.14}$$

$$M_{\mathrm{Spin}} = \begin{cases} -\dfrac{\mathrm{i}}{2}\displaystyle\sum_{\rho\in\Delta_+} g_\rho\rho^2\frac{1}{(\rho\cdot\bar{q})^2}\hat{\mathcal{P}}_\rho(\hat{s}_\rho - I) \\ -\dfrac{\mathrm{i}}{2}\displaystyle\sum_{\rho\in\Delta_+} g_\rho\rho^2\frac{1}{\sin^2(\rho\cdot\bar{q})}\hat{\mathcal{P}}_\rho(\hat{s}_\rho - I) \end{cases} \tag{3.15}$$

the Heisenberg equations of motion for the trigonometric spin exchange model can be written in a matrix form

$$\mathrm{i}[\mathcal{H}_{\mathrm{Spin}}, L_{\mathrm{Spin}}] = [L_{\mathrm{Spin}}, M_{\mathrm{Spin}}]. \tag{3.16}$$

Since the M_{Spin} matrix satisfies a *sum up to zero* condition,

$$\sum_{\mu\in\mathcal{R}}(M_{\mathrm{Spin}})_{\mu\nu} = \sum_{\nu\in\mathcal{R}}(M_{\mathrm{Spin}})_{\mu\nu} = 0 \tag{3.17}$$

one obtains conserved quantities via the *total sum* of L^k_{Spin}:

$$\left[\mathcal{H}_{\mathrm{Spin}}, \mathrm{Ts}\left(L^k_{\mathrm{Spin}}\right)\right] = 0 \qquad \mathrm{Ts}\left(L^k_{\mathrm{Spin}}\right) \equiv \sum_{\mu,\nu\in\mathcal{R}}\left(L^k_{\mathrm{Spin}}\right)_{\mu\nu} \qquad k = 3, \ldots. \tag{3.18}$$

These are necessary ingredients for complete integrability.

The rational spin exchange model needs some modification similar to those for the Calogero systems. We define

$$L^\pm_{\mathrm{Spin}} = L_{\mathrm{Spin}} \pm \mathrm{i}\omega\bar{Q} \qquad \bar{Q} = \bar{q}\cdot\hat{H}. \tag{3.19}$$

Then the Heisenberg equations of motion in a matrix form read

$$\mathrm{i}\big[\mathcal{H}_{\mathrm{Spin}}, L^{+}_{\mathrm{Spin}}L^{-}_{\mathrm{Spin}}\big] = \big[L^{+}_{\mathrm{Spin}}L^{-}_{\mathrm{Spin}}, M_{\mathrm{Spin}}\big] \tag{3.20}$$

and conserved quantities are given by

$$\mathrm{Ts}\left(\left(L^{+}_{\mathrm{Spin}}L^{-}_{\mathrm{Spin}}\right)^{k}\right) \equiv \sum_{\mu,\nu\in\mathcal{R}} \left(L^{+}_{\mathrm{Spin}}L^{-}_{\mathrm{Spin}}\right)^{k}_{\mu\nu} \qquad k = 3, \ldots .$$

Let us emphasize that the current definition of completely integrable spin exchange models is universal, in the sense that it applies to any root system Δ and to an arbitrary choice of the set of vectors $\mathcal{R}$. It contains all the known examples of spin exchange models as subcases. For the A_r root system and for the set of vector weights, $\mathcal{R} = \mathbf{V}$ (A.12), the trigonometric spin exchange model reduces to the well-known Haldane–Shastry model [16], the rational spin exchange model reduces to the so-called Polychronakos model [17]. For the BC_r root systems with trigonometric interactions, a spin model has been proposed with $\mathcal{R}$ chosen to be the set of short roots Δ_S, or rather, to be more precise, its r-dimensional degeneration. In this case, complete integrability is known only for three different values of the coupling ratio g_S/g_L [18]. For BC_r root systems with rational interactions, a spin model with r sites has been proposed [19].

As is clear from the formulation, the dynamics of spin exchange models depends on the details of the classical potential V_C or W at the *equilibrium point* and on $\mathcal{R}$. It is quite natural to expect that the highly organized spectra of the known spin exchange models [16–21] are correlated with the remarkable properties of the Lax matrices at the equilibrium point—for example, the integer eigenvalues and their high degeneracies. These will be explored in the following two sections.

A determination of the energy spectrum of specific spin exchange models is not pursued in the present paper.

4. Classical data I: rational potential

Next, we will obtain various data of the classical Calogero systems extracted from the potentials, pre-potentials, Lax matrices etc, near the equilibrium point. First, we will derive universal properties which are valid in any root system. Those results depending on specific root systems will be discussed afterwards.

4.1. Minimum energy

Let us consider equations (3.2) and (3.3) for determining the classical equilibrium, which for the rational case reads

$$\left.\frac{\partial V_C}{\partial q_j}\right|_{\bar{q}} = 0 \quad\Rightarrow\quad \sum_{\rho\in\Delta_+} g_\rho^2 \frac{\rho^2\rho_j}{(\rho\cdot\bar{q})^3} = \omega^2\bar{q}_j \qquad j = 1, \ldots, r \tag{4.1}$$

$$\left.\frac{\partial W}{\partial q_j}\right|_{\bar{q}} = 0 \quad\Rightarrow\quad \sum_{\rho\in\Delta_+} g_\rho \frac{\rho_j}{(\rho\cdot\bar{q})} = \omega\bar{q}_j \qquad j = 1, \ldots, r. \tag{4.2}$$

By multiplying $\bar{q}_j$ with both equations, we obtain the virial theorem for the classical potential V_C,

$$\sum_{\rho\in\Delta_+} g_\rho^2 \frac{\rho^2}{(\rho\cdot\bar{q})^2} = \omega^2\bar{q}^2 \tag{4.3}$$

and a relationship

$$\omega\bar{q}^2 = \sum_{\rho\in\Delta_+} g_\rho \frac{\rho\cdot\bar{q}}{(\rho\cdot\bar{q})} = \sum_{\rho\in\Delta_+} g_\rho. \tag{4.4}$$

By combining these, we arrive at the minimal value of the classical potential (2.13):

$$V_C(\bar{q}) = \omega^2\bar{q}^2 = \omega\left(\sum_{\rho\in\Delta_+} g_\rho\right) = \tilde{\mathcal{E}}_0. \tag{4.5}$$

As stated before, this is the ground state energy $\mathcal{E}_0$ minus the zero point energy $\omega r/2$. Although it is a classical quantity, it has the general structure of a *coupling constant*(s) times an *integer*:

$$\tilde{\mathcal{E}}_0 = \begin{cases} \omega g \times \#\Delta/2 & \text{simply laced} \\ \omega(g_L \times \#\Delta_L + g_S \times \#\Delta_S)/2 & \text{non-simply laced.} \end{cases} \tag{4.6}$$

Here, $\#\Delta$ is the total number of roots, $\#\Delta_L$ ($\#\Delta_S$) is the number of long (short) roots, and $\#\Delta = \#\Delta_L + \#\Delta_S$.

4.2. Determination of the equilibrium point and eigenvalues of $\widetilde{W}$

Once the equilibrium position $\bar{q} = (\bar{q}_1, \ldots, \bar{q}_r)$ of the pre-potential W is determined, one can define a Coxeter invariant polynomial of one variable, say x, to encode the data. For A_r it is $\prod_{j=1}^r (x - \bar{q}_j)$ and for B_r (D_r) it is $\prod_{j=1}^r \left(x - \bar{q}_j^2\right)$, since the set of $\{\bar{q}_1, \ldots, \bar{q}_r\}$ and $\left\{\bar{q}_1^2, \ldots, \bar{q}_r^2\right\}$ (or rather $\{\pm\bar{q}_1, \ldots, \pm\bar{q}_r\}$) are invariant under the Weyl group of A_r and B_r, respectively. As shown below, these are classical orthogonal polynomials for the classical root systems (after suitably scaling x): the Hermite polynomials for A_r [4, 13], and the associated Laguerre polynomials for B_r (C_r and D_r) [4]. For an arbitrary root system Δ, such polynomials can be defined through a Lax matrix for a proper choice of $\mathcal{R}$ by $\det(yI - \bar{Q})$, in which $\bar{Q} = \bar{q}\cdot\hat{H}$ is the diagonal matrix Q (2.39) at equilibrium. In fact, for A_r and the choice of vector weights, this is the Hermite polynomial $H_{r+1}(x)$ (with $x = y$), and for B_r (D_r) and the set of short roots (vector weights), it is the Laguerre polynomial $L_r^{(\alpha)}(x)$ (with $x = y^2$). For the exceptional and non-crystallographic root systems, the polynomials have not to the best of our knowledge been identified or named. We strongly believe and have several pieces of numerical evidence that the polynomials for non-classical root systems have 'integer coefficients' like the Hermite and Laguerre polynomials. We have not been able to determine these polynomials exactly, except in the case $I_2(m)$ with special coupling ratios $g_e = g_o = g$, (4.32)–(4.34).

After determining $\bar{q}$ we will evaluate the eigenvalues of $\widetilde{W} \equiv W''|_{\bar{q}}$ and $\tilde{V} \equiv V''|_{\bar{q}}$ in this subsection and various Lax pair matrices $\bar{L}$, $\bar{M}$, $\mathcal{L}_k$ etc in section 4.3.

With the integer spaced quantum spectrum (2.16), apart from the $\mathcal{E}_0$ term, one could simply associate the following *effective quadratic potential*,

$$V_{\text{eff}} = \frac{1}{2}\sum_{j=1}^r (\omega f_j)^2 \tilde{q}_j^2 \tag{4.7}$$

in certain normal coordinates $\tilde{q}_j$. We will show later that the classical potential V_C has the same behaviour as above when expanded at the equilibrium point, in other words,

$$\text{Spec}(\tilde{V}) = \omega^2\left\{f_1^2, \ldots, f_r^2\right\}. \tag{4.8}$$

Considering the relation $\tilde{V} = \widetilde{W}^2$ (3.10), this is equivalent to showing

$$\text{Spec}(\widetilde{W}) = -\omega\{f_1, \ldots, f_r\} = -\omega\{1 + e_1, \ldots, 1 + e_r\}. \tag{4.9}$$

Since the exponents $\{e_j\}$ satisfy the relation

$$\sum_{j=1}^{r} e_j = \#\Delta/2 = hr/2 \tag{4.10}$$

where h is the Coxeter number, we have a simple sum rule (see footnote 4)

$$\mathrm{Tr}(\widetilde{W}) = -\omega(r + \#\Delta/2) = -\omega r(1 + h/2). \tag{4.11}$$

4.2.1. A_r. Calogero and collaborators discussed this problem about quarter of a century ago [4, 13]. In these cases, the root vectors embedded in $\mathbf{R}^{r+1}$ are given by

$$A_r = \{\boldsymbol{e}_j - \boldsymbol{e}_k,\ j,k = 1,\ldots,r+1 | \boldsymbol{e}_j \in \mathbf{R}^{r+1}, \boldsymbol{e}_j \cdot \boldsymbol{e}_k = \delta_{jk}\}. \tag{4.12}$$

The equations (4.2) read

$$\sum_{k\neq j}^{r+1} \frac{1}{\bar{q}_j - \bar{q}_k} = \frac{\omega}{g}\bar{q}_j \qquad j = 1,\ldots,r+1. \tag{4.13}$$

These determine $\left\{\bar{x}_j = \sqrt{\frac{\omega}{g}}\bar{q}_j\right\}$, $j = 1,\ldots,r+1$ to be the zeros of the Hermite polynomial $H_{r+1}(x)$ [26].

The matrix $\widetilde{W}$ is given by

$$\widetilde{W}_{jk} = -\left(\omega + g\sum_{l\neq j}\frac{1}{(\bar{q}_j - \bar{q}_l)^2}\right)\delta_{jk} + g\frac{1}{(\bar{q}_j - \bar{q}_k)^2} \tag{4.14}$$

and is equal to $(-\omega I + \mathrm{i}\bar{M})_{jk}$ for the representation of the Lax matrix $\bar{M}$ (2.23) in terms of the A_r vector weights (A.12). From the general result in section 4.3 (4.43), we obtain

$$A_r: \quad \mathrm{Spec}(\widetilde{W}) = -\omega\{1, 2, \ldots, r+1\}. \tag{4.15}$$

The general result on the relationship between $\bar{q}$ and $\tilde{\mathcal{E}}_0$ (4.5) translates, in this case, to the classical result

$$\sum_{j=1}^{r+1} \bar{x}_j^2 = \frac{r(r+1)}{2} \tag{4.16}$$

in which $\{\bar{x}_j\}$, $j = 1,\ldots,r+1$ are the zeros of $H_{r+1}(x)$. These are some of the earliest results concerning integer eigenvalues associated with the Calogero–Moser classical equilibrium points. The original results [13, 15] depended heavily on specific properties of Hermite polynomials. Here we have emphasized the universal structure rather than particular properties of specific systems.

4.2.2. B_r (D_r). In this case, the root vectors are expressed neatly in terms of an orthonormal basis of $\mathbf{R}^r$ by

$$B_r = \{\pm\boldsymbol{e}_j \pm \boldsymbol{e}_k, \pm\boldsymbol{e}_j,\ j,k = 1,\ldots,r | \boldsymbol{e}_j \in \boldsymbol{R}^r, \boldsymbol{e}_j \cdot \boldsymbol{e}_k = \delta_{jk}\}. \tag{4.17}$$

Let us note that the rational C_r and BC_r systems are identical with the B_r system. Assuming $\bar{q}_j \neq 0$, equations (4.2) read

$$\sum_{k\neq j}^{r} \frac{1}{\bar{q}_j^2 - \bar{q}_k^2} + \frac{g_S/2g_L}{\bar{q}_j^2} = \frac{\omega}{2g_L} \qquad j = 1,\ldots,r \tag{4.18}$$

and determine $\{\bar{q}_j^2\}$, $j = 1, \ldots, r$, as the zeros of the associated Laguerre polynomial $L_r^{(\alpha)}(cx)$, with $\alpha = g_S/g_L - 1$, $c = \omega/g_L$, [4, 26]. The general result on the relationship between $\bar{q}$ and $\tilde{\mathcal{E}}_0$ (4.5) translates in this case to the classical result

$$\sum_{j=1}^{r} \bar{x}_j = r(r+\alpha) \tag{4.19}$$

in which $\{\bar{x}_j\}$, $j = 1, \ldots, r$, are the zeros of $L_r^{(\alpha)}(x)$. The subcase with $g_S = 0$, that is D_r, $\{\bar{q}_j^2\}$, $j = 1, \ldots, r$, are the zeros of the associated Laguerre polynomial [4, 26],

$$rL_r^{(-1)}(cx) = -cxL_{r-1}^{(1)}(cx) \tag{4.20}$$

for which one of the $\bar{q}_j$ is zero. (This also means that the $\{\bar{q}_j\}$ of B_r for $g_S/g_L = 2$ or $\alpha = 1$ are the same as the non-vanishing $\{\bar{q}_j\}$ of D_{r+1}. This can be understood easily from the Dynkin diagram folding $D_{r+1} \to B_r$. Let us note that another Weyl invariant of D_r, $\bar{q}_1 \cdots \bar{q}_r$, is trivial (zero) in the present case.) By summing (4.18) over j, we obtain another sum rule for the inverse square of the zeros,

$$g_S \sum_{j=1}^{r} \frac{1}{\bar{q}_j^2} = r\omega \qquad \text{or} \qquad \sum_{j=1}^{r} \frac{1}{\bar{x}_j} = \frac{r}{\alpha+1}. \tag{4.21}$$

This formula implies that the D_r limit or $g_S \to 0$ ($\alpha \to -1$) limit is singular. In other words, in this limit one of the $\bar{q}_j$ must vanish, otherwise the left-hand side of (4.21) goes to zero, whereas the right-hand side is a constant. This singularity explains the difference of the spectrum of $\widetilde{W}$ for B_r and D_r in units of $-\omega$,

Δ	Spec($\widetilde{W}$)
B_r	$2, 4, 6, \ldots, 2r-2, 2r$
D_r	$2, 4, 6, \ldots, 2r-2, r$

(4.22)

verified by direct computation. This is to be compared with table 1. It is easy to understand via a scaling the coupling constant independence of the spectrum of $\widetilde{W}$ for A_r and D_r root systems. However, for the non-simply laced root systems, B_r, F_4 and $I_2(even)$, with two independent coupling constants, g_L and g_S, the coupling independence of the spectrum of $\widetilde{W}$ is rather non-trivial.

In fact, Szegö derived equations (4.13) and (4.18) while tackling the problem of maximizing $\phi_0 = e^W$ (theorem 6.7.2 [26]) in a slightly different notation and setting—without V_C, quantum mechanics or the Lax pairs. However, he did not mention (4.21).

4.2.3. Exceptional root systems (F_4 and E_r, $r = 6, 7, 8$). In each of these cases, we have calculated the equilibrium position numerically, and evaluated the spectrum of $\widetilde{W}$. For F_4, various ratios of g_S/g_L have been tried and we have verified that the spectrum of $\widetilde{W}$ is independent of the coupling ratio. The results are tabulated in units of $-\omega$:

Δ	r	Spec($\widetilde{W}$)
F_4	4	2, 6, 8, 12
E_6	6	2, 5, 6, 8, 9, 12
E_7	7	2, 6, 8, 10, 12, 14, 18
E_8	8	2, 8, 12, 14, 18, 20, 24, 30

(4.23)

4.2.4. $I_2(m)$. In this case, the root vectors are given by

$$I_2(m) = \{\sqrt{2}(\cos((j-1)\pi/m), \sin((j-1)\pi/m)) \in \mathbf{R}^2,\ j = 1, \ldots, 2m\}. \tag{4.24}$$

The complete set of quantum eigenfunctions are obtained by separation of variables in terms of two-dimensional polar coordinates [4, 10],

$$(q_1, q_2) = r(\sin\varphi, \cos\varphi) \tag{4.25}$$

which of course separate the pre-potential and the ground state wavefunction:

$$W = \begin{cases} g(m\log r + \log\sin m\varphi) - \frac{\omega}{2}r^2 & m \text{ odd} \\ g_e(m/2\log r + \log\sin(m\varphi/2)) + g_o(m/2\log r + \log\cos(m\varphi/2)) - \frac{\omega}{2}r^2 & m \text{ even.} \end{cases} \tag{4.26}$$

The equilibrium points are easily obtained,

$$\bar{r}^2 = \frac{mg}{\omega} \quad \bar{\varphi} = \frac{\pi}{2m} \quad m \text{ odd} \qquad \bar{r}^2 = \frac{m(g_e+g_o)}{2\omega} \quad \tan\frac{m\bar{\varphi}}{2} = \sqrt{\frac{g_e}{g_o}} \qquad m \text{ even} \tag{4.27}$$

together with the values of the second derivatives:

$$\left.\frac{\partial^2 W}{\partial r^2}\right|_{\bar{q}} = -2\omega \qquad \left.\frac{\partial^2 W}{\partial\varphi^2}\right|_{\bar{q}} = -m^2 g \qquad m \text{ odd} \tag{4.28}$$

$$\left.\frac{\partial^2 W}{\partial r^2}\right|_{\bar{q}} = -2\omega \qquad \left.\frac{\partial^2 W}{\partial\varphi^2}\right|_{\bar{q}} = -m^2(g_e+g_o)/2 \qquad m \text{ even.} \tag{4.29}$$

These translate into the spectrum of $\widetilde{W}$ in Cartesian coordinates:

$$I_2(m): \quad \mathrm{Spec}(\widetilde{W}) = -\omega\{2, m\} \qquad m \text{ odd or even.} \tag{4.30}$$

The case G_2 is the $m = 6$ dihedral root system treated above.

It is relatively easy to derive the explicit form the Coxeter invariant polynomials for $I_2(m)$ when

$$g_e = g_o = g. \tag{4.31}$$

In these cases, the pre-potential W and the equations for the equilibrium position look the same for even or odd m. If we choose for $\mathcal{R}$ the set of vertices R_m of the regular m-gon associated with $I_2(m)$, (A.36), we obtain a degree m polynomial in y, $\det(yI - \bar{Q})$. By scaling $y = \sqrt{\frac{2mg}{\omega}}x$ we obtain

$$\det(yI - \bar{Q}) \propto \left\{\begin{matrix} \prod_{k=1}^{m}\left(x - \sin\left[\frac{2k\pi}{m} + \frac{\pi}{2m}\right]\right) & m \text{ even} \\ \prod_{k=1}^{m}\left(x - \sin\left[\frac{2k\pi}{m} + \frac{\pi}{m}\right]\right) & m \text{ odd} \end{matrix}\right\} \propto T_m(x) \tag{4.32}$$

in which T_m is the Chebyshev polynomial of the first kind (5.49),

$$T_m(x) = \cos m\varphi \qquad x = \cos\varphi. \tag{4.33}$$

They satisfy the orthogonality

$$\int_{-1}^{1} \frac{T_m(x)T_l(x)}{\sqrt{1-x^2}}\,\mathrm{d}x \propto \delta_{ml}. \tag{4.34}$$

The general theory of Coxeter invariant polynomials for the arbitrary coupling case $g_e \neq g_o$ will be published elsewhere.

4.2.5. H_r. In this case, we have evaluated the equilibrium points numerically and verified the following:

$$H_3 \quad \mathrm{Spec}(\widetilde{W}) = -\omega\{2, 6, 10\} \tag{4.35}$$

$$H_4 \quad \mathrm{Spec}(\widetilde{W}) = -\omega\{2, 12, 20, 30\}. \tag{4.36}$$

In all the cases, from section 4.2.1 to 4.2.5 the 1 part in the spectrum of $\widetilde{W}$, i.e. $f_j = 1 + e_j$, is always due to the confining harmonic potential $-\omega q^2/2$ in the pre-potential W.

4.3. Eigenvalues of Lax matrices

The Lax pair operators $L, M, L^{\pm}$ etc (2.22), (2.23), (2.39) are $D \times D$ matrices if a set of vectors $\mathcal{R}$ forming a single orbit of the reflection (Coxeter) group with D elements is chosen.

4.3.1. Universal spectrum of M. Let us denote by $\boldsymbol{v}_0$ a special vector in $\mathbf{R}^D$ with each element unity:

$$\boldsymbol{v}_0 = (1, 1, \ldots, 1)^T \in \mathbf{R}^D \qquad D = \#\mathcal{R} \quad \text{or} \quad \boldsymbol{v}_{0\mu} = 1 \qquad \forall \mu \in \mathcal{R}. \tag{4.37}$$

Let us note that the condition for classical equilibrium (4.2) can be written simply in terms of L^- as

$$\sum_{\nu\in\mathcal{R}} (\bar{L}^-)_{\mu\nu} = 0 \qquad \bar{L}^- \equiv L^-(0, \bar{q})$$

since $\sum_{\nu\in\mathcal{R}} (\hat{s}_\rho)_{\mu\nu} = 1$. Similarly, from (2.23) we obtain *sum up to zero* conditions

$$\sum_{\nu\in\mathcal{R}} M_{\mu\nu} = 0 \qquad \sum_{\mu\in\mathcal{R}} M_{\mu\nu} = 0.$$

It should be stressed that the above two conditions are essential for deriving the *quantum conserved quantities* [8, 10]. These can be expressed neatly in matrix–vector notation as

$$\bar{L}^- \boldsymbol{v}_0 = 0 \qquad \boldsymbol{v}_0^T \bar{L}^+ = 0 \qquad \bar{M}\boldsymbol{v}_0 = 0 \qquad \boldsymbol{v}_0^T \bar{M} = 0 \tag{4.38}$$

inspiring the idea that $\boldsymbol{v}_0$ is the classical (Coxeter invariant) *ground* state of a matrix counterpart of the Hamiltonian ($\bar{M}$) and $\bar{L}^-$ is an *annihilation* operator. The analogy goes further when we evaluate the Lax equation for $L^{\pm}$ (2.38) at the classical equilibrium to obtain

$$[\bar{M}, \bar{L}^{\pm}] = \pm \mathrm{i}\omega \bar{L}^{\pm}. \tag{4.39}$$

However, the commutator of L^+ and L^- does not produce $\bar{M}$ but the constant matrix K (see (2.40) and the appendix),

$$[\bar{L}^+, \bar{L}^-] = [\bar{L} + \mathrm{i}\omega\bar{Q}, \bar{L} - \mathrm{i}\omega\bar{Q}] = -2\omega K \tag{4.40}$$

together with the relation

$$[\bar{M}, [\bar{L}^+, \bar{L}^-]] = 0 \tag{4.41}$$

since K and $\bar{M}$ commute (A.5). Relation (4.39) simply means that the eigenvalues of $\bar{M}$ are integer spaced in units of $\mathrm{i}\omega$. We obtain

$$\bar{M}\boldsymbol{v}_0 = 0 \qquad \bar{M}\bar{L}^+\boldsymbol{v}_0 = \mathrm{i}\omega\bar{L}^+\boldsymbol{v}_0, \ldots, \bar{M}(\bar{L}^+)^n\boldsymbol{v}_0 = \mathrm{i}n\omega(\bar{L}^+)^n\boldsymbol{v}_0 \tag{4.42}$$

implying $\bar{L}^+$ is a corresponding *creation* operator. This also means that there is a *universal* formula,

$$\mathrm{Spec}(\bar{M}) = \mathrm{i}\omega\{0, 1, 2, \ldots,\} \tag{4.43}$$

with possible degeneracies. The following sum rules (trace formulae) (4.45) and (4.46) are useful. Let us note the simple formula

$$(\hat{s}_\rho - I)_{\mu\mu} = \begin{cases} -1 & \rho \cdot \mu \neq 0 \\ 0 & \rho \cdot \mu = 0 \end{cases} \tag{4.44}$$

and the fact that, for a fixed ρ, the number of μ in $\mathcal{R}$ such that $\rho \cdot \mu \neq 0$ is almost independent of ρ, depending only on its orbit ($|\rho|$) and $\mathcal{R}$. Let us denote this number by $F_\rho^{\mathcal{R}}$. On taking the trace of $\bar{M}$, we obtain

$$\mathrm{Tr}(\bar{M}) = \frac{\mathrm{i}}{2} \sum_{\rho \in \Delta_+} g_\rho F_\rho^{\mathcal{R}} \frac{\rho^2}{(\rho \cdot \bar{q})^2}. \tag{4.45}$$

This formula simplifies for the simply laced root systems. In those cases, we arrive at a simple relation between $\mathrm{Tr}(\widetilde{W})$, which is independent of $\mathcal{R}$, and $\mathrm{Tr}(\bar{M})$, which depends on the choice of $\mathcal{R}$, by comparing with (3.6),

$$\mathrm{Tr}(\bar{M}) = -F^{\mathcal{R}} \frac{\mathrm{i}}{2}(\omega r + \mathrm{Tr}(\widetilde{W})) = \frac{\mathrm{i}}{4} \omega r h F^{\mathcal{R}} \qquad \Delta : \text{simply laced} \tag{4.46}$$

in which (4.11) is used. This formula provides a non-trivial check for the numerical evaluation of the eigenvalues of $\bar{M}$, since the right-hand side (except for the factor $\mathrm{i}\omega/4$) is an integer determined by Δ and $\mathcal{R}$.

For A_r (B_r) with vector weights (or with short roots) $\bar{M}$ has no degeneracy but high multiplicities occur for the D_r vector or spinor weights. Here is the summary of the spectrum of $\bar{M}$ (in units of $\mathrm{i}\omega$) with [multiplicity] for the classical root systems:

Δ	$\mathcal{R}$	D	$\mathrm{Spec}(\bar{M})$
A_r	**V**	$r+1$	$0, 1, \ldots, r-1, r$
B_r	Δ_S	$2r$	$0, 1, 2, \ldots, 2r-1$
D_r	**V**	$2r$	$0, 1, 2, \ldots, r-1[2], \ldots, 2r-2$
D_4	**S**	8	0, 1, 2, 3[2], 4, 5, 6
D_5	**S**	16	0, 1, 2, 3[2], 4[2], 5[2], 6[2], 7[2], 8, 9, 10
D_6	**S**	32	0, 1, 2, 3[2], 4[2], 5[3], 6[3], 7[3], 8[3], 9[3], 10[3], 11[2], 12[2], 13, 14, 15

(4.47)

For the minimal weights of the exceptional root systems E_r, we obtain

Δ	$\mathcal{R}$	D	$\mathrm{Spec}(\bar{M})$
E_6	**27**	27	0, 1, 2, 3, 4[2], 5[2], 6[2], 7[2], 8[3], 9[2], 10[2], 11[2], 12[2], 13, 14, 15, 16
E_7	**56**	56	0, 1, 2, 3, 4, 5[2], 6[2], 7[2], 8[2], 9[3], 10[3], 11[3], 12[3], 13[3], 14[3], 15[3], 16[3], 17[3], 18[3], 19[2], 20[2], 21[2], 22[2], 23, 24, 25, 26, 27

(4.48)

These are consistent with the trace formula for $\bar{M}$ (4.46), since

$$F^{\mathbf{V}}(A_r) = 2 \qquad F^{\mathbf{V}}(D_r) = 4 \qquad F^{\mathbf{S}}(D_r) = 2^{r-2} \qquad F^{\mathbf{27}}(E_6) = 12 \qquad F^{\mathbf{56}}(E_7) = 24. \tag{4.49}$$

For example, $F^{\mathbf{V}}(A_r) = 2$ can be seen easily. For the typical choice $\rho = \boldsymbol{e}_1 - \boldsymbol{e}_2$, $\mu = \boldsymbol{e}_1$ and $\mu = \boldsymbol{e}_2$ are the only two vectors having non-vanishing scalar product with ρ.

Let us define the *height* of a vector $\mu \in \mathbf{R}^r$ by its scalar product with the *Weyl vector*, i.e. (2.11),

$$\delta \cdot \mu \in \mathbf{R}. \tag{4.50}$$

The set of heights of all vectors in $\mathcal{R}$, denoted by $\delta \cdot \mathcal{R}$,

$$\delta \cdot \mathcal{R} = \{\delta \cdot \mu | \mu \in \mathcal{R}\} \tag{4.51}$$

together with its maximum $h_{\max} \equiv \max(\delta \cdot \mathcal{R})$, are independent of the choice of the positive roots. The above results on the spectrum of the Lax matrix $\bar{M}$ at equilibrium (defined by $\mathcal{R}$) (4.47)–(4.48) are summarized neatly by the set of heights of the vectors in $\mathcal{R}$ shifted by $h_{\max}$:

$$\mathrm{Spec}(\bar{M}) = \mathrm{i}\omega\{\delta \cdot \mu + h_{\max} | \mu \in \mathcal{R}\}. \tag{4.52}$$

The eigenvalues and multiplicities of $\bar{M}$ in the root-type Lax pairs of simply laced crystallographic root systems can also be understood as the *height* and multiplicities of Δ. These are mirror symmetric with respect to the midpoint, including multiplicity. The maximum multiplicity is always the rank r occurring twice in the middle. The highest eigenvalue (in units of $\mathrm{i}\omega$) is $2h - 3$, where h is the Coxeter number. If the lower half is shifted by $-(h-1)$ ($h-1$ is the maximal height) and the higher half by $-(h-2)$, then the eigenvalues range from $-(h-1)$ to $h-1$, precisely the range of the *height* of the roots. The multiplicities of $\bar{M}$ are just the numbers of roots with that height. The following tables summarize the spectrum of $\bar{M}$ in the root-type Lax pairs for the simply laced classical root systems obtained by direct computation:

Δ	h	$\mathcal{R}$	D	$\mathrm{Spec}(\bar{M})$
A_r	$r+1$	Δ	$r(r+1)$	$0, 1[2], \ldots, r-1[r], r[r], r+1[r-1], \ldots, 2r-2[2], 2r-1$
D_4	6	Δ	24	0, 1, 2[3], 3[3], 4[4], 5[4], 6[3], 7[3], 8, 9,
D_5	8	Δ	40	0, 1, 2[2], 3[3], 4[4], 5[4], 6[5], 7[5], 8[4], 9[4], 10[3], 11[2], 12, 13,
D_6	10	Δ	60	0, 1, 2[2], 3[2], 4[4], 5[4], 6[5], 7[5], 8[6], 9[6], 10[5], 11[5], 12[4], 13[4], 14[2], 15[2], 16, 17
D_7	12	Δ	84	0, 1, 2[2], 3[2], 4[3], 5[4], 6[5], 7[5], 8[6], 9[6], 10[7], 11[7], 12[6], 13[6], 14[5], 15[5], 16[4], 17[3], 18[2], 19[2], 20, 21,

(4.53)

The spectrum of $\bar{M}$ with the long roots of the B_r is very interesting. The highest eigenvalue is $2h - 5 = 2h^\vee - 3$, in which $h^\vee$ is the *dual Coxeter number* and the highest multiplicity is $r-1$, the number of the long simple roots. The spectrum is mirror symmetric with respect to the midpoint. If the lower half is shifted by $-(h^\vee - 1)$ and the higher half by $-(h^\vee - 2)$, the eigenvalues range from $-(h^\vee - 1)$ to $h^\vee - 1$, which is the range of the *height* of the roots. Thus we conclude that the spectrum of $\bar{M}$ with the long roots is again the same as the distribution of the B_r long roots with respect to the *height*:

Δ	h	$h^\vee$	$\mathcal{R}$	D	$\mathrm{Spec}(\bar{M})$
B_4	8	7	Δ_L	24	0, 1, 2[2], 3[2], 4[3], 5[3], 6[3], 7[3], 8[2], 9[2], 10, 11
B_5	10	9	Δ_L	40	0, 1, 2[2], 3[2], 4[3], 5[3], 6[4], 7[4], 8[4], 9[4], 10[3], 11[3], 12[2], 13[2], 14, 15
B_6	12	11	Δ_L	60	0, 1, 2[2], 3[2], 4[3], 5[3], 6[4], 7[4], 8[5], 9[5], 10[5], 11[5], 12[4], 13[4], 14[3], 15[3], 16[2], 17[2], 18, 19
B_7	14	13	Δ_L	84	0, 1, 2[2], 3[2], 4[3], 5[3], 6[4], 7[4], 8[5], 9[5], 10[6], 11[6], 12[6], 13[6], 14[5], 15[5], 16[4], 17[4], 18[3], 19[3], 20[2], 21[2], 22, 23

(4.54)

The spectrum of root-type $\bar{M}$ (4.53), (4.54), (4.56), (4.57) can be expressed succinctly in terms of the Weyl vector δ,

$$\mathrm{Spec}(\bar{M}) = \left\{ \begin{matrix} \delta \cdot \mu + h_{\max}, & \text{for } \delta \cdot \mu < 0 \\ \delta \cdot \mu + h_{\max} - 1, & \text{for } \delta \cdot \mu > 0 \end{matrix} \middle| \mu \in \Delta(\Delta_L) \right\} \tag{4.55}$$

in which as before $h_{\max} \equiv \max(\delta \cdot \Delta)$ or $\max(\delta \cdot \Delta_L)$. The spectra for F_4 in terms of Δ_L and Δ_S are the same, reflecting the self-duality of the F_4 root system. The situation is about the same as in the B_r cases. The highest multiplicity is 2, which is the number of long (short) simple roots:

Δ	h	$h^\vee$	$\mathcal{R}$	D	Spec($\bar{M}$)
F_4	12	9	Δ_L	24	0, 1, 2, 3, 4[2], 5[2], 6[2], 7[2], 8[2], 9[2], 10[2], 11[2], 12, 13, 14, 15,
F_4	12	9	Δ_S	24	0, 1, 2, 3, 4[2], 5[2], 6[2], 7[2], 8[2], 9[2], 10[2], 11[2], 12, 13, 14, 15,

(4.56)

Δ	h	$\mathcal{R}$	D	Spec($\bar{M}$)
E_6	12	Δ	72	0, 1, 2, 3[2], 4[3], 5[3], 6[4], 7[5], 8[5], 9[5], 10[6], 11[6], 12[5], 13[5], 14[5], 15[4], 16[3], 17[3], 18[2], 19, 20, 21
E_7	18	Δ	126	0, 1, 2, 3, 4[2], 5[2], 6[3], 7[3], 8[4], 9[4], 10[5], 11[5], 12[6], 13[6], 14[6], 15[6], 16[7], 17[7], 18[6], 19[6], 20[6], 21[6], 22[5], 23[5], 24[4], 25[4], 26[3], 27[3], 28[2], 29[2], 30, 31, 32, 33
E_8	30	Δ	240	0, 1, 2, 3, 4, 5, 6[2], 7[2], 8[2], 9[2], 10[3], 11[3], 12[4], 13[4], 14[4], 15[4], 16[5], 17[5], 18[6], 19[6], 20[6], 21[6], 22[7], 23[7], 24[7], 25[7], 26[7], 27[7], 28[8], 29[8], 30[7], 31[7], 32[7], 33[7], 34[7], 35[7], 36[6], 37[6], 38[6], 39[6], 40[5], 41[5], 42[4], 43[4], 44[4], 45[4], 46[3], 47[3], 48[2], 49[2], 50[2], 51[2], 52, 53, 54, 55, 56, 57

(4.57)

The eigenvalue (the *height* of the root) where the multiplicity changes corresponds to the exponent. When the multiplicity changes by two units, which occurs only in D_{even}, there are two equal exponents. We do not have analytic proofs of these facts.

The situation for the non-crystallographic root systems is different since the 'integral heights' are not defined for the roots. The highest eigenvalue is not $2h - 3$. The places where the multiplicity changes, counted from the centre of the spectrum, are not the exponents but 3, 5 and 7 ($3 + 7 = 10 = 5 + 5 = h$ for H_3) and 7, 13, 17 and 23 ($7 + 23 = 13 + 17 = 30 = h$ for H_4). It is known that H_3 (H_4) is obtained from D_6 (E_8) by 'folding'. The above integers are the exponents of D_6 and E_8. The rest of the exponents of D_6 (E_8) are inherited by H_3 (H_4). The pair D_6 and H_3 (E_8 and H_4) share the same Coxeter number h. For other aspects of the $\bar{M}$ spectra of root-type Lax pairs of H_r, we do not have an explanation to offer. Here is the summary of the spectrum of $\bar{M}$ for the root-type Lax pairs of H_r:

Δ	h	$\mathcal{R}$	D	Spec($\bar{M}$)
H_3	10	Δ	30	0, 1, 2[2], 3[2], 4[3], 5[3], 6[3], 7[3], 8[3], 9[3], 10[2], 11[2], 12, 13
H_4	30	Δ	120	0, 1, 2, 3, 4, 5, 6[2], 7[2], 8[2], 9[2], 10[3], 11[3], 12[3], 13[3], 14[3], 15[3], 16[4], 17[4], 18[4], 19[4], 20[4], 21[4], 22[4], 23[4], 24[4], 25[4], 26[4], 27[4], 28[4], 29[4], 30[3], 31[3], 32[3], 33[3], 34[3], 35[3], 36[2], 37[2], 38[2], 39[2], 40, 41, 42, 43, 44, 45

(4.58)

In all these root-type cases, the highest multiplicity is equal to the rank r. The spectra of $\bar{M}$ for the simply laced root systems are consistent with the trace formula for $\bar{M}$ (4.46), since

$$F^{\Delta}(A_r) = 2(2r-1) \qquad F^{\Delta}(D_r) = 8r - 14 \qquad F^{\Delta}(E_6) = 42$$
$$F^{\Delta}(E_7) = 66 \qquad F^{\Delta}(E_8) = 114 \qquad F^{\Delta}(H_3) = 26 \qquad F^{\Delta}(H_4) = 90. \tag{4.59}$$

For crystallographic root systems, i.e. A_r, D_r and E_r, $F^{\Delta} = 4h - 6$ and F^{Δ} is twice the maximal eigenvalue of $\bar{M}$ for all the cases listed above.

Finally, for $I_2(m)$ in the m-dimensional representation (A.36):

Δ	h	$\mathcal{R}$	D	Spec($\bar{M}$)
$I_2(2n+1)$	$2n+1$	R_{2n+1}	$2n+1$	$0, 1, \ldots, 2n-1, 2n$
$I_2(2n)$	$2n$	R_{2n}	$2n$	$0, 1, \ldots, 2n-2, 2n-1$

(4.60)

4.3.2. Spectrum of $\bar{\mathcal{L}}_1$ and $\bar{\mathcal{L}}_2$. Next let us consider the spectra of $\bar{\mathcal{L}}_1 = \bar{L}^+\bar{L}^-$ and $\bar{\mathcal{L}}_2 = \bar{L}^-\bar{L}^+$, the generators of the conserved quantities (2.42). Note first that a classical analogue of the creation–annihilation operator commutation relation of a harmonic oscillator reads $[L^+, L^-] = -2\omega K$, see (4.40). By using the information on K in the appendix, we can derive the spectrum of $\bar{\mathcal{L}}_1 = \bar{L}^+\bar{L}^-$ and $\bar{\mathcal{L}}_2 = \bar{L}^-\bar{L}^+$ for specific choices of $\mathcal{R}$.

Let us explain the method using the simplest examples. First, A_r with vector weights embedded in $\mathbf{R}^{r+1}$ (A.12). The K matrix has the following form,

$$K = g\left(\boldsymbol{v}_0\boldsymbol{v}_0^T - I\right)$$

with the highest eigenvalue at $\boldsymbol{v}_0$ (A.15), (A.6):

$$K\boldsymbol{v}_0 = gr\boldsymbol{v}_0.$$

Since $\bar{\mathcal{L}}_{1,2}$ are simultaneously diagonalizable with $\bar{M}$ (3.4), it is natural to assume that $\{(\bar{L}^+)^m\boldsymbol{v}_0\}$ form the eigenvectors for $\bar{\mathcal{L}}_{1,2}$. In fact, we have

$$\begin{aligned} &\bar{\mathcal{L}}_1\boldsymbol{v}_0 = 0 \qquad \bar{\mathcal{L}}_1\bar{L}^+\boldsymbol{v}_0 = \bar{L}^+([\bar{L}^-, \bar{L}^+] + \bar{L}^+\bar{L}^-)\boldsymbol{v}_0 = 2\omega\bar{L}^+K\boldsymbol{v}_0 = 2\omega gr\bar{L}^+\boldsymbol{v}_0 \\ &\bar{\mathcal{L}}_2\boldsymbol{v}_0 = ([\bar{L}^-, \bar{L}^+] + \bar{L}^+\bar{L}^-)\boldsymbol{v}_0 = 2\omega K\boldsymbol{v}_0 = 2\omega gr\boldsymbol{v}_0 \end{aligned} \tag{4.61}$$

and we arrive at

$$A_r(\mathbf{V}): \quad \mathrm{Spec}(\bar{\mathcal{L}}_1) = 2g\omega\{0, r, r-1, \ldots, 1\} \tag{4.62}$$

$$A_r(\mathbf{V}): \quad \mathrm{Spec}(\bar{\mathcal{L}}_2) = 2g\omega\{r, r-1, \ldots, 1, 0\}. \tag{4.63}$$

In this case, it is easy to see that $\bar{L}^2 + \omega^2\bar{Q}^2$ also has integer eigenvalues.

Next, let us consider D_r with vector weights (A.22), or B_r with the short roots (A.19). In these cases, we have (A.24) and (A.21),

$$D_r(\mathbf{V}): K = g\left(\boldsymbol{v}_0\boldsymbol{v}_0^T - I - {}_SI\right) \qquad B_r(\Delta_S): K = g_L\left(\boldsymbol{v}_0\boldsymbol{v}_0^T - I - {}_SI\right) + 2g_S\, {}_SI$$

in which ${}_SI$ is the second identity matrix. It is 1 for the elements $(\boldsymbol{e}_j, -\boldsymbol{e}_j)$, $(-\boldsymbol{e}_j, \boldsymbol{e}_j)$, $j = 1, \ldots, r$ and 0 otherwise. The $L^{\pm}$ satisfy simple commutation relation with ${}_SI$, and

$${}_SI(L^{\pm})^m = (-1)^m(L^{\pm})^m\, {}_SI. \tag{4.64}$$

We have (in units of $2g\omega$ for the simply laced root systems)

Δ	$\mathcal{R}$	D	Spec($\bar{\mathcal{L}}_1$)
A_r	$\mathbf{V}$	$r+1$	$0, r, r-1, \ldots, 2, 1$
D_r	$\mathbf{V}$	$2r$	$0[2], 2(r-1)[2], \ldots, 2[2]$
B_r	Δ_S	$2r$	$0, 2(r-1)g_L + 2g_S, 2(r-1)g_L, 2(r-2)g_L + 2g_S,$ $2(r-2)g_L, \ldots, 2g_L + 2g_S, 2g_L, 2g_S$

(4.65)

In these cases, the spectrum of $\bar{L}^2 + \omega^2\bar{Q}^2$ also consists of integer eigenvalues.

It is interesting to note that for other cases the spectrum of $\bar{\mathcal{L}}_1$ does not always consist of integers; for example, the spinor weights of D_r, the set of roots for A_r, D_r etc and for the exceptional E_r and non-crystallographic root systems H_r. Here we list only the integer eigenvalues of $\bar{\mathcal{L}}_1$ in units of $2g\omega$ (the total number of integer eigenvalues including multiplicity is denoted by $\#I$):

Δ	$\mathcal{R}$	D	$\#I$	Spec($\bar{\mathcal{L}}_1$)
A_3	Δ	12	10	0[3], 2, 4[2], 6[3], 8
A_4	Δ	20	14	0[4], 4[3], 5, 6[3], 9, 10[2]
A_5	Δ	30	20	0[5], 4[3], 6[6], 8[2], 10[2], 12[2]
A_6	Δ	42	30	0[6], 3, 4[5], 6[5], 7, 8[3], 10[4], 14[3], 15[2]

(4.66)

Δ	$\mathcal{R}$	D	$\#I$	Spec($\bar{\mathcal{L}}_1$)
D_4	**S**	8	8	0[2], 2[2], 4[2], 6[2]
D_5	**S**	16	8	0[2], 2, 4, 6, 10[2], 12
D_6	**S**	32	16	0[3], 1[2], 3, 4[2], 5, 7, 8, 11, 12, 15, 19, 20
D_7	**S**	64	14	0[5], 2, 3, 5, 6, 9, 15, 21, 30, 31
D_4	Δ	24	24	0[4], 2[3], 4[4], 6[4], 8[3], 10, 12[4], 16
D_5	Δ	40	26	0[5], 2[2], 4[3], 6[6], 8[2], 10[3], 12, 14, 16[2], 18
D_6	Δ	60	38	0[6], 2, 4[4], 6[7], 8[2], 10[7], 12[3], 14[2], 16[2], 20[2], 24[2]
D_7	Δ	84	49	0[7], 2, 4[5], 6[8], 8[3], 10[7], 12[3], 14[4], 16[2], 18[2], 20[3], 24[3], 30

(4.67)

The results for the exceptional root systems are in units of 2ω for F_4:

Δ	$\mathcal{R}$	D	$\#I$	Spec($\bar{\mathcal{L}}_1$)
F_4	Δ_L	24	12	$0[2], 6g_L, 2(g_L+2g_S)[2], 4(2g_L+g_S), 4(g_L+2g_S), 2(5g_L+4g_S), 8(2g_L+g_S), 12(g_L+g_S)[3]$
F_4	Δ_S	24	12	$0[2], 2g_L+g_S[2], 3g_S, 2(2g_L+g_S), 2(g_L+2g_S), 2g_L+5g_S, 6(g_L+g_S)[3], 4(g_L+2g_S)$

(4.68)

For the simply laced E_r in units of $2g\omega$:

Δ	$\mathcal{R}$	D	$\#I$	Spec($\bar{\mathcal{L}}_1$)
E_6	**27**	27	15	0[3], 2[3], 4, 6, 8, 10, 16[3], 18, 20
E_7	**56**	56	23	0[3], 1[2], 3, 4[2], 5, 7, 8[2], 9, 11, 12, 15, 16, 18, 20, 27, 32, 35, 36
E_6	Δ	72	29	0[6], 6[9], 12[8], 18[2], 24, 30[2], 36
E_7	Δ	126	31	0[7], 6[8], 8, 10, 12[3], 14[2], 16[2], 18[2], 24, 36, 48, 50, 56
E_8	Δ	240	55	0[8], 6[11], 12[6], 18[3], 24[5], 30[9], 36[4], 54, 60[2], 84[3], 90[2], 96

(4.69)

The results for the non-crystallographic root systems are

Δ	$\mathcal{R}$	D	$\#I$	Spec($\bar{\mathcal{L}}_1$)
H_3	Δ	30	15	0[3], 2, 3, 5[6], 8[2], 10[2]
H_4	Δ	120	48	0[4], 5[4], 10[7], 15[18], 20[3], 25[2], 30[10]

(4.70)

All the eigenvalues are 'integers' for $I_2(m)$ in the m-dimensional representation (A.36):

Δ	$\mathcal{R}$	D	$\#I$	Spec($\bar{\mathcal{L}}_1$)
$I_2(2n+1)$	R_{2n+1}	$2n+1$	$2n+1$	$0, 4n+2[2n-1], 8n+4$
$I_2(2n)$	R_{2n}	$2n$	$2n$	$0, 8g_en, 8g_on, 4(g_e+g_o)n[2n-4], 8(g_e+g_o)n$

(4.71)

5. Classical data II: trigonometric potential

5.1. Minimum energy

Let us start this subsection by recalling that the classical minimum energy $2\varrho^2$ (2.9) is, in fact, 'quantized'. In this section, we discuss only the crystallographic root system Δ with which a Lie algebra $\mathfrak{g}_\Delta$ is associated. If all the coupling constants are unity $g_\rho = 1, \varrho = \delta$, and the Freudenthal-de Vries ('strange') formula leads to

$$2\varrho^2 = \frac{\dim(\mathfrak{g}_\Delta)\rho_h^2 h^\vee}{12} \tag{5.1}$$

in which $\dim(\mathfrak{g}_\Delta)$ is the dimension of the Lie algebra $\mathfrak{g}_\Delta$, ρ_h is the highest root and $h^\vee$ is the dual Coxeter number. This gives the classical minimum energy formula for the simply laced root systems (in the unit of g^2 and with $\alpha^2 = 2$):

Δ	$\mathcal{E}_0$	Δ	$\mathcal{E}_0$	Δ	$\mathcal{E}_0$	Δ	$\mathcal{E}_0$	Δ	$\mathcal{E}_0$
A_r	$r(r+1)(r+2)/6$	D_r	$r(r-1)(2r-1)/3$	E_6	156	E_7	399	E_8	1240

(5.2)

For the non-simply laced root systems, the classical minimum energy formula is given by

Δ	$\mathcal{E}_0$
B_r	$r\big(2g_L^2 + 4r^2g_L^2 - 6g_Lg_S + 3g_S^2 + r(-6g_L^2 + 6g_Lg_S)\big)/6$
C_r	$r\big(g_S^2 - 6g_Sg_L + 6g_L^2 - 3g_S^2r + 6g_Sg_Lr + 2g_S^2r^2\big)/3$
F_4	$28g_L^2 + 36g_Lg_S + 14g_S^2$
G_2	$4g_L^2 + 4g_Lg_S + 4g_S^2/3$

(5.3)

in which long roots have $\rho_L^2 = 2$, except for the C_r case where a different normalization $\rho_L^2 = 4$ is chosen.

By taking the trace of $\widetilde{W}$ (3.6), we obtain

$$\mathrm{Tr}(\widetilde{W}) = -\sum_{\rho\in\Delta_+} \frac{g_\rho\rho^2}{\sin^2(\rho\cdot\bar{q})}. \tag{5.4}$$

For the simply laced root systems, this is related to $V_C(\bar{q})$ (2.3) and thus to $\mathcal{E}_0$ (2.13):

$$\mathrm{Tr}(\widetilde{W}) = -2V_C(\bar{q})/g = -2\mathcal{E}_0/g = -4\varrho^2/g \qquad \Delta : \text{simply laced.} \tag{5.5}$$

As in the Calogero systems (4.46), $\mathrm{Tr}(\bar{M})$ is related to $\mathrm{Tr}(\widetilde{W})$. By taking the trace of $\bar{M}$, we obtain

$$\mathrm{Tr}(\bar{M}) = \frac{\mathrm{i}}{2}\sum_{\rho\in\Delta_+} g_\rho F_\rho^{\mathcal{R}} \frac{\rho^2}{\sin^2(\rho\cdot\bar{q})^2} \tag{5.6}$$

on recalling the earlier definition of $F_\rho^{\mathcal{R}}$ (4.44). This formula simplifies for the simply laced root systems to

$$\mathrm{Tr}(\bar{M}) = -\frac{\mathrm{i}}{2}F^{\mathcal{R}}\,\mathrm{Tr}(\widetilde{W}) = 2\mathrm{i}F^{\mathcal{R}}\varrho^2/g \qquad \Delta : \text{simply laced.} \tag{5.7}$$

As in the Calogero case, this formula provides a non-trivial check for the numerical evaluation of the eigenvalues of $\bar{M}$. Since the Lax matrix $\bar{L}$ is off-diagonal, $(\bar{L})_{\mu\mu} = 0$ and we have a trivial trace formula:

$$\mathrm{Tr}(\bar{L}) = 0. \tag{5.8}$$

5.2. Determination of the equilibrium point and eigenvalues of $\widetilde{W}$

Since the quantum energy levels of the Sutherland systems are not integers (time a constant) spaced but (2.20)

$$\mathcal{E}_{\lambda_{\vec{n}}} = 2\left(\lambda_{\vec{n}}^2 + \varrho^2 + 2\sum_{j=1}^{r} n_j\lambda_j \cdot \varrho\right)$$

it is not obvious what to expect for the eigenfrequencies of the small oscillations near the equilibrium point. In other words, what are the corresponding spectra of $\tilde{V}$ or equivalently of $\widetilde{W}$? An educated guess would be that, just as in the rational potential situation, we assume the parts of the spectra which are *linear* in the integer labels $\vec{n}$ correspond to the eigenfrequencies of the small oscillations near the equilibrium point. That is, we expect

$$\mathrm{Spec}(\tilde{V}) = \{(4\lambda_1 \cdot \varrho)^2, \ldots, (4\lambda_r \cdot \varrho)^2\} \tag{5.9}$$

and

$$\mathrm{Spec}(\tilde{W}) = -\{4\lambda_1 \cdot \varrho, \ldots, 4\lambda_r \cdot \varrho\} \tag{5.10}$$

which we will show presently. For the simply laced root systems, we have a simple relation

$$\varrho = \frac{g}{2}\sum_{\rho\in\Delta_+}\rho = g\sum_{j=1}^{r}\lambda_j \tag{5.11}$$

which implies a simple sum rule

$$\mathrm{Tr}(\tilde{W}) = -\frac{4\varrho^2}{g} = -\frac{2\mathcal{E}_0}{g} \qquad \Delta : \text{simply laced} \tag{5.12}$$

which has been derived before (5.5) via a different route. The equations determining the equilibrium position (3.3) read

$$\sum_{\rho\in\Delta_+} g_\rho \cot(\rho\cdot\bar{q})\rho_j = 0 \qquad j = 1, \ldots, r$$

and can be expressed in terms of the L matrix at equilibrium:

$$\bar{L}\boldsymbol{v}_0 = 0 = \boldsymbol{v}_0^T\bar{L}. \tag{5.13}$$

The 'ground state' $\boldsymbol{v}_0$ (4.37) is also annihilated by $\bar{M}$:

$$\bar{M}\boldsymbol{v}_0 = 0 = \boldsymbol{v}_0^T\bar{M}.$$

These relations are valid for any $\mathcal{R}$. As in the Calogero case, the equilibrium positions $\bar{q} = (\bar{q}_1, \ldots, \bar{q}_r)$ can be easily identified for the classical root systems. For the exceptional root systems, the equilibrium positions are determined numerically. We shall discuss each case in turn.

5.2.1. A_r. In this case, the equilibrium position and the eigenvalues of the Lax matrices can be obtained explicitly. This is the reason why the Haldane–Shastry model is better understood than other spin exchange models. The equations determining the equilibrium position (3.2) and (3.3) read

$$\sum_{k\neq j}^{r+1}\frac{\cos[\bar{q}_j - \bar{q}_k]}{\sin^3[\bar{q}_j - \bar{q}_k]} = 0 \qquad \sum_{k\neq j}^{r+1}\cot[\bar{q}_j - \bar{q}_k] = 0 \qquad j = 1, \ldots, r+1$$

and the equilibrium position is '*equally spaced*',

$$\bar{q} = \pi(0, 1, \ldots, r-1, r)/(r+1) + \xi \boldsymbol{v}_0 \qquad \xi \in \mathbf{R} : \text{arbitrary} \tag{5.14}$$

due to the well-known trigonometric identities,

$$\sum_{k\neq j}^{r+1} \frac{\cos[\pi(j-k)/(r+1)]}{\sin^3[\pi(j-k)/(r+1)]} = 0 \qquad \sum_{k\neq j}^{r+1} \cot[\pi(j-k)/(r+1)] = 0 \qquad j = 1, \ldots, r+1.$$

This enables us to calculate most quantities exactly. For example, we have

$$\widetilde{W}_{jk} = g\frac{(1-\delta_{jk})}{\sin^2[(j-k)\pi/(r+1)]} - g\delta_{jk}\sum_{l\neq j}\frac{1}{\sin^2[(j-l)\pi/(r+1)]} \qquad j,k = 1, \ldots, r+1 \tag{5.15}$$

and

$$A_r : \quad \mathrm{Spec}(\widetilde{W}) = -2g\{r, (r-1)2, \ldots, (r+1-j)j, \ldots, 2(r-1), r\} \tag{5.16}$$

in which the trivial eigenvalue 0, coming from the translational invariance, is removed. This agrees with the general formula (5.10) of the $\widetilde{W}$ spectrum (i.e. the jth entry is $4\lambda_j \cdot \varrho$, and obviously satisfies the above sum rule (5.2), (5.12)). The spectrum (5.16) is symmetric with respect to the middle point, $\lambda_j \leftrightarrow \lambda_{r+1-j}$, reflecting the symmetry of the A_r Dynkin diagram. It is easy to see that $\widetilde{W}$ is essentially the same as the Lax matrix $\bar{M}$ with the vector weights ($\mathcal{R} = \mathbf{V}$, see (A.12)):

$$\bar{M} = -\mathrm{i}\widetilde{W}. \tag{5.17}$$

(This is consistent with (5.7), since $F^{\mathbf{V}} = 2$, see (4.49)).

A_r Universal Lax pair (**V**). The other Lax matrix with the vector weights reads ($j, k = 1, \ldots, r+1$)

$$(\bar{L})_{jk} = \mathrm{i}g(1-\delta_{jk})\cot[\pi(j-k)/(r+1)] \tag{5.18}$$

$$A_r(\mathbf{V}) : \quad \mathrm{Spec}(\bar{L}) = g\begin{Bmatrix} 0[2], \pm 2, \pm 4, \ldots, \pm(r-1) & r : \text{odd} \\ 0, \pm 1, \pm 3, \ldots, \pm(r-1) & r : \text{even} \end{Bmatrix} \tag{5.19}$$

with the common eigenvectors ($h = r+1$)

$$u^{(a)}, \ (u^{(a)})_j = \mathrm{e}^{2\mathrm{i}aj\pi/h} \qquad a = 0, 1, \ldots, r \quad u^{(0)} \equiv \boldsymbol{v}_0, \tag{5.20}$$

satisfying

$$\bar{L}u^{(a)} = g\lambda_a u^{(a)} \qquad \lambda_a = \begin{cases} 0 & a = 0 \\ r+1-2a & a \neq 0 \end{cases} \tag{5.21}$$

$$\bar{M}u^{(a)} = \mathrm{i}g\mu_a u^{(a)} \qquad \mu_a = 2a(r+1-a). \tag{5.22}$$

These are well-known results [4, 14].

A_r minimal-type Lax pair (**V**). The minimal Lax pair matrices in the vector weights read ($j, k = 1, \ldots, r+1$)

$$(\bar{L}_m)_{jk} = \mathrm{i}g(1-\delta_{jk})/\sin[\pi(j-k)/(r+1)] \tag{5.23}$$

$$(\bar{M}_m)_{jk} = \mathrm{i}g\frac{(1-\delta_{jk})}{\sin^2[(j-k)\pi/(r+1)]} - \mathrm{i}g\delta_{jk}\sum_{l\neq j}\frac{\cos[(j-l)\pi/(r+1)]}{\sin^2[(j-l)\pi/(r+1)]}. \tag{5.24}$$

They have common eigenvectors with integer eigenvalues ($h = r+1$):

$$v^{(a,\pm)},\ (v^{(a,\pm)})_j = \mathrm{e}^{\pm\mathrm{i}aj\pi/h} \qquad a = 1, 3, 5, \ldots, \leqslant h \tag{5.25}$$

$$\bar{L}_m v^{(a,\pm)} = \pm g(h-a)v^{(a,\pm)} \tag{5.26}$$

$$\bar{M}_m v^{(a,\pm)} = \mathrm{i}g(ah - (a^2+1)/2)v^{(a,\pm)}. \tag{5.27}$$

The above spectrum of $\bar{M}_m$ can be derived easily from the following relation between $\bar{L}_m$ and $\bar{M}_m$ (see equation (5.8) of [23]),

$$R^{1/2}\bar{M}_m R^{-1/2} - R^{-1/2}\bar{M}_m R^{1/2} = -\mathrm{i}(R^{1/2}\bar{L}_m R^{-1/2} + R^{-1/2}\bar{L}_m R^{1/2}) \tag{5.28}$$

in which $R \equiv \mathrm{e}^{2\mathrm{i}\hat{Q}}$. We note that $R^{\pm a/2}\boldsymbol{v}_0 = v^{(a,\pm)}$ and use the spectrum of $\bar{L}_m$. The above relationship is a special case of the general formulae which are valid in any root systems having minimal weights,

$$R^{-1/2}\bar{L}_m R^{1/2} = \bar{L} + K \qquad R^{1/2}\bar{L}_m R^{-1/2} = \bar{L} - K \tag{5.29}$$

$$R^{1/2}\bar{M}_m R^{-1/2} = \bar{M} - \mathrm{i}R^{1/2}\bar{L}_m R^{-1/2} \tag{5.30}$$

$$R^{-1/2}\bar{M}_m R^{1/2} = \bar{M} + \mathrm{i}R^{-1/2}\bar{L}_m R^{1/2} \tag{5.31}$$

in which the constant matrix K is defined in (2.40). These mean, for example, that the spectrum of $\bar{L}_m$ and $\bar{L} \pm K$ are the same and those of $\bar{M}$ and $\bar{M}_m \pm \mathrm{i}\bar{L}_m$ are the same. We will see many examples later.

A_r root-type Lax pair. The $\bar{L}$-matrices of the A_r root-type Lax pair do not have integer eigenvalues, although the quantities $\bar{L}^2$ do. Let us tentatively say that $\bar{L}$ has $\sqrt{\text{integer}}$ eigenvalues. (Recall that $\mathrm{Tr}(L^2)$ is proportional to the Hamiltonian.) However, a new type of L-matrix having all integer eigenvalues can be defined by

$$\bar{L}_K = \bar{L} + \tilde{K} \qquad \tilde{K} = \sum_{\rho\in\Delta_+} g_\rho|\rho\cdot\hat{H}|\hat{s}_\rho \qquad [\tilde{K}, \bar{M}] = 0 \tag{5.32}$$

in which $\tilde{K}$ is a non-negative matrix closely related to the K-matrix defined by (2.40). The absolute value in the definition of $\tilde{K}$ means $\tilde{K}_{\mu\nu} = \sum_{\rho\in\Delta_+} g_\rho|\rho\cdot\mu|(\hat{s}_\rho)_{\mu\nu}$, $\mu, \nu \in \mathcal{R}$. This type of Lax matrix has been obtained (see section 8.3, equation (8.22) in [10]) by incorporating a spectral parameter (ξ) into the Lax pair and taking a limit (say, $\xi \to -\mathrm{i}\infty$). For $\mathcal{R} = \Lambda$ {set of minimal weights}, we have $\tilde{K} \equiv K$ and $\bar{L}_K$ has the same spectrum as the minimal-type $\bar{L}_m$ due to relation (5.29). The spectra of $\bar{L}_K$ are very simple, whereas those of $\bar{M}$ of the root type are sums of those of $\widetilde{W}$, i.e. $4\lambda_j\cdot\varrho$, with varied multiplicities:

$$A_r(\Delta): \quad \mathrm{Spec}(\bar{L}_K) = g\{\pm 2[r], \pm 4[r-1], \ldots, \pm 2(r-1)[2], \pm r\} \tag{5.33}$$

Δ	$\mathcal{R}$	D	Spec($\bar{M}$)
A_3	Δ	12	0, 6[4], 8[3], 12[2], 14[2]
A_4	Δ	20	0, 8[4], 12[6], 16[2], 20[6], 24
A_5	Δ	30	0, 10[4], 16[6], 18[3], 20[2], 26[6], 28[4], 32[2], 34[2]
A_6	Δ	42	0, 12[4], 20[6], 24[8], 32[6], 36[8], 40[2], 44[6], 48
A_7	Δ	56	0, 14[4], 24[6], 28[2], 30[6], 32[3], 38[6], 44[8], 46[4], 48[2], 54[6], 56[4], 60[2], 62[2]
A_8	Δ	72	0, 16[4], 28[6], 32[2], 36[6], 40[6], 44[6], 52[8], 56[10], 64[6], 68[8], 72[2], 76[6], 80

(5.34)

The eigenvalues of $\bar{M}$ are of the form i $\sum_{j=1}^{r} a_j(4\varrho \cdot \lambda_j)$, in which $a_j = 0, 1$. The relation between $\mathrm{Tr}(\widetilde{W})$ and $\mathrm{Tr}(\bar{M})$ (5.7) is satisfied, since $F^{\Delta}(A_r) = 2(2r-1)$, see (4.59).

5.2.2. BC_r and D_r. The analytical treatment of the classical equilibrium position of the BC_r and D_r Sutherland system has not been reported, to the best of our knowledge, except for the aforementioned three cases when the coupling ratio g_S/g_L takes special values [18, 20]. We will show in this subsection, that the equilibrium position is given in terms of the zeros of Jacobi polynomials. The Jacobi polynomials $P_r^{(\alpha,\beta)}$ are known to reduce to elementary trigonometric polynomials, Chebyshev polynomials etc for three cases,

$$(i)\,\alpha = \beta = -1/2 \qquad (ii)\,\alpha = \beta = 1/2 \qquad (iii)\,\alpha = 1/2 \qquad \beta = -1/2 \tag{5.35}$$

which will be identified later with the three cases discussed in [18, 20].

Let us start from the pre-potential of the BC_r Sutherland system,

$$W = g_M \sum_{j<k}^{r} \log\left[\sin(q_j - q_k)\sin(q_j + q_k)\right] + \sum_{j=1}^{r}\{g_S \log\sin q_j + g_L \log\sin 2q_j\} \tag{5.36}$$

$$= g_M \sum_{j<k}^{r} \log\left[(-1/2)(\cos 2q_j - \cos 2q_k)\right] + \sum_{j=1}^{r}\{g_S \log\sin q_j + g_L \log\sin 2q_j\} \tag{5.37}$$

which depends on three independent coupling constants, g_L, g_M and g_S, for the long, middle and short roots, respectively. Here we have adopted the following representation of the BC_r roots in terms of an orthonormal basis of $\mathbf{R}^r$:

$$BC_r = \{\pm \boldsymbol{e}_j \pm \boldsymbol{e}_k, \pm \boldsymbol{e}_j, \pm 2\boldsymbol{e}_j,\ j,k = 1,\ldots,r \,|\, \boldsymbol{e}_j \in \mathbf{R}^r, \boldsymbol{e}_j \cdot \boldsymbol{e}_k = \delta_{jk}\}. \tag{5.38}$$

We look for the solutions $\{\bar{q}_j\}$ of (3.3),

$$\frac{\partial W}{\partial q_j} = 0 \qquad j = 1,\ldots,r$$

which read

$$-2g_M \sum_{k\neq j}^{r} \frac{\sin 2\bar{q}_j}{\cos 2\bar{q}_j - \cos 2\bar{q}_k} + g_S \frac{\cos\bar{q}_j}{\sin\bar{q}_j} + 2g_L \frac{\cos 2\bar{q}_j}{\sin 2\bar{q}_j} = 0 \qquad j = 1,\ldots,r. \tag{5.39}$$

For non-vanishing g_S and g_L, $\sin 2\bar{q}_j = 0$ cannot satisfy the above equation. Thus by dividing by $\sin 2\bar{q}_j$ we obtain

$$\sum_{k\neq j}^{r} \frac{1}{\bar{x}_j - \bar{x}_k} + \frac{g_S + g_L}{2g_M}\frac{1}{\bar{x}_j - 1} + \frac{g_L}{2g_M}\frac{1}{\bar{x}_j + 1} = 0 \qquad j = 1,\ldots,r \tag{5.40}$$

in which

$$\bar{x}_j \equiv \cos 2\bar{q}_j. \tag{5.41}$$

These are the equations satisfied by the zeros $\{\bar{x}_j\}$ of Jacobi polynomial $P_r^{(\alpha,\beta)}(x)$ [26] with

$$\alpha = (g_L + g_S)/g_M - 1 \qquad \beta = g_L/g_M - 1. \tag{5.42}$$

The solution (the equilibrium position) is shown to be unique.

Next, let us consider the D_r case; the pre-potential is simply

$$W = g\sum_{j<k}^{r} \log\left[(-1/2)(\cos 2q_j - \cos 2q_k)\right] \tag{5.43}$$

and the equations for its equilibrium point read

$$\sin 2\bar{q}_j \sum_{k\neq j}^{r} \frac{1}{\cos 2\bar{q}_j - \cos 2\bar{q}_k} = 0 \qquad j = 1, \ldots, r. \tag{5.44}$$

These can be decomposed into two parts,

$$\sin 2\bar{q}_1 = 0 = \sin 2\bar{q}_r \qquad \Longleftrightarrow \qquad \cos 2\bar{q}_1 = 1 \quad \cos 2\bar{q}_r = -1 \tag{5.45}$$

and

$$\sum_{k=2,\neq j}^{r-1} \frac{1}{\bar{x}_j - \bar{x}_k} + \frac{1}{\bar{x}_j - 1} + \frac{1}{\bar{x}_j + 1} = 0 \qquad j = 2, \ldots, r-1 \tag{5.46}$$

in which $\{\bar{x}_j\}$, $j = 2, \ldots, r-1$ are defined as before (5.41). The latter part (5.46) are the equations that the zeros $\{\bar{x}_j\}$ of the Jacobi polynomial $P_{r-2}^{(1,1)}(x)$ or the Gegenbauer polynomial $C_{r-2}^{3/2}(x)$ satisfy.

Note, the problem of finding the maximal point of the D_r pre-potential W is the same as the classical problem of maximizing the van der Monde determinant

$$V\,\mathrm{d}M(x_1, \ldots, x_r) = \prod_{j<k}^{r} (x_j - x_k) \tag{5.47}$$

under the boundary conditions

$$1 = x_1 > x_2 > \cdots > x_{r-1} > x_r = -1. \tag{5.48}$$

Now let us show that the three special cases (5.35) are also characterized by *equally spaced* $\bar{q}_j$, that is $\bar{q}_j - \bar{q}_{j+1}$ is independent of j.

(i) For $\alpha = \beta = -1/2 \Leftrightarrow g_L/g_M = 1/2$, $g_S = 0$, which is a special case of C_r obtained from the Dynkin diagram folding $A_{2r-1} \to C_r$. Jacobi polynomial $P_r^{(-1/2,-1/2)}(x)$ is known to be proportional to Chebyshev polynomial of the first kind $T_r(x)$, which can be expressed as

$$T_r(x) = \cos r\varphi \qquad x = \cos\varphi. \tag{5.49}$$

The zeros are *equally spaced* in φ:

$$\bar{\varphi}_j = \frac{(2j-1)\pi}{2r} \quad \Leftrightarrow \quad \cos 2\bar{q}_j = \cos\frac{(2j-1)\pi}{2r} \quad \Leftrightarrow \quad \bar{q}_j = \frac{(2j-1)\pi}{4r} \qquad j = 1, \ldots, r. \tag{5.50}$$

The Dynkin diagram folding $A_{2r-1} \to C_r$ explains this situation neatly. By imposing the following restrictions on the dynamical variables

$$q_j = -q_{2r+1-j} \qquad j = 1, \ldots, r \tag{5.51}$$

in the pre-potential of A_{2r-1} Sutherland system,

$$W_{A_{2r-1}} = g \sum_{j<k}^{2r} \log \sin(q_j - q_k)$$

it reduces to that of C_r with the coupling relation $g_L/g_M = 1/2$:

$$W_{A_{2r-1}\to C_r} = 2g \sum_{j<k}^{r} \log\left[\sin(q_j - q_k)\sin(q_j + q_k)\right] + g \sum_{j=1}^{r} \log \sin 2q_j. \tag{5.52}$$

The equilibrium point of the above A_{2r-1} pre-potential is given in general by

$$\bar{q}_j = \frac{j\pi}{2r} + \xi \qquad j = 1, \ldots, 2r \tag{5.53}$$

in which ξ is an arbitrary real constant, reflecting the 'translational invariance' of the pre-potential. By imposing the restrictions (5.51) on the above equilibrium point, we find $\xi = -(2r+1)\pi/4r$, which turns the general A_{2r-1} equilibrium point (5.53) to that of C_r (5.50). It is interesting to note that the above equilibrium point (5.50) is given by the deformed Weyl vector (2.10) with $g_M = \pi/2r$, $g_L = \pi/4r$ and a choice of positive roots $\{\boldsymbol{e}_j - \boldsymbol{e}_k\}$ for $j < k$.

(ii) For $\alpha = \beta = 1/2 \Leftrightarrow g_L/g_M = 3/2$, $g_S = 0$, which is also a special case of C_r. Jacobi polynomial $P_r^{(1/2,1/2)}(x)$ is known to be proportional to Chebyshev polynomial of the second kind $U_r(x)$ which has a simple expression as a trigonometric polynomial:

$$U_r(x) = \frac{\sin(r+1)\varphi}{\sin\varphi} \qquad x = \cos\varphi. \tag{5.54}$$

The zeros are *equally spaced* in φ:

$$\bar{\varphi}_j = \frac{j\pi}{r+1} \quad \Leftrightarrow \quad \cos 2\bar{q}_j = \cos\frac{j\pi}{r+1} \quad \Leftrightarrow \quad \bar{q}_j = \frac{j\pi}{2(r+1)}$$
$$j = 1, \ldots, r. \tag{5.55}$$

(iii) For $\alpha = 1/2$, $\beta = -1/2 \Leftrightarrow g_L/g_M = 1/2$, $g_S/g_M = 1$. In this case, we have

$$P_r^{(1/2,-1/2)}(x) \propto \frac{\sin[(2r+1)\varphi/2]}{\sin[\varphi/2]} \qquad x = \cos\varphi.$$

The zeros are *equally spaced* in φ:

$$\bar{\varphi}_j = \frac{2j\pi}{2r+1} \quad \Leftrightarrow \quad \cos 2\bar{q}_j = \cos\frac{2j\pi}{2r+1} \quad \Leftrightarrow \quad \bar{q}_j = \frac{j\pi}{2r+1}$$
$$j = 1, \ldots, r. \tag{5.56}$$

This equilibrium point is also obtained as a deformed Weyl vector (2.10) for $g_M = \pi/(2r+1)$. For $\alpha = -1/2$, $\beta = 1/2$, Jacobi polynomial $P_r^{(\alpha,\beta)}$ is proportional to another trigonometric polynomial. But this case is not compatible with *positive* coupling constants g_ρ and will not be discussed here.

In this connection, let us remark on the dynamical implications of another well-known Dynkin diagram folding, $D_{r+1} \to B_r$. By restricting one of the dynamical variables of D_{r+1} Sutherland system to its equilibrium position

$$q_{r+1} = 0 \tag{5.57}$$

its pre-potential,

$$W_{D_{r+1}} = g \sum_{j<k}^{r+1} \log\left[\sin(q_j - q_k)\sin(q_j + q_k)\right]$$

reduces to that of B_r with the coupling relation $g_S/g_M = 2$:

$$W_{D_{r+1}\to B_r} = g\sum_{j<k}^{r}\log\left[\sin(q_j-q_k)\sin(q_j+q_k)\right] + 2g\sum_{j=1}^{r}\log\sin q_j. \tag{5.58}$$

This means that the equilibrium position of the reduced B_r system (5.57), the zeros of $P_r^{(1,-1)}(x)$, is given by that of the original D_{r+1} system, i.e. the zeros of $P_{r-1}^{(1,1)}(x)$ plus $x=-1$. In other words, the following identities hold,

$$(r+1)(x+1)P_{r-1}^{(1,1)}(x) = 2rP_r^{(1,-1)}(x) \qquad r = 1, 2, \ldots \tag{5.59}$$

which are trigonometric counterparts of (4.20).

In the following, we summarize the spectra of $\widetilde{W}$, $\bar{L}$, $\bar{M}$, $\bar{L}_m$, $\bar{M}_m$ of the D_r Sutherland system which are evaluated numerically, with the vector weights $\mathbf{V}$ and the roots Δ. The spectra are all 'integer valued', except for $\bar{L}$. The combinations (5.32) $\bar{L}_K = \pm\tilde{K} + \bar{L}$ are integer valued having the same spectra as $\bar{L}_m$ for $\mathcal{R} = \mathbf{V}$, see (5.29). It is interesting to note that $\bar{L}^2$ is integer valued for $\mathcal{R} = \mathbf{V}$, but the eigenvalues are not all integers for $\mathcal{R} = \Delta$. The spectrum of $\widetilde{W}$ is

$$D_r: \quad \mathrm{Spec}(\widetilde{W}) = -g\{4(r-1), 4(2r-3), \ldots, 2j(2r-1-j), \ldots, 2(r-2)(r+1), r(r-1)[2]\} \tag{5.60}$$

which agrees with the general formula (5.10) of the $\widetilde{W}$ spectrum, i.e. the jth entry is $4\lambda_j\cdot\varrho$, and obviously satisfies the sum rule (5.2), (5.12). The two-fold degeneracy reflects the Dynkin diagram symmetry corresponding to the spinor and anti-spinor fundamental weights, $\lambda_S \leftrightarrow \lambda_{\bar{S}}$.

D_r universal Lax pair ($\mathbf{V}$). The spectrum of $\bar{M}$ is

$$D_r(\mathbf{V}): \quad \mathrm{Spec}(\bar{M}) = \mathrm{i}g\{0, 4(r-1)[2], 4(2r-3)[2], \ldots, 2j(2r-1-j)[2], \ldots, 2(r-2)(r+1)[2], r(r-1)[2], 2r(r-1)\} \tag{5.61}$$

which is essentially the duplication of that of $\widetilde{W}$, except for the lowest, i.e. 0, and the highest eigenvalues, $2r(r-1)$. The latter is exactly twice the eigenvalue of those belonging to λ_S ($\lambda_{\bar{S}}$). Let us note that the identity between the traces of $\widetilde{W}$ and $\bar{M}$ (5.7) is also satisfied, since $F^{\mathbf{V}}(D_r) = 4$, see (4.49). As in the A_r vector weight case (5.17), these can be understood by the close relationship between $\widetilde{W}$ and $\bar{M}$:

$$\bar{M} = \mathrm{i}\begin{pmatrix} A & B \\ B & A \end{pmatrix} \qquad \widetilde{W} = -A + B. \tag{5.62}$$

The $r\times r$ matrices A and B are

$$A_{jj} = g\sum_{k\neq j}^{r}\left(\frac{1}{\sin^2(\bar{q}_j-\bar{q}_k)} + \frac{1}{\sin^2(\bar{q}_j+\bar{q}_k)}\right) = -\widetilde{W}_{jj} \qquad B_{jj} = 0 \tag{5.63}$$

$$A_{jk} = -g\frac{1}{\sin^2(\bar{q}_j-\bar{q}_k)} \qquad B_{jk} = -g\frac{1}{\sin^2(\bar{q}_j+\bar{q}_k)}. \tag{5.64}$$

Thus to each eigenvector v of $\widetilde{W}$ with eigenvalue $-\lambda$, $\widetilde{W}v = (-A+B)v = -\lambda v$, corresponds to an eigenvector V with eigenvalue $\mathrm{i}\lambda$:

$$V = \begin{pmatrix} v \\ -v \end{pmatrix} \qquad \bar{M}V = \mathrm{i}\begin{pmatrix} A & B \\ B & A \end{pmatrix}\begin{pmatrix} v \\ -v \end{pmatrix} = \mathrm{i}\lambda V. \tag{5.65}$$

The $\bar{L}$ matrix with the vector weights has the following decomposition,

$$\bar{L} = \begin{pmatrix} C & D \\ -D & -C \end{pmatrix} \qquad {}_SI\bar{L} = -\bar{L}{}_SI \qquad {}_SI = \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix} \tag{5.66}$$

in which I_r is the $r \times r$ identity matrix and

$$C_{jk} = \mathrm{i}g(1-\delta_{jk})\cot(\bar q_j - \bar q_k) \qquad D_{jk} = \mathrm{i}g(1-\delta_{jk})\cot(\bar q_j + \bar q_k) \qquad j,k = 1,\ldots,r. \tag{5.67}$$

Since $\bar L$ commutes with $\bar M$, $\bar L V$ provides another independent eigenvector with the same eigenvalue $\bar M(\bar L V) = \mathrm{i}\lambda(\bar L V)$,

$$\bar L V = \begin{pmatrix} C & D \\ -D & -C \end{pmatrix}\begin{pmatrix} v \\ -v \end{pmatrix} = (C-D)v\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

except for the duplicated eigenvalue $r(r-1)$ and the lowest and the highest. The zero mode (the eigenvector corresponding to the lowest eigenvalue) is v_0 which is annihilated by $\bar L$. The eigenvectors of $\widetilde W$ belonging to the duplicated eigenvalue $-r(r-1)$ are

$$v_s = (1,0,\ldots,0)^T \qquad v_{\bar s} = (0,0,\ldots,1)^T$$

corresponding to the conditions $\cos 2q_1 = 1$ and $\cos 2q_r = -1$. The corresponding eigenvectors of $\bar M$ are both annihilated by $\bar L$,

$$\bar L\begin{pmatrix} v_s \\ -v_s \end{pmatrix} = 0 \qquad \bar L\begin{pmatrix} v_{\bar s} \\ -v_{\bar s} \end{pmatrix} = 0.$$

The spectrum of $\bar L$ is $\sqrt{\text{integer}}$:

$$D_r(\mathbf{V}): \quad \mathrm{Spec}(\bar L) = g\{0[4], \pm 2\sqrt{2}, \pm 2\sqrt{6}, \pm 4\sqrt{3}, \ldots, \pm 2\sqrt{(j-1)(j-2)}, \ldots, \pm 2\sqrt{(r-1)(r-2)}\}. \tag{5.68}$$

D_r minimal-type Lax pair (**V**). The minimal Lax pairs have integer spectrum,

$$D_r: \quad \mathrm{Spec}(\bar L_m) = g\{0[2], \pm 2, \pm 4, \ldots, \pm 2(r-2), \pm 2(r-1)\} \tag{5.69}$$

and

Δ	h	$\mathcal{R}$	D	$\mathrm{Spec}(\bar M_m)$
D_4	6	**V**	8	6[2], 12[2], 16[2], 22[2]
D_5	8	**V**	10	8[2], 20[2], 22[2], 32[2], 38[2]
D_6	10	**V**	12	10[2], 28[2], 30[2], 42[2], 52[2], 58[2]
D_7	12	**V**	14	12[2], 34[2], 42[2], 52[2], 66[2], 76[2], 82[2]
D_8	14	**V**	16	14[2], 40[2], 56[2], 62[2], 80[2], 94[2], 104[2], 110[2]

(5.70)

The lowest eigenvalue is h, the Coxeter number, and the highest eigenvalue is $rh-2$. The two-fold degenerate eigenvalues of $\widetilde W$, $r(r-1)$ are always contained.

D_r root-type Lax pair. The $\bar L_K$ matrices have simple spectra. They are mirror symmetric with respect to zero. The highest multiplicity is the rank r and the highest (lowest) eigenvalue is $2(h-1)$, with interval 2. Thus the multiplicity distribution of the eigenvalues of $\bar L_K$ of the root-type Lax matrix is the number of roots having the specified (2 times the) height. We have encountered the same distributions (shifted parallelly) in the eigenvalues of $\bar M$ in Calogero systems.

Δ	h	$\mathcal{R}$	D	$\mathrm{Spec}(\bar L_K)$
D_4	6	Δ	24	±2[4], ±4[3], ±6[3], ±8, ±10
D_5	8	Δ	40	±2[5], ±4[4], ±6[4], ±8[3], ±10[2], ±12, ±14
D_6	10	Δ	60	±2[6], ±4[5], ±6[5], ±8[4], ±10[4], ±12[2], ±14[2], ±16, ±18
D_7	12	Δ	84	±2[7], ±4[6], ±6[6], ±8[5], ±10[5], ±12[4], ±14[3], ±16[2], ±18[2], ±20, ±22
D_8	14	Δ	112	±2[8], ±4[7], ±6[7], ±8[6], ±10[6], ±12[5], ±14[5], ±16[3], ±18[3], ±20[2], ±22[2], ±24, ±26

(5.71)

Here is the summary of the spectra of the $\bar{L}_K$ (L_m) matrices (5.26), (5.33), (5.69), (5.71). The eigenvalues are 2 times the 'height' which is determined by the deformed Weyl vector ϱ:

$$\mathrm{Spec}(\bar{L}_K) = \{2\varrho\cdot\mu|\mu\in\mathcal{R}\}. \tag{5.72}$$

This formula applies to all the other $\bar{L}_K$ (L_m) matrices, (5.78), (5.79), (5.85), (5.86), (5.89), (5.92), (5.95), (5.98). This is to be compared with the formulae for the spectra of $\bar{M}$ matrices of Calogero system (4.52), (4.55), in which the 'height' is determined by the Weyl vector δ. The difference is visible in the non-simply laced root systems (5.78), (5.79), (5.85), (5.86), (5.95), (5.98).

Δ	$\mathcal{R}$	D	$\mathrm{Spec}(\bar{M})$
D_4	Δ	24	0, 12[6], 20[6], 24[6], 32[3], 36[2]
D_5	Δ	40	0, 16[2], 20[4], 28[5], 36[10], 40[2], 44, 48[4], 52[4], 56[4], 64, 68[2]
D_6	Δ	60	0, 20[2], 30[4], 36[5], 48[5], 50[4], 56[7], 60[2], 66[4], 68[4], 76[4], 78[4], 80[2], 84, 86[2], 92[4], 96[2], 104, 108[2]
D_7	Δ	84	0, 24[2], 42[4], 44[5], 60[5], 66[4], 68, 72[5], 80[6], 84[6], 86[4], 96[4], 102[4], 104[5], 108[2], 114[4], 116[4], 122[2], 124[4], 128[2], 132, 140[4], 144[2], 152, 156[2]
D_8	Δ	112	0, 28[2], 52[5], 56[4], 72[5], 80, 84[4], 88[5], 100[9], 108[10], 112[2], 116[4], 124, 128[8], 136[4], 140[6], 144[4], 152[4], 156[4], 160[5], 164[4], 172[4], 180[4], 184[2], 188, 196[4], 200[2], 208, 212[2]

(5.73)

The relation between $\mathrm{Tr}(\widetilde{W})$ and $\mathrm{Tr}(\bar{M})$ (5.7) is satisfied, since $F^{\Delta}(D_r) = 8r - 14$, see (4.59).

Next, let us summarize the spectra of $\widetilde{W}$, $\bar{M}$ and $\bar{L}$ of the B_r Sutherland system which are evaluated numerically. The set of short roots Δ_S is chosen for the Lax pairs. They are all 'integer valued', except for $\bar{L}$. As in the D_r case $\bar{L}^2$ is integer valued. The spectrum of $\widetilde{W}$ is

$$\begin{aligned} B_r:\quad \mathrm{Spec}(\widetilde{W}) = -\{&4(r-1)g_L + 2g_S, 4((2r-3)g_L+g_S),\ldots,\\ &2j((2r-1-j)g_L+g_S),\ldots,2(r-1)(rg_L+g_S),\\ &r((r-1)g_L+g_S)\}\end{aligned} \tag{5.74}$$

which agrees with the general formula (5.10) of the $\widetilde{W}$ spectrum, i.e. the jth entry is $4\lambda_j\cdot\varrho$, and obviously satisfies the sum rule (5.2), (5.12). The last piece corresponds to the spinor fundamental weight.

B_r root-type Lax pair (Δ_S). The spectrum of $\bar{M}$ is

$$\begin{aligned} B_r(\Delta_S):\quad \mathrm{Spec}(\bar{M}) = \mathrm{i}\{&0, 4(r-1)g_L + 2g_S[2], 4((2r-3)g_L+g_S)[2],\ldots,\\ &2j((2r-1-j)g_L+g_S)[2],\ldots,2(r-1)(rg_L+g_S)[2],\\ &r((r-1)g_L+g_S)\}\end{aligned} \tag{5.75}$$

$$= \mathrm{i}4\varrho\cdot\{0,\lambda_1[2],\lambda_2[2],\ldots,\lambda_{r-1}[2],\lambda_r\} \tag{5.76}$$

which is essentially the duplication of that of $\widetilde{W}$, except for the lowest, i.e. 0, and the eigenvalue belonging to the spinor weight. Let us note that the identity between the traces of $\widetilde{W}$ and $\bar{M}$ (5.7) is not satisfied, since B_r is not simply laced. It is simply the lack of the contribution from the 'anti-spinor weight' which is removed by the Dynkin diagram folding (5.57). The eigenvectors of $\widetilde{W}$ belonging to the degenerate eigenvalue $-r((r-1)g_L+g_S)$ are

$$v_s = (1,0,\ldots,0)^T$$

corresponding to the condition $\cos 2q_1 = 1$. The explanation of the duplication of the $\bar{M}$ spectrum is essentially the same as in the D_r case. The spectrum of $\bar{L}$ is $\sqrt{\text{integer}}$:

$$B_r(\Delta_S): \quad \mathrm{Spec}(\bar{L}) = \sqrt{g_L}\{0[2], \pm 2\sqrt{g_S}, \pm 2\sqrt{2(g_L+g_S)}, \ldots, \\ \pm 2\sqrt{j((j-1)g_L+g_S)}, \ldots, \pm 2\sqrt{(r-1)((r-2)g_L+g_S)}\}. \tag{5.77}$$

To be more precise, $\sqrt{\text{integer}}$ means that the spectrum of $\bar{L}^2$ is a quadratic polynomial in g_L and g_S with integer coefficients. For $g_L = g_S = g$ it reduces to that of minimal Lax matrix $\bar{L}_m$ of D_r (5.69). For $g_S = 0$ it reduces to that of the Lax matrix $\bar{L}$ of D_r (5.68). The modified Lax matrix $\bar{L}_K$ (see (5.32)) has simple integer spectrum, see formula (5.72):

$$B_r(\Delta_S): \quad \mathrm{Spec}(\bar{L}_K) = \{\pm g_S, \pm(2g_L+g_S), \pm(4g_L+g_S), \ldots, \\ \pm(2(r-1)g_L+g_S)\}. \tag{5.78}$$

B_r root-type Lax pair (Δ_L). The $\bar{L}_K$ matrices have simple spectrum:

Δ	$\mathcal{R}$	D	$\mathrm{Spec}(\bar{L}_K)$
B_4	Δ_L	24	$\pm 2g_L[3], \pm 4g_L[2], \pm 6g_L, \pm 2(g_L+g_S), \pm 2(2g_L+g_S),$ $\pm 2(3g_L+g_S)[2], \pm 2(4g_L+g_S), \pm 2(5g_L+g_S),$
B_5	Δ_L	40	$\pm 2g_L[4], \pm 4g_L[3], \ldots, \pm 8g_L, \pm 2(g_L+g_S), \pm 2(2g_L+g_S),$ $\pm 2(3g_L+g_S)[2], \pm 2(4g_L+g_S)[2], \pm 2(5g_L+g_S)[2], \pm 2(6g_L+g_S)$ $\pm 2(7g_L+g_S)$
B_6	Δ_L	60	$\pm 2g_L[5], \pm 4g_L[4], \ldots, \pm 10g_L, \pm 2(g_L+g_S), \pm 2(2g_L+g_S),$ $\pm 2(3g_L+g_S)[2], \pm 2(4g_L+g_S)[2], \pm 2(5g_L+g_S)[3], \pm 2(6g_L+g_S)[2]$ $\pm 2(7g_L+g_S)[2], \pm 2(8g_L+g_S), \pm 2(9g_L+g_S)$

(5.79)

The spectra of $\bar{M}$ can be expressed succinctly in terms of the fundamental weights $\{\lambda_j\}$, whose expression in terms of the coupling constants g_L and g_S can be found in (5.74). The entry λ_j means that the corresponding eigenvalue is $\mathrm{i}4\varrho\cdot\lambda_j$ etc:

Δ	$\mathcal{R}$	D	$\mathrm{Spec}(\bar{M})$
B_4	Δ_L	24	$0, \lambda_1[2], \lambda_4[2], \lambda_2[5], \lambda_3[6], (\lambda_1+\lambda_2), (\lambda_2+\lambda_4)[2], (\lambda_1+\lambda_3)[4],$ $(\lambda_2+\lambda_3)$
B_5	Δ_L	40	$0, \lambda_1[2], \lambda_5[2], \lambda_2[5], \lambda_3[5], (\lambda_1+\lambda_5)[2], \lambda_4[4], (\lambda_1+\lambda_2),$ $(\lambda_2+\lambda_5)[2], (\lambda_1+\lambda_3)[4], (\lambda_1+\lambda_4)[4], (\lambda_3+\lambda_5)[2], (\lambda_2+\lambda_3),$ $(\lambda_2+\lambda_4)[4], (\lambda_3+\lambda_4)$
B_6	Δ_L	60	$0, \lambda_1[2], \lambda_6[2], \lambda_2[5], \lambda_3[5], (\lambda_1+\lambda_6)[2], (\lambda_1+\lambda_2), \lambda_4[5], \lambda_5[4],$ $(\lambda_2+\lambda_6)[2], (\lambda_1+\lambda_3)[4], (\lambda_1+\lambda_4)[4], (\lambda_1+\lambda_5)[4], (\lambda_2+\lambda_3),$ $(\lambda_3+\lambda_6)[2], (\lambda_4+\lambda_6)[2], (\lambda_2+\lambda_4)[4], (\lambda_2+\lambda_5)[4], (\lambda_3+\lambda_4),$ $(\lambda_3+\lambda_5)[4], (\lambda_4+\lambda_5)$
B_7	Δ_L	84	$0, \lambda_1[2], \lambda_2[5], \lambda_7[2], \lambda_3[5], (\lambda_1+\lambda_2), (\lambda_1+\lambda_7)[2], \lambda_4[5], \lambda_5[5],$ $(\lambda_1+\lambda_3)[4], \lambda_6[4], (\lambda_2+\lambda_7)[2], (\lambda_1+\lambda_4)[4], (\lambda_2+\lambda_3), (\lambda_3+\lambda_7)[2],$ $(\lambda_1+\lambda_5)[4], (\lambda_1+\lambda_6)[4], (\lambda_2+\lambda_4)[4], (\lambda_4+\lambda_7)[2], (\lambda_2+\lambda_5)[4],$ $(\lambda_5+\lambda_7)[2], (\lambda_2+\lambda_6)[4], (\lambda_3+\lambda_4), (\lambda_3+\lambda_5)[4], (\lambda_3+\lambda_6)[4],$ $(\lambda_4+\lambda_5), (\lambda_4+\lambda_6)[4], (\lambda_5+\lambda_6)$

(5.80)

C_r Lax pair (**V**). Here let us summarize the spectra of $\widetilde{W}$, $\bar{M}$ and $\bar{L}$ of the C_r Sutherland system which are evaluated numerically. The set of vector weights **V** is chosen for the Lax pairs. They are all 'integer valued', except for $\bar{L}$. As in the B_r and D_r case $\bar{L}^2$ is integer valued.

The spectrum of $\widetilde{W}$ is

$$C_r: \quad \mathrm{Spec}(\widetilde{W}) = -\{4((r-1)g_M + g_L), 4((2r-3)g_M + 2g_L), \ldots, 2j((2r-1-j)g_M + 2g_S), \ldots, 2(r-1)(rg_M + 2g_L), 2r((r-1)g_M + 2g_L)\} \tag{5.81}$$

which agrees with the general formula (5.10) of the $\widetilde{W}$ spectrum, i.e. the jth entry is $4\lambda_j \cdot \varrho$, and obviously satisfies the sum rule (5.2), (5.12). The spectrum of $\bar{M}$ is

$$C_r(\mathbf{V}): \quad \mathrm{Spec}(\bar{M}) = \mathrm{i}\{0, 4((r-1)g_M + g_L)[2], 4((2r-3)g_M + 2g_L)[2], \ldots, 2j((2r-1-j)g_M + 2g_S)[2], \ldots, 2(r-1)(rg_M + 2g_L)[2], 2r((r-1)g_M + 2g_L)\} \tag{5.82}$$

$$= \mathrm{i}4\varrho \cdot \{0, \lambda_1[2], \lambda_2[2], \ldots, \lambda_{r-1}[2], \lambda_r\}, \tag{5.83}$$

which is essentially the duplication of that of $\widetilde{W}$, except for the lowest, i.e. 0, and the highest eigenvalue corresponding to the fundamental weight of the long simple root. This degeneracy pattern reflects the Dynkin diagram folding $A_{2r-1} \to C_r$. Let us note that the identity between the traces of $\widetilde{W}$ and $\bar{M}$ (5.7) is not satisfied, since C_r is not simply laced. The spectrum of $\bar{L}$ is $\sqrt{\text{integer}}$:

$$C_r(\mathbf{V}): \quad \mathrm{Spec}(\bar{L}) = \sqrt{g_M}\{0[2], \pm 2\sqrt{2g_L}, \pm 2\sqrt{2(g_M + 2g_L)}, \ldots, \pm 2\sqrt{j((j-1)g_M + 2g_L)}, \ldots, \pm 2\sqrt{(r-1)((r-2)g_M + 2g_L)}\}. \tag{5.84}$$

The modified Lax matrices $\bar{L}_K$ (see (5.32)) have simple integer spectra:

$$C_r(\mathbf{V}): \quad \mathrm{Spec}(\bar{L}_K) = \{\pm 2g_L, \pm(2g_M + 2g_L), \pm(4g_M + 2g_L), \ldots, \pm(2(r-1)g_M + 2g_L)\}. \tag{5.85}$$

C_r Root-type Lax pair (Δ_L). The $\bar{L}_K$ matrices have simple spectrum,

Δ	$\mathcal{R}$	D	$\mathrm{Spec}(\bar{L}_K)$
C_4	Δ_M	24	$\pm 2g_M[3], \pm 4g_M[2], \pm 6g_M, \pm 2(g_M + 2g_L), \pm 2(2g_M + 2g_L), \pm 2(3g_M + 2g_L)[2], \pm 2(4g_M + 2g_L), \pm 2(5g_M + 2g_L),$
C_5	Δ_M	40	$\pm 2g_M[4], \pm 4g_M[3], \ldots, \pm 8g_M, \pm 2(g_M + 2g_L), \pm 2(2g_M + 2g_L), \pm 2(3g_M + 2g_L)[2], \pm 2(4g_M + 2g_L)[2], \pm 2(5g_M + 2g_L)[2], \pm 2(6g_M + 2g_L), \pm 2(7g_M + 2g_L),$
C_6	Δ_M	60	$\pm 2g_M[5], \pm 4g_M[4], \ldots, \pm 10g_M, \pm 2(g_M + 2g_L), \pm 2(2g_M + 2g_L), \pm 2(3g_M + 2g_L)[2], \pm 4(2g_M + g_L)[2], \pm 2(5g_M + 2g_L)[3], \pm 4(3g_M + g_L)[2], \pm 2(7g_M + 2g_L)[2], \pm 2(8g_M + 2g_L), \pm 2(9g_M + 2g_L)$
C_7	Δ_M	84	$\pm 2g_M[6], \pm 4g_M[5], \ldots, \pm 12g_M, \pm 2(g_M + 2g_L), \pm 2(2g_M + 2g_L), \pm 2(3g_M + 2g_L)[2], \pm 2(4g_M + 2g_L)[2], \pm 2(5g_M + 2g_L)[3], \pm 2(6g_M + 2g_L)[3], \pm 2(7g_M + 2g_L)[3], \pm 2(8g_M + 2g_L)[2], \pm 2(9g_M + 2g_L)[2], \pm 2(10g_M + 2g_L), \pm 2(11g_M + 2g_L),$

(5.86)

The interpretation of the root-type $\bar{L}_K$ eigenvalues in terms of the 'height' of the roots is also valid for C_r. The spectra of $\bar{M}$ can be expressed succinctly in terms of the fundamental weights $\{\lambda_j\}$, whose expression in terms of the coupling constants g_L and g_M can be found in (5.81).

The entry λ_j means that the corresponding eigenvalue is $i4\varrho \cdot \lambda_j$ etc:

Δ	$\mathcal{R}$	D	Spec($\bar{M}$)
C_4	Δ_M	24	$0, \lambda_1[2], \lambda_2[5], \lambda_3[4], \lambda_4[2], (\lambda_1+\lambda_2), (\lambda_1+\lambda_3)[4], (\lambda_1+\lambda_4)[2], (\lambda_2+\lambda_3), (\lambda_2+\lambda_4)[2]$
C_5	Δ_M	40	$0, \lambda_1[2], \lambda_2[5], \lambda_3[5], \lambda_4[4], (\lambda_1+\lambda_2), \lambda_5[2], (\lambda_1+\lambda_3)[4], (\lambda_1+\lambda_4)[4], (\lambda_1+\lambda_5)[2], (\lambda_2+\lambda_3), (\lambda_2+\lambda_4)[4], (\lambda_2+\lambda_5)[2], (\lambda_3+\lambda_4), (\lambda_3+\lambda_5)[2]$
C_6	Δ_M	60	$0, \lambda_1[2], \lambda_2[5], \lambda_3[5], (\lambda_1+\lambda_2), \lambda_4[5], \lambda_5[4], (\lambda_1+\lambda_3)[4], \lambda_6[2], (\lambda_1+\lambda_4)[4], (\lambda_1+\lambda_5)[4], (\lambda_2+\lambda_3), (\lambda_1+\lambda_6)[2], (\lambda_2+\lambda_4)[4], (\lambda_2+\lambda_5)[4], (\lambda_2+\lambda_6)[2], (\lambda_3+\lambda_4), (\lambda_3+\lambda_5)[4], (\lambda_2+\lambda_6)[2], (\lambda_4+\lambda_5), (\lambda_4+\lambda_6)[2]$
C_7	Δ_M	84	$0, \lambda_1[2], \lambda_2[5], \lambda_3[5], (\lambda_1+\lambda_2), \lambda_4[5], \lambda_5[5], (\lambda_1+\lambda_3)[4], \lambda_6[4], \lambda_7[2], (\lambda_1+\lambda_4)[4], (\lambda_2+\lambda_3), (\lambda_1+\lambda_5)[4], (\lambda_1+\lambda_6)[4], (\lambda_1+\lambda_7)[2], (\lambda_2+\lambda_4)[4], (\lambda_2+\lambda_5)[4], (\lambda_2+\lambda_6)[4], (\lambda_3+\lambda_4), (\lambda_2+\lambda_7)[2], (\lambda_3+\lambda_5)[4], (\lambda_3+\lambda_6)[4], (\lambda_3+\lambda_7)[2], (\lambda_4+\lambda_5), (\lambda_4+\lambda_6)[4], (\lambda_4+\lambda_7)[2], (\lambda_5+\lambda_6), (\lambda_5+\lambda_7)[2]$

(5.87)

In the rest of this section, we list the results on the exceptional root systems using tables since most of the methods and concepts have now been explained.

5.2.3. E_r. First we list the eigenvalues of $\widetilde{W}$ (the coupling constant and minus sign removed):

Δ	r	Spec($\widetilde{W}$)
E_6	6	32[2], 44, 60[2], 84
E_7	7	54, 68, 98, 104, 132, 150, 192
E_8	8	116, 184, 228, 272, 336, 364, 440, 540

(5.88)

Of course they are equal to $\{4\varrho \cdot \lambda_j\}$. The degeneracies in E_6 spectrum reflect the symmetry in Dynkin diagram. First, let us show the eigenvalues of the Lax matrices for the set of minimal weights **27** of E_6 and **56** of E_7. In these cases, the spectrum of minimal-type L matrix and that of modified L_K are the same:

Δ	$\mathcal{R}$	D	Spec($\bar{L}_m$)
E_6	**27**	27	$0[3], \pm2[2], \pm4[2], \pm6[2], \pm8[2], \pm10, \pm12, \pm14, \pm16,$
E_7	**56**	56	$\pm1[3], \pm3[3], \pm5[3], \pm7[3], \pm9[3], \pm11[2], \pm13[2], \pm15[2], \pm17[2], \pm19, \pm21, \pm23, \pm25, \pm27,$

(5.89)

The eigenvalues of minimal $\bar{M}_m$ and those of $\bar{M}$ are slightly different. The latter are $4\varrho\cdot\lambda_j$ and their sums but those of the former are different:

Δ	$\mathcal{R}$	D	Spec($\bar{M}_m$)
E_6	**27**	27	0[3], 16[2], 32, 38[2], 46[2], 52[2], 60, 62[2], 68[2], 72[2], 80[2], 88[2], 92, 94[2], 110[2], 118[2]
E_7	**56**	56	27[2], 61[2], 7[2], 83[2], 101[2], 113[2], 115[2], 127[4], 141[2], 149[2], 151[2], 161[2], 171[2], 179[4], 189[2], 197[2], 203[2], 205[2], 211[2], 227[2], 241[2], 245[2], 247[2], 271[2], 289[2], 299[2]

(5.90)

Δ	$\mathcal{R}$	D	$\mathrm{Spec}(\bar{M})$
E_6	**27**	27	0, 32[4], 44[2], 60[6], 64, 76[2], 84[4], 92[2], 104[2], 116[2], 120,
E_7	**56**	56	0, 54[3], 68[2], 98[3], 104[4], 122[2], 132[4], 150[5], 152, 166[2], 172[2], 186[2], 192[6], 202[2], 204, 218[2], 230[2], 236[2], 246[2], 248, 260[2], 282[2], 296[2], 302

(5.91)

The relation between $\mathrm{Tr}(\widetilde{W})$ and $\mathrm{Tr}(\bar{M})$ (5.7) is satisfied, since $F^{\mathbf{27}} = 12$ in E_6 and $F^{\mathbf{56}} = 24$ in E_7, see (4.49).

Root-type Lax pair

Δ	h	$\mathcal{R}$	D	$\mathrm{Spec}(\bar{L}_K)$
E_6	12	Δ	72	$\pm2[6], \pm4[5], \pm6[5], \pm8[5], \pm10[4], \pm12[3], \pm14[3], \pm16[2]$ $\pm18, \pm20, \pm22$
E_7	18	Δ	126	$\pm2[7], \pm4[6], \pm6[6], \pm8[6], \pm10[6], \pm12[5], \pm14[5], \pm16[4], \pm18[4], \pm20[3], \pm2[3], \pm24[2], \pm26[2], \pm28, \pm30, \pm32, \pm34$
E_8	30	Δ	240	$\pm2[8], \pm4[7], \pm6[7], \pm8[7], \pm10[7], \pm12[7], \pm14[7], \pm16[6], \pm18[6], \pm20[6], \pm22[6], \pm24[5], \pm26[5], \pm28[4], \pm30[4], \pm32[4], \pm34[4], \pm36[3], \pm38[3], \pm40[2], \pm42[2], \pm44[2], \pm46[2], \pm48, \pm50, \pm52, \pm54, \pm56, \pm58$

(5.92)

In all cases, the highest multiplicity of $\bar{L}_K$ is the rank r and the highest eigenvalue is $2(h-1)$ with interval 2. Thus the multiplicity distribution of the eigenvalues of $\bar{L}_K$ of the root-type Lax matrix is the number of roots having the specified (2 times the) height. As in all the other cases, the eigenvalues of $\bar{M}$ are of the form $\mathrm{i}\sum_{j=1}^{r} a_j(4\varrho\cdot\lambda_j)$, in which $a_j = 0, 1$:

Δ	h	$\mathcal{R}$	D	$\mathrm{Spec}(\bar{M})$
E_6	12	Δ	72	0, 32[4], 44[4], 60[10], 64[2], 76[4], 84[8], 92[10], 104[4], 108[2], 116[8], 120[4], 128, 136[4], 144[2], 148[2], 164[2]
E_7	18	Δ	126	0, 54[2], 68[4], 98[4], 104[5], 122[2], 132[7], 150[7], 152[4], 158, 166[2], 172[4], 186[4], 192[10], 200, 202[4], 204[4], 218[4], 220[2], 230[2], 236[6], 246[6], 248[4], 254, 260[4], 270[2], 272[2], 282[4], 284[2], 290, 296[6], 314[2], 316[2], 324, 334[2], 336[2], 342, 364[2], 380[2]
E_8	30	Δ	240	0, 116[4], 184[4], 228[7], 272[6], 300[4], 336[9], 344, 364[9], 388[4], 412[6], 440[11], 452[4], 456[4], 480[6], 500[6], 520[6], 540[14], 548, 556[6], 564, 572[2], 592[8], 608[6], 624[6], 636[6], 65[8], 668[6], 684[2], 700[8], 712[4], 724[8], 740[2], 752[2], 768[10], 776, 792[2], 804[6], 812, 816[2], 828[2], 840[2], 852[2], 864[2], 876[8], 896[2], 904, 920[2], 940[2], 952[2], 972[2], 980, 992[2], 1032[2], 1060[2], 1076[2]

(5.93)

The relation between $\mathrm{Tr}(\widetilde{W})$ and $\mathrm{Tr}(\bar{M})$ (5.7) is satisfied, since $F^{\Delta}(E_6) = 42$, $F^{\Delta}(E_7) = 66$ and $F^{\Delta}(E_8) = 114$, see (4.59).

5.2.4. F_4. The eigenvalues of $\widetilde{W}$ are

$$F_4: \quad \mathrm{Spec}(\widetilde{W}) = -\{20g_L + 12g_S, 36g_L + 24g_S, 24g_L + 18g_S, 12g_L + 10g_S\}$$
$$= -4\varrho\cdot\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \}. \tag{5.94}$$

The eigenvalues of the modified Lax matrix $\bar{L}_K$ are simple:

Δ	$\mathcal{R}$	D	Spec($\bar{L}_K$)
F_4	Δ_L	24	$\pm(10g_L+6g_S), \pm(8g_L+6g_S), \pm(6g_L+6g_S), \pm(6g_L+4g_S), \pm(4g_L+4g_S), \pm(6g_L+2g_S), \pm(2g_L+4g_S), \pm(4g_L+2g_S), \pm 4g_L, \pm(2g_L+2g_S), \pm 2g_L[2]$
F_4	Δ_S	24	$(6g_L+5g_S), \pm(6g_L+4g_S), \pm(6g_L+3g_S), \pm(4g_L+3g_S), \pm(4g_L+2g_S), \pm(2g_L+3g_S), \pm(4g_L+g_S), \pm(2g_L+2g_S), \pm(2g_L+g_S), \pm 2g_S, \pm g_S[2]$

(5.95)

The interpretation of the root-type $\bar{L}_K$ eigenvalues in terms of the 'height' of the roots is also valid for F_4. The eigenvalues of $\bar{M}$ can be expressed succinctly in terms of the fundamental weights $\{\lambda_j\}$, which are listed in (5.94). The entry λ_j means that the corresponding eigenvalue is $\mathrm{i}4\varrho\cdot\lambda_j$ etc:

Δ	$\mathcal{R}$	D	Spec($\bar{M}$)
F_4	Δ_L	24	$0, \lambda_4[2], \lambda_1[4], \lambda_3[4], (\lambda_1+\lambda_4)[2], \lambda_2[6], (\lambda_1+\lambda_3)[2], (\lambda_2+\lambda_4)[2], (\lambda_1+\lambda_2)$
F_4	Δ_S	24	$0, \lambda_4[4], \lambda_1[2], \lambda_3[6], (\lambda_1+\lambda_4)[2], \lambda_2[4], (\lambda_3+\lambda_4), (\lambda_1+\lambda_3)[2], (\lambda_2+\lambda_4)[2]$

(5.96)

5.2.5. G_2. The eigenvalues of $\widetilde{W}$ are

$$\begin{aligned} G_2: \quad \mathrm{Spec}(\widetilde{W}) &= -\{4g_L+8g_S/3, 8g_L+4g_S,\} \\ &= -4\varrho\cdot\{\lambda_1,\lambda_2\}. \end{aligned} \tag{5.97}$$

The eigenvalues of the modified Lax matrix $\bar{L}_K$ are simple:

Δ	$\mathcal{R}$	D	Spec($\bar{L}_K$)
G_2	Δ_L	6	$\pm(4g_L+2g_S), \pm(2g_L+2g_S), \pm 2g_L$
G_2	Δ_S	6	$\pm(2g_L+(4g_S/3)), \pm(2g_L+(2g_S/3)), \pm 2g_S/3$

(5.98)

The eigenvalues of $\bar{M}$ can be expressed succinctly in terms of the fundamental weights $\{\lambda_j\}$, which are listed in (5.97). The entry λ_j means that the corresponding eigenvalue is $\mathrm{i}4\varrho\cdot\lambda_j$ etc:

Δ	$\mathcal{R}$	D	Spec($\bar{M}$)
G_2	Δ_L	6	$0, \lambda_1[2], \lambda_2[3],$
G_2	Δ_S	6	$0, \lambda_1[3], \lambda_2[2],$

(5.99)

6. Comments and discussion

We have shown that the classical Calogero and Sutherland systems at their equilibrium points have very interesting properties. The equilibrium point is related to the zeros of classical polynomials of Hermite, Laguerre and Jacobi types. The second derivatives of the potential have 'integer eigenvalues', and various Lax matrices also have 'integer eigenvalues' at the equilibrium point. Most of these results are obtained by numerical evaluation and it remains a real challenge to derive these 'integer eigenvalues' analytically.

In this connection, it is interesting to compare with the situation of another well-known set of integrable multiparticle dynamical systems based on crystallographic root systems—the

Toda systems. Since the non-affine Toda molecule systems do not have a finite equilibrium point, we only consider the affine Toda molecule of the root system Δ,

$$H = \frac{1}{2}p^2 + V_{\text{Toda}}(q) \qquad V_{\text{Toda}}(q) = \frac{1}{\beta^2}\sum_{j=0}^{r} n_j\, e^{\beta\alpha_j\cdot q} \tag{6.1}$$

in which $\{\alpha_1, \ldots, \alpha_r\}$ are the simple roots of Δ and

$$\alpha_0 = -\sum_{j=1}^{r} n_j\alpha_j \qquad n_0 = 1 \tag{6.2}$$

is the Euclidean part of the additional affine simple root. The integers $\{n_j\}$, $j = 1, \ldots, r$ are called Coxeter labels and β is the real coupling constant. The above potential is so chosen as to have the equilibrium point

$$\bar{q} = (0, 0, \ldots, 0). \tag{6.3}$$

The eigenvalues of the second derivatives of the potential

$$V''_{\text{Toda}}(0) = \sum_{j=0}^{r} n_j\alpha_j \otimes \alpha_j \tag{6.4}$$

are not integers but so-called affine Toda masses $\{m_1^2, \ldots, m_r^2\}$, corresponding to the Perron–Frobenius eigenvector of the incidence matrix of the root system Δ, [28]. Since the Lax pair of the Toda molecules is expressed in terms of the coordinates q and the Lie algebra generators corresponding to Δ [29], the eigenvalues of the Lax pair matrices at the equilibrium point are completely determined by the chosen representation of the Lie algebra.

Note added in proof. The equilibrium positions of BC_r Sutherland systems and Jacobi polynomials were discussed in [30]. We thank A Perelomov for this information.

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Appendix. Eigenvalues of the *K* matrix

Here we show that the constant matrix K defined in (2.40),

$$K \equiv \sum_{\rho\in\Delta_+} g_\rho(\rho\cdot\hat{H})(\rho^\vee\cdot\hat{H})\hat{s}_\rho \tag{A.1}$$

has a remarkable property that its eigenvalues are all *integer* $\times$ *coupling constant*. The $\tilde{K}$ matrix (5.32) has a similar property. This matrix plays an important role in the theory of classical r-matrix of Calogero–Moser systems [27]. First, we note that it is Coxeter invariant and symmetric,

$$\hat{s}_\sigma K\hat{s}_\sigma = K \qquad \forall\sigma\in\Delta \quad K^T = K \tag{A.2}$$

implying that the eigenvalues are real, and the eigenvectors span representation spaces of the Weyl group whose dimensions are the multiplicities given in the tables below. As simple examples, we indicate, for the A_r root system, the decomposition of $\mathcal{R}$ into the irreducible representations of the Weyl group, which is the symmetric group. The diagonal elements of K are all vanishing,

$$K_{\mu\mu} = \sum_{\rho\in\Delta_+} g_\rho(\rho\cdot\mu)(\rho^\vee\cdot\mu)\delta_{\mu,s_\rho(\mu)} = 0 \tag{A.3}$$

since $\mu - s_\rho(\mu) = (\rho^\vee\cdot\mu)\rho = 0$ is necessary for the Kronecker delta to be non-vanishing. Thus it is traceless,

$$\mathrm{Tr}\, K = 0 \tag{A.4}$$

which is also obvious from the definition as a commutator (2.40). Another important property is that it commutes with M,

$$[K, M] = 0 \tag{A.5}$$

at the general position q for both Calogero and Sutherland systems. All the matrix elements of K are non-negative and the eigenvector for the highest eigenvalue (the Perron–Frobenius eigenvector) is in fact $\boldsymbol{v}_0$ (4.37), which is a singlet representation of the Weyl group:

$$K\boldsymbol{v}_0 = \lambda_{PF}\boldsymbol{v}_0 \qquad \lambda_{PF} = 2\mu^2\left(\sum_{\rho\in\Delta_+} g_\rho\right)\Bigg/ r = 2\mu^2\tilde{\mathcal{E}}_0/\omega r. \tag{A.6}$$

Other important eigenvectors of K are given by

$$Q\boldsymbol{v}_0 \tag{A.7}$$

in which Q is defined by (2.39) and $\boldsymbol{v}_0$ is the above Perron–Frobenius eigenvector introduced in (4.37). For all possible values of the coordinates $q = (q_1, \ldots, q_r)$, it is always an eigenvector of K,

$$KQ\boldsymbol{v}_0 = \lambda_Q Q\boldsymbol{v}_0 \tag{A.8}$$

in which the eigenvalue λ_Q is expressed by boldface fonts in the formulae from (A.13) to (A.37). This eigenvalue is usually r (rank) fold degenerate and the corresponding eigenvectors form an ever present r-dimensional irreducible representation of the Weyl group. Exceptional situations of additional degeneracies occur in A_7 root-type (A.17), D_4, D_6 and E_6 root-type (A.26), (A.31) and H_3 and H_4 root-type (A.34). For the cases when $\mathcal{R}$ is the set of minimal weights, (A.13), (A.23) and (A.29), λ_Q is related to λ_{PF} by the Coxeter number h:

$$\lambda_Q = \lambda_{PF} - gh. \tag{A.9}$$

For the crystallographic simply laced root-type cases, (A.17), (A.26) and (A.31), we have

$$\lambda_Q = (h-6)g. \tag{A.10}$$

If the set $\mathcal{R}$ consists of minimal weights (2.32), all the matrix elements of K are either 1 or 0 times the coupling constant. If the set $\mathcal{R}$ coincides with the set of all roots Δ for a crystallographic simply laced root system, the matrix element of K is characterized by the inner products of the roots (the roots are normalized as $\alpha^2 = 2$),

$$K_{\alpha\beta} = g\begin{cases} 4 & \text{if} \quad \alpha\cdot\beta = -2 \text{ i.e. } \alpha = -\beta \\ 1 & \text{if} \quad \alpha\cdot\beta = 1 \\ 0 & \text{otherwise} \end{cases} \tag{A.11}$$

and a similar statement holds for non-simply laced crystallographic root systems.

We list the spectrum of K, i.e. set of eigenvalues with [multiplicity] for all Δ and for typical choices of the set of single Coxeter (Weyl) orbits $\mathcal{R}$ for which the typical Lax pairs are known. For the simply laced root systems, we omit the coupling constant g in the spectrum. In these formulae h denotes the Coxeter number.

(1) A_r with vector weights embedded in $\mathbf{R}^{r+1}$, i.e.

$$\mathcal{R} = \mathbf{V} = \{\boldsymbol{e}_j, j = 1, \ldots, r+1 | \boldsymbol{e}_j \in \mathbf{R}^{r+1}, \boldsymbol{e}_j \cdot \boldsymbol{e}_k = \delta_{jk}\} \tag{A.12}$$

Δ	h	$\mathcal{R}$	D	μ^2	Spec(K)
A_r	$r+1$	$\mathbf{V}$	$r+1$	1	$r, -\mathbf{1}[r]$

(A.13)

corresponding to the following decomposition into the irreducible representation of the Weyl group:

$$(1+r) = 1 \oplus r. \tag{A.14}$$

In this case K has a very simple expression in terms of $\boldsymbol{v}_0$:

$$K = g\left(\boldsymbol{v}_0\boldsymbol{v}_0^T - I\right) \qquad I : \text{identity matrix.} \tag{A.15}$$

The matrix elements of K are also characterized by the inner products:

$$K_{\mu\nu} = g\begin{cases}1 & \text{if} \quad \mu\cdot\nu = 0\\ 0 & \text{otherwise}\end{cases} \qquad \mu, \nu \in \mathbf{V}. \tag{A.16}$$

(2) A_r with roots $\mathcal{R} = \Delta$,

Δ	h	$\mathcal{R}$	D	μ^2	Spec(K)
A_2	3	Δ	6	2	$6, 3[2], -\mathbf{3}[2], -6$
A_3	4	Δ	12	2	$8, 4[3], 2[2], -\mathbf{2}[3], -6[3]$
A_4	5	Δ	20	2	$10, 5[4], 2[5], -\mathbf{1}[4], -6[6]$
A_5	6	Δ	30	2	$12, 6[5], 2[9], \mathbf{0}[5], -6[10]$
A_6	7	Δ	42	2	$14, 7[6], 2[14], \mathbf{1}[6], -6[15]$
A_7	8	Δ	56	2	$16, 8[7], \mathbf{2}[27], -6[21]$
A_8	9	Δ	72	2	$18, 9[8], \mathbf{3}[8], 2[27], -6[28]$
A_9	10	Δ	90	2	$20, 10[9], \mathbf{4}[9], 2[35], -6[36]$
A_{10}	11	Δ	110	2	$22, 11[10], \mathbf{5}[10], 2[44], -6[45]$
A_{11}	12	Δ	132	2	$24, 12[11], \mathbf{6}[11], 2[54], -6[55]$
A_{12}	13	Δ	156	2	$26, 13[12], \mathbf{7}[12], 2[65], -6[66]$
A_r	$r+1$	Δ	$r(r+1)$	2	$2h, h[r], (\mathbf{h}-\mathbf{6})[r],$ $2[(r+1)(r-2)/2], -6[r(r-1)/2]$

(A.17)

corresponding to the following decomposition into the irreducible representations of A_r Weyl group,

$$r(r+1) = 1 \oplus r \oplus r' \oplus (r+1)(r-2)/2 \oplus r(r-1)/2 \tag{A.18}$$

in which r and r' are two distinct r-dimensional irreducible representations.

(3) B_r with short roots:

$$\mathcal{R} = \Delta_S = \{\pm\boldsymbol{e}_j, j = 1, \ldots, r | \boldsymbol{e}_j \in \mathbf{R}^r, \boldsymbol{e}_j \cdot \boldsymbol{e}_k = \delta_{jk}\} \tag{A.19}$$

Δ	h	$\mathcal{R}$	D	μ^2	Spec(K)
B_r	$2r$	Δ_S	$2r$	1	$2g_L(r-1)+2g_S, -\mathbf{2g_S}[r], -2g_L+2g_S[r-1]$

(A.20)

The C_r in the vector representation $\mathcal{R} = \mathbf{V} = \{\pm\boldsymbol{e}_j\}$, $j = 1, \ldots, r$ is the same as above if the coupling constants are interchanged, $g_S \leftrightarrow g_L$. Similarly to the A_r vector weight case (A.15), we have

$$K = g_L\left(\boldsymbol{v}_0\boldsymbol{v}_0^T - I - {}_SI\right) + 2g_{SS}I \tag{A.21}$$

in which ${}_SI$ is the second identity matrix. It is 1 for the elements $(\boldsymbol{e}_j, -\boldsymbol{e}_j)$, $(-\boldsymbol{e}_j, \boldsymbol{e}_j)$, $j = 1, \ldots, r$ and 0 otherwise.

(4) D_r with the vector weights:

$$\mathcal{R} = \mathbf{V} = \{\pm e_j | j = 1, \ldots, r\} \qquad \text{(A.22)}$$

Δ	h	$\mathcal{R}$	D	μ^2	Spec(K)
D_r	$2(r-1)$	**V**	$2r$	1	$2(r-1)$, $\mathbf{0}[r]$, $-2[r-1]$

(A.23)

Similarly to the C_r vector weight case (A.21), we have an expression

$$K = g\left(v_0 v_0^T - I - {}_S I\right). \qquad \text{(A.24)}$$

This K matrix is also characterized by the inner product as in (A.16).

(5) D_r with the (anti) spinor weights:

$$\mathcal{R} = \mathbf{S} = \tfrac{1}{2}\{\pm e_1 \pm \cdots \pm e_r\} \qquad \text{with even (odd) number of } - \text{ signs} \qquad \text{(A.25)}$$

Δ	h	$\mathcal{R}$	D	μ^2	Spec(K)
D_4	6	**S**	8	1	6, **0**[4], −2[3]
D_5	8	**S**	16	5/4	10, **2**[5], −2[10]
D_6	10	**S**	32	3/2	15, **5**[6], −1[15], −3[10]
D_7	12	**S**	64	7/4	21, **9**[7], 1[21], −3[35]
D_8	14	**S**	128	2	28, **14**[8], 4[28], −2[56], −4[35]
D_9	16	**S**	256	9/4	36, **20**[9], 8[36], 0[84], −4[126]
D_{10}	18	**S**	512	5/2	45, **27**[10], 13[45], 3[120], −3[210], −5[126]
D_{11}	20	**S**	1024	11/4	55, **35**[11], 19[55], 7[165], −1[330], −5[462]

(A.26)

The characterization of the spinor K matrix is a bit different:

$$K_{\mu\nu} = g \begin{cases} 1 & \text{if} \quad \mu \cdot \nu = (r-4)/4 \\ 0 & \text{otherwise} \end{cases} \qquad \mu, \nu \in \mathbf{S}. \qquad \text{(A.27)}$$

(6) D_r with the roots i.e. $\mathcal{R} = \Delta$:

Δ	h	$\mathcal{R}$	D	μ^2	Spec(K)
D_4	6	Δ	24	2	12, 4[9], **0**[6], −6[8]
D_5	8	Δ	40	2	16, 6[4], 4[10], **2**[5], 0[5], −6[15]
D_6	10	Δ	60	2	20, 8[5], **4**[21], 0[9], −6[24]
D_7	12	Δ	84	2	24, 10[6], **6**[7], 4[21], 0[14], −6[35]
D_8	14	Δ	112	2	28, 12[7], **8**[8], 4[28], 0[20], −6[48]
D_9	16	Δ	144	2	32, 14[8], **10**[9], 4[36], 0[27], −6[63]
D_{10}	18	Δ	180	2	36, 16[9], **12**[10], 4[45], 0[35], −6[80]
D_{11}	20	Δ	220	2	40, 18[10], **14**[11], 4[55], 0[44], −6[99]
D_r	$2(r-1)$	Δ	$2r(r-1)$	2	$2h$, $2(h-1)\ [r-1]$, $(\mathbf{h-6})[r]$, $4[r(r-1)/2]$, $0[r(r-3)/2]$, $-6[r(r-2)]$

(A.28)

(7) E_r with the minimal weights:

Δ	h	$\mathcal{R}$	D	μ^2	Spec(K)
E_6	12	**27**	27	4/3	6, **4**[6], −2[20]
E_7	18	**56**	56	3/2	27, **9**[7], −1[27], −3[21]

(A.29)

These K matrices are characterized by

$$K_{\mu\nu} = g\begin{cases}1 & \text{if} \quad \mu\cdot\nu = 1/3 \\ 0 & \text{otherwise}\end{cases} \qquad \mu,\nu\in\mathbf{27}$$
$$g\begin{cases}1 & \text{if} \quad \mu\cdot\nu = 1/2 \\ 0 & \text{otherwise}\end{cases} \qquad \mu,\nu\in\mathbf{56}. \tag{A.30}$$

(8) E_r with the roots i.e. $\mathcal{R} = \Delta$:

Δ	h	$\mathcal{R}$	D	μ^2	Spec(K)
E_6	12	Δ	72	2	24, **6**[26], 0[15], −6[30]
E_7	18	Δ	126	2	36, **12**[7], 8[27], 0[35], −6[56]
E_8	30	Δ	240	2	60, **24**[8], 12[35], 0[84], −6[112]
E_r	h	Δ	D	2	$2h$, $(\mathbf{h}-\mathbf{6})[r]$, $(h/3+2)[(r-1)(r+2)/2]$, $0[(D-r(r+1))/2]$, $-6[D/2-r]$

(A.31)

(9) F_4 with the long roots i.e. $\mathcal{R} = \Delta_L$:

Δ	h	$\mathcal{R}$	D	μ^2	Spec(K)
F_4	12	Δ_L	24	2	$12(g_L+g_S)$, $12g_S[2]$, $4(g_L-g_S)[9]$, $\mathbf{0}[4]$, $-6g_L[8]$

(A.32)

(10) G_2 with the long roots i.e. $\mathcal{R} = \Delta_L$:

Δ	h	$\mathcal{R}$	D	μ^2	Spec(K)
G_2	6	Δ_L	6	2	$6(g_L+g_S)$, $3(g_L-g_S)[2]$, $-\mathbf{3}(g_L+g_S)[2]$, $6(-g_L+g_S)$

(A.33)

(11) H_r with the roots i.e. $\mathcal{R} = \Delta$:

Δ	h	$\mathcal{R}$	D	μ^2	Spec(K)
H_3	10	Δ	30	1	10, 4[5], 3[3], 0[9], −**2**[7], −5[5]
H_4	30	Δ	120	1	30, 15[4], 10[9], **0**[70], −5[36]

(A.34)

These K matrices for H_3 and H_4 are characterized by

$$K_{\alpha\beta} = g\begin{cases}1 & \text{if} \quad \alpha\cdot\beta = -1 \text{ i.e. } \alpha = -\beta \\ 1/2 & \text{if} \quad \alpha\cdot\beta = 1/2 \\ (3\pm\sqrt{5})/4 & \text{if} \quad \alpha\cdot\beta = (1\mp\sqrt{5})/4 \\ 0 & \text{otherwise.}\end{cases} \tag{A.35}$$

(12) $I_2(m)$ in the m-dimensional representation consisting of the vertices of the regular m-gon $\mathcal{R} = R_m$:

$$R_m = \{\sqrt{2}(\cos(2k\pi/m+t_0), \sin(2k\pi/m+t_0)) \in \mathbf{R}^2 | k = 1,\ldots,m\}$$
$$t_0 = 0, (\pi/2m) \qquad \text{for} \quad m \text{ even (odd)}. \tag{A.36}$$

Δ	h	$\mathcal{R}$	D	μ^2	Spec(K)
$I_2(2n+1)$	$2n+1$	R_{2n+1}	$2n+1$	2	$2(2n+1)$, $0[2n-2]$, $-(\mathbf{2n}+\mathbf{1})[2]$
$I_2(2n)$	$2n$	R_{2n}	$2n$	2	$2n(g_o+g_e)$, $(-1)^n n(g_o-g_e)[2]$, $0[2n-6]$, $-\mathbf{n}(g_o+g_e)[2]$, $-(-1)^n 2n(g_o-g_e)$

(A.37)

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